

ARTICLE

Non-Gaussian score-driven conditionally heteroskedastic models with a macroeconomic application

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Abstract

We contribute to the literature on empirical macroeconomic models with time-varying conditional moments, by introducing a heteroskedastic score-driven model with Student's t -distributed innovations, named the heteroskedastic score-driven t -QVAR (quasi-vector autoregressive) model. The t -QVAR model is a robust nonlinear extension of the VARMA (VAR moving average) model. As an illustration, we apply the heteroskedastic t -QVAR model to a dynamic stochastic general equilibrium model, for which we estimate Gaussian-ABCD and t -ABCD representations. We use data on economic output, inflation, interest rate, government spending, aggregate productivity, and consumption of the USA for the period of 1954 Q3 to 2022 Q1. Due to the robustness of the heteroskedastic t -QVAR model, even including the period of the coronavirus disease of 2019 (COVID-19) pandemic and the start of the Russian invasion of Ukraine, we find a superior statistical performance, lower policy-relevant dynamic effects, and a higher estimation precision of the impulse response function for US gross domestic product growth and US inflation rate, for the heteroskedastic score-driven t -ABCD representation rather than for the homoskedastic Gaussian-ABCD representation.

Keywords: Dynamic conditional score; generalized autoregressive score; score-driven location plus score-driven scale models; heteroskedastic score-driven t -QVAR model; Maximum likelihood

1. Introduction

This paper contributes to the literature on empirical macroeconomic models with time-varying conditional moments, by introducing the heteroskedastic score-driven state-space model with Student's t -distributed innovations, in which both location and scale are score-driven, and illustrating the model performance in an empirical macroeconomic model applied to a well-known dynamic stochastic general equilibrium (DSGE) model [Kydland and Prescott (1982), Long and Plosser (1983)].

From a methodological perspective, this paper contributes to the well-developed literature on modeling time variation in multivariate econometric models. In the works of Canova (1993), Sims (1993), Stock and Watson (1996), and Cogley and Sargent (2001), vector autoregressive (VAR) models with drifting coefficients and homoskedastic errors are used. By using multivariate stochastic volatility models [Harvey, et al. (1994)] for the innovations, the works of Cogley and Sargent (2005) and Primiceri (2005) present VAR models with drifting coefficients and stochastic volatility. In the work of Chan and Eisenstat (2018), the performances of VAR models with drifting coefficients and stochastic volatilities, and VAR models with time-invariant coefficients and stochastic volatilities are compared. Those authors find that, given conditional heteroskedasticity,

there is no need for drifting coefficients in the VAR model. The empirical macroeconomic model of our paper fits into that literature. In the heteroskedastic score-driven state-space model with Student's t -distributed innovations [also named heteroskedastic t -QVAR, quasi-VAR, model], the coefficients are time-invariant and the error terms are conditionally heteroskedastic, similar to the best-performing model of Chan and Eisenstat (2018). The t -QVAR model is a nonlinear extension of the VARMA (VAR moving average) model.

Score-driven models [Creal et al. (2008, 2011, 2013); Harvey and Chakravarty (2008), Harvey (2013)] are observation-driven models [Cox (1981)], in which the filters are updated by the scaled partial derivatives of the log-likelihood (LL) of the dependent variables with respect to dynamic parameters (i.e., scaled score functions). The following statistical advantages of score-driven time series models motivate their use in our paper: (i) score-driven models are robust to outliers and missing data [Harvey (2013), Blazsek and Escribano (2016, 2022), Ayala et al. (2022)]. (ii) Score-driven models are generalizations of observation-driven time series models [e.g., ARMA model; generalized AR conditional heteroskedasticity model; VARMA model] [see in the work of Harvey (2013)]. (iii) In the work of Blasques et al. (2015) it is shown for score-driven models that a score-driven update locally reduces the Kullback–Leibler divergence in expectation at every step asymptotically.

We use the heteroskedastic t -QVAR model to estimate a heteroskedastic score-driven t -ABCD representation of the DSGE model of An and Schorfheide (2007). The t -ABCD representation is an extension of the Gaussian-ABCD representation [Fernández-Villaverde et al. (2007)] and the heteroskedastic score-driven Gaussian-ABCD representation [Angelini and Gorgi (2018)]. We report results by using data on economic output, inflation, interest rate, government spending, aggregate productivity, and consumption of the USA for the period of 1954 Q3 to 2022 Q1. We find a superior statistical performance, lower policy-relevant dynamic effects, and a higher estimation precision of the impulse response function (IRF) for US gross domestic product (GDP) growth and US inflation rate, for the heteroskedastic score-driven t -ABCD representation rather than for the homoskedastic Gaussian-ABCD representation. This is due to the robustness of the heteroskedastic t -QVAR model.

The remainder of this paper is organized as follows: Section 2 presents the econometric model. Section 3 presents the empirical application. Section 4 concludes.

2. Score-driven empirical macroeconomic models with heteroskedastic innovations

2.1. A Gaussian linear state-space model

For the purpose of explanation, we start with a Gaussian linear state-space model, which is extended to the heteroskedastic t -QVAR model in this paper. The observable variables are collected into the vector Y_t ($\mathcal{N} \times 1$) and the state variables are collected into the vector X_t ($\mathcal{N} \times 1$). The first representation of the Gaussian linear state-space model is the following system of equations:

$$X_t = AX_{t-1} + B\epsilon_t \tag{1}$$

$$Y_t = CX_{t-1} + D\epsilon_t \tag{2}$$

where ϵ_t ($\mathcal{N} \times 1$) has an i.i.d. multivariate normal distribution $\epsilon_t \sim N(0, \Omega\Omega')$ and a diagonal covariance matrix $\Omega\Omega'$. The parameter matrices are A , B , C , and D , which are square matrices with dimensions $(\mathcal{N} \times \mathcal{N})$. If matrix D is non-singular, then, by using the reduced-form error $v_t \equiv D\epsilon_t$, an equivalent representation of the system of Equations (1) and (2) is

$$X_t = AX_{t-1} + BD^{-1}v_t \tag{3}$$

$$Y_t = CX_{t-1} + v_t \tag{4}$$

where $v_t \sim N(0, \Sigma) \equiv N[0, D\Omega\Omega'D']$ is the multivariate i.i.d. reduced-form error term. Moreover, if C is non-singular, then we get the following Gaussian-VARMA(1,1) representation:

$$Y_t = CAC^{-1}Y_{t-1} + (CBD^{-1} - CAC^{-1})v_{t-1} + v_t \tag{5}$$

The reduced-form representation of Equations (3) and (4) is extended to the heteroskedastic t -QVAR model according to the following points. In the heteroskedastic t -QVAR model, (i) v_t has a multivariate t -distribution with degrees of freedom ν , (ii) v_t is conditionally heteroskedastic, and (iii) the updating term v_t in Equation (3) is replaced by the score function u_t , for which we have that $u_t \rightarrow_p v_t$ if $\nu \rightarrow \infty$. The resulting heteroskedastic t -QVAR model is presented in the following section.

2.2. Heteroskedastic score-driven t -QVAR model

The first representation of the heteroskedastic t -QVAR model is

$$X_t = AX_{t-1} + BD^{-1}u_t \tag{6}$$

$$Y_t = CX_{t-1} + D\epsilon_t \tag{7}$$

where D is non-singular and $\epsilon_t|\mathcal{F}_{t-1} \sim t(0, \Omega_t\Omega'_t, \nu)$ is a multivariate t -distribution with scale matrix $\Omega_t\Omega'_t$ and degrees of freedom $\nu > 2$, where the sigma algebra is $\mathcal{F}_{t-1} = \sigma(Y_1, \dots, Y_{t-1}, X_1, \Omega_1)$. We use parameter vector X_0 , which is jointly estimated with the rest of the time-invariant parameters, to initialize the state variables X_t . Technical details for the updating term u_t (i.e., scaled score function) are presented in Section 2.3. By using the reduced-form error $v_t \equiv D\epsilon_t$, the second representation is

$$X_t = AX_{t-1} + BD^{-1}u_t \tag{8}$$

$$Y_t = CX_{t-1} + v_t \tag{9}$$

where $v_t|\mathcal{F}_{t-1} \sim t(0, \Sigma_t, \nu) \equiv t(0, D\Omega_t\Omega'_tD', \nu)$. By comparing the models of Equations (8) and (9) and Equations (3) and (4), the mathematical formulas indicate that v_t of Equation (3) is replaced by u_t , and the error term with normal distribution of Equation (4) is replaced by an error term with t -distribution. If C is non-singular, then we get a nonlinear t -VARMA(1,1) with heteroskedastic errors representation:

$$Y_t = CAC^{-1}Y_{t-1} + CBD^{-1}u_{t-1} - CAC^{-1}v_{t-1} + v_t \tag{10}$$

2.3. Score functions

The observation-driven updates of filter X_t and the scale matrix $\Omega_t\Omega'_t$ of the t -distribution are defined as follows, respectively: First, for the multivariate t -distribution, the log conditional density of Y_t is

$$\begin{aligned} \ln f(Y_t|\mathcal{F}_{t-1}, \Theta) &= \ln \Gamma\left(\frac{\nu + \mathcal{N}}{2}\right) - \ln \Gamma\left(\frac{\nu}{2}\right) - \frac{\mathcal{N}}{2} \ln(\pi\nu) \\ &\quad - \frac{1}{2} \ln |\Sigma_t| - \frac{\nu + \mathcal{N}}{2} \ln \left(1 + \frac{v'_t \Sigma_t^{-1} v_t}{\nu}\right) \end{aligned} \tag{11}$$

where $v_t = Y_t - CX_{t-1}$ and Θ is the vector of the time-invariant parameters with dimension S . The derivative of the log conditional density, with respect to the conditional mean CX_{t-1} , is [Harvey (2013)]:

$$\frac{\partial \ln f(Y_t|\mathcal{F}_{t-1}, \Theta)}{\partial [CX_{t-1}]} = \frac{\nu + \mathcal{N}}{\nu} \Sigma_t^{-1} \times \left(1 + \frac{v'_t \Sigma_t^{-1} v_t}{\nu}\right)^{-1} v_t \equiv \frac{\nu + \mathcal{N}}{\nu} \Sigma_t^{-1} \times u_t \tag{12}$$

where the last equality defines the scaled score function u_t with respect to location, that is,

$$u_t = \left(1 + \frac{v_t' \Sigma_t^{-1} v_t}{\nu} \right)^{-1} v_t = \left[1 + \frac{\epsilon_t' D' \Sigma_t^{-1} D \epsilon_t}{\nu} \right]^{-1} D \epsilon_t \tag{13}$$

which, asymptotically and at the true values of parameters Θ_0 , is an uncorrelated white noise vector; the conditions of which are presented in the Supplementary Material.

Second, the diagonal of the scale matrix $\Omega_t \Omega_t'$ is driven by score function e_t . For the heteroskedastic t -QVAR model, $\Omega_t \Omega_t'$ is formulated as follows:

$$\Omega_t \Omega_t' = \begin{bmatrix} \exp(2\lambda_{1,t}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \exp(2\lambda_{\mathcal{N},t}) \end{bmatrix} \tag{14}$$

where the filters within the diagonal of $\Omega_t \Omega_t'$ are score-driven, and specified as follows:

$$\lambda_{i,t} = \omega_i + \beta_i \lambda_{i,t-1} + \alpha_i e_{i,t-1} + \alpha_i^* \text{sgn}(-\epsilon_{i,t-1})(e_{i,t-1} + 1) \equiv \omega_i + \beta_i \lambda_{i,t-1} + g(e_{i,t-1}) \tag{15}$$

for $i = 1, \dots, \mathcal{N}$, where $\text{sgn}(x)$ is the signum function, and α_i^* measures asymmetric effects in log-scale. The specification of each element in the diagonal of Ω_t is the Beta- t -EGARCH(1,1) with leverage effects model [Harvey and Chakravarty (2008)]. The score function $e_{i,t}$ is defined in what follows.

The marginal distribution of the multivariate t -distribution is the univariate Student's t -distribution, where the degrees of freedom parameters coincide. The partial derivatives of the log marginal densities of the univariate Student's t -distributions $\ln f_i(Y_{i,t} | \mathcal{F}_{t-1}, \Theta)$, with respect to $\lambda_{i,t}$, for $i = 1, \dots, \mathcal{N}$, are the following score functions with respect to log-scale [Harvey and Chakravarty (2008)]:

$$e_{i,t} = \frac{\partial \ln f_i(Y_{i,t} | \mathcal{F}_{t-1}, \Theta)}{\partial \lambda_{i,t}} = \frac{(\nu + 1)v_{i,t}^2}{\nu \exp(2\lambda_{i,t}) + v_{i,t}^2} - 1 \tag{16}$$

for $i = 1, \dots, \mathcal{N}$. These score functions update $\lambda_{i,t}$ within the scale matrix $\Omega_t \Omega_t'$. The updating terms $e_{i,t}$ and $g(e_{i,t})$ for $i = 1, \dots, \mathcal{N}$ in Equation (15), asymptotically and at the true values of parameters Θ_0 , are uncorrelated white noise; the conditions of which are presented in the Supplementary Material.

If $\nu \rightarrow \infty$, then $e_{i,t}$ of Equation (16) converges in probability to

$$e_{i,t} = \frac{\partial \ln f_i(Y_{i,t} | \mathcal{F}_{t-1}, \Theta)}{\partial \lambda_{i,t}} = \frac{v_{i,t}^2}{\exp(2\lambda_{i,t})} - 1 \tag{17}$$

for $i = 1, \dots, \mathcal{N}$, which are quadratic transformations of $v_{i,t}$ as in the GARCH model [Bollerslev (1986)].

If $\nu \rightarrow \infty$, then we also have that $u_t \rightarrow_p v_t$, $\epsilon_t \sim t(0, \Omega_t \Omega_t', \nu) \rightarrow_d N(0, \Omega_t \Omega_t')$, and $v_t \sim t(0, \Sigma_t, \nu) \rightarrow_d N(0, \Sigma_t)$. Moreover, if we also assume homoskedastic innovations, that is $\Sigma_t = \Sigma$, then we obtain a classical Gaussian linear state-space model (Section 2.1).

2.4. Impulse responses

For the reduced-form error term $\text{Var}(v_t | \mathcal{F}_{t-1}) = \Sigma_t \times \nu / (\nu - 2)$, which is factorized as

$$\text{Var}(v_t | \mathcal{F}_{t-1}) = \Sigma_t \times \frac{\nu}{\nu - 2} = \left(\frac{\nu}{\nu - 2} \right)^{1/2} \times D \Omega_t \Omega_t' D' \times \left(\frac{\nu}{\nu - 2} \right)^{1/2} \tag{18}$$

For the IRF analysis, the multivariate i.i.d. structural-form error term $\tilde{\epsilon}_t$ is introduced as

$$\tilde{\epsilon}_t = \left(\frac{\nu}{\nu - 2}\right)^{1/2} D^{-1}\Omega_t^{-1}v_t = \left(\frac{\nu}{\nu - 2}\right)^{1/2} D^{-1}\Omega_t^{-1}D\epsilon_t \tag{19}$$

where $\tilde{\epsilon}_t \sim t[0, I_{\mathcal{N}} \times (\nu - 2)/\nu, \nu]$ is an i.i.d. multivariate t -distribution with zero mean and an identity covariance matrix. From Equations (12) to (19), the $\tilde{\epsilon}_t$ error term-based representation of u_t is

$$u_t = [(\nu - 2)\nu]^{1/2}D\Omega_t \times \frac{\tilde{\epsilon}_t}{\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t} \tag{20}$$

The nonlinear MA(∞) representation of Y_t is

$$Y_t = \left\{ \sum_{j=1}^{\infty} CA^{j-1}B[(\nu - 2)\nu]^{1/2}\Omega_{t-j} \frac{\tilde{\epsilon}_{t-j}}{\nu - 2 + \tilde{\epsilon}'_{t-j}\tilde{\epsilon}_{t-j}} \right\} + \left(\frac{\nu}{\nu - 2}\right)^{1/2} D\Omega_t\tilde{\epsilon}_t \tag{21}$$

The IRFs for the heteroskedastic t -QVAR model are

$$\frac{\partial Y_{t+j}}{\partial \tilde{\epsilon}_t} = \left(\frac{\nu}{\nu - 2}\right)^{1/2} D\Omega_t \quad \text{for } j = 0 \tag{22}$$

$$\frac{\partial Y_{t+j}}{\partial \tilde{\epsilon}_t} = CA^{j-1}B[(\nu - 2)\nu]^{1/2}\Omega_t\tilde{D}_t \quad \text{for } j = 1, \dots, \infty \tag{23}$$

where \tilde{D}_t is

$$\begin{aligned} \tilde{D}_t &= \frac{\partial \frac{\tilde{\epsilon}_t}{\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t}}{\partial \tilde{\epsilon}_t} = \begin{bmatrix} d_{11,t} & \dots & d_{1\mathcal{N},t} \\ \dots & \dots & \dots \\ d_{\mathcal{N}1,t} & \dots & d_{\mathcal{N}\mathcal{N},t} \end{bmatrix} = \\ &= \begin{bmatrix} \frac{\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t - 2\tilde{\epsilon}_{1t}^2}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} & \frac{-2\tilde{\epsilon}_{1t}\tilde{\epsilon}_{2t}}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} & \dots & \frac{-2\tilde{\epsilon}_{1t}\tilde{\epsilon}_{\mathcal{N}t}}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} \\ \frac{-2\tilde{\epsilon}_{2t}\tilde{\epsilon}_{1t}}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} & \frac{\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t - 2\tilde{\epsilon}_{2t}^2}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{-2\tilde{\epsilon}_{\mathcal{N}t}\tilde{\epsilon}_{1t}}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} & \dots & \dots & \frac{\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t - 2\tilde{\epsilon}_{\mathcal{N}t}^2}{(\nu - 2 + \tilde{\epsilon}'_t\tilde{\epsilon}_t)^2} \end{bmatrix} \end{aligned} \tag{24}$$

Motivated by the work of Herwartz and Lütkepohl (2000), we replace the matrices Ω_t and $\Omega_t\tilde{D}_t$ by $E(\Omega_t)$ and $E(\Omega_t\tilde{D}_t)$ in Equations (22) and (23), respectively, which are estimated by using sample averages of the full sample period. In the literature, there are alternative approaches for the estimation of nonlinear IRFs [Lütkepohl (2005)]. Future empirical applications could consider other nonlinear IRF estimation approaches, which may be more appropriate than our approach for specific policy analyses.

2.5. Statistical inference

We use the maximum likelihood (ML) method [Creal et al. (2013), Harvey (2013), Blasques et al. (2018, 2022)]:

$$\hat{\Theta} = \arg \max_{\Theta} LL(Y_1, \dots, Y_T, \Theta) = \arg \max_{\Theta} \sum_{t=1}^T \ln f(Y_t | \mathcal{F}_{t-1}, \Theta) \tag{25}$$

where the log conditional density $\ln f(Y_t | \mathcal{F}_{t-1}, \Theta)$ is Equation (11). The LL function is maximized numerically, by using alternative start values of parameters. The standard errors are estimated using the inverse information matrix: $\{(1/T) \sum_{t=1}^T [G_t(\hat{\Theta})' G_t(\hat{\Theta})]\}^{-1}$, where $G_t(\hat{\Theta})$ is the gradient.

With respect to the sufficient conditions of the asymptotic properties of the ML estimator for score-driven location plus score-driven scale models, such as the heteroskedastic t -QVAR model, we refer to the work of Blazsek et al. (2022), which is directly related to the present work from the perspective of statistical inference. The heteroskedastic t -QVAR model is an extension of the bivariate t -QVAR model of Blazsek et al. (2022), because in the present paper we have multivariate location plus multivariate scale filters, while in the work of those authors the model includes univariate location plus univariate scale filters. Although most of the ML assumptions coincide for the two models, due to the different levels of technical complexity at some points of model formulation, we report the asymptotic theory for the heteroskedastic t -QVAR model in the Supplementary Material.

In the remainder of this section, we present some of those conditions for covariance stationarity and invertibility. For the covariance stationarity of X_t , asymptotically and at the true values of parameters Θ_0 , it is required that the maximum modulus of the eigenvalues of A , that is Stat_μ , is less than one. For the invertibility of X_t , that is X_t converges almost surely (a.s.) to a unique strictly stationary vector sequence for all $\Theta \in \tilde{\Theta}$ where $\tilde{\Theta}$ is the parameter set, one of the conditions of invertibility of X_t is

$$\text{Inv}_\mu = \inf_{n \geq 1} \left\{ n^{-1} E \left(\ln \left\| \prod_{t=1}^n A + BD^{-1} \frac{\partial u_{t-1}}{\partial (X_{t-1})'} \right\|_2 \right) \right\} < 0 \tag{26}$$

where the matrix norm $\|\cdot\|_2$ is the spectral norm. We present empirical estimates of Stat_μ and Inv_μ in the application.

Covariance stationarity of $\lambda_{i,t}$, for $i = 1, \dots, \mathcal{N}$, asymptotically and at the true values of parameters Θ_0 , requires $|\beta_i| < 1$ for $i = 1, \dots, \mathcal{N}$, respectively, which are denoted by $\text{Stat}_{\lambda,i}$ for $i = 1, \dots, \mathcal{N}$. Hence, $E(\lambda_{i,t}) = \omega_i / (1 - \beta_i)$, for $i = 1, \dots, \mathcal{N}$, which is applied to initialize $\lambda_{i,t}$ for $i = 1, \dots, \mathcal{N}$, respectively. For the invertibility of $\lambda_{i,t}$ for $i = 1, \dots, \mathcal{N}$, that is $\lambda_{i,t}$ for $i = 1, \dots, \mathcal{N}$ converge a.s. to a unique strictly stationary sequences for all $\Theta \in \tilde{\Theta}$, one of the conditions of the invertibility of $\lambda_{i,t}$ is

$$\text{Inv}_{\lambda,i} = \inf_{n \geq 1} \left\{ n^{-1} E \left[\ln \left| \prod_{t=1}^n \beta_i + [\alpha_i + \alpha_i^* \text{sgn}(-v_{i,t})] \frac{\partial e_{i,t}}{\partial \lambda_{i,t}} \right| \right] \right\} \tag{27}$$

for $i = 1, \dots, \mathcal{N}$. For the invertibility of filters $\lambda_{i,t}$, it is required that $\text{Inv}_{\lambda,i} < 0$ for $i = 1, \dots, \mathcal{N}$. We present the estimates of $\text{Stat}_{\lambda,i}$ and $\text{Inv}_{\lambda,i}$, for $i = 1, \dots, \mathcal{N}$, in the empirical application.

3. Empirical application

3.1. The DSGE model of An and Schorfheide (2007)

In this paper, we assume that the number of observable variables Y_t and the number of shocks ϵ_t coincide. If Equations (1) and (2) are the minimal square ABCD representation of a DSGE model, then Equations (6) and (7) represent the nonlinear extension of that minimal square ABCD representation. We apply the heteroskedastic t -QVAR model for the estimation of an analytically tractable identified DSGE model with as many shocks as variables. For this purpose, we use the DSGE model of An and Schorfheide (2007), which has a minimal square ABCD representation [see Morris (2014)]. We recognize that this DSGE model is a very specific example of DSGE models. We use the model of An and Schorfheide (2007) to illustrate the performance of the heteroskedastic t -QVAR model.

The DSGE model of our application represents dynamic interactions among economic output, inflation, interest rate, government spending, aggregate productivity, and consumption in a system of nonlinear equations. The variables of the DSGE model of An and Schorfheide (2007) are as follows: y_t is the difference between current output and steady state output; π_t is the difference between current inflation and steady state inflation; r_t is the difference between current interest rate and steady-state interest rate; g_t is the difference between current government spending and steady state government spending; z_t is the error term in $\ln A_t = \gamma + \ln A_{t-1} + z_t$, where A_t is aggregate productivity; and c_t is the difference between current consumption and steady-state consumption.

We present the following specification of the DSGE model, which in a log-linearized form around the steady state is the following system of expectational difference equations [see Giacomini (2013)]:

$$z_t = \rho_z z_{t-1} + \epsilon_{z,t} \tag{28}$$

$$g_t = \rho_g g_{t-1} + \epsilon_{g,t} \tag{29}$$

$$r_t = \rho_r r_{t-1} + (1 - \rho_r) \psi_1 \pi_t + (1 - \rho_r) \psi_2 (y_t - g_t) + \epsilon_{r,t} \tag{30}$$

$$y_t = E_t(y_{t+1}) + g_t - E_t(g_{t+1}) - \frac{1}{\tau} [r_t - E_t(\pi_{t+1}) - E_t(z_{t+1})] \tag{31}$$

$$\pi_t = \zeta E_t(\pi_{t+1}) + \frac{\tau(1 - \xi)}{\xi \bar{\pi}^2 \phi} (y_t - g_t) = \zeta E_t(\pi_{t+1}) + \kappa (y_t - g_t) \tag{32}$$

$$c_t = y_t - g_t \tag{33}$$

where E_t denotes expected value, which is conditional on the values of all variables in the system of equations until period t (t included), and $\bar{\pi}$ is steady-state inflation.

In the literature on DSGE models, an i.i.d. $\epsilon_t \sim N(0, \Omega\Omega')$ with diagonal covariance matrix is used for $\epsilon_t = (\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{r,t})'$. For the heteroskedastic t -QVAR model, we extend the probability distribution of $\epsilon_t = (\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{r,t})'$, by using $\epsilon_t | \mathcal{F}_{t-1} \sim t(0, \Omega_t \Omega_t', \nu)$ with score-driven conditional scale matrix.

3.2. Heteroskedastic t -ABCD representation

We present the heteroskedastic t -ABCD representation which is adapted to the DSGE model of An and Schorfheide (2007). We specify the ABCD representation according to Morris (2014), as follows:

$$\underbrace{\begin{bmatrix} z_t \\ g_t \\ r_t \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} \rho_z & 0 & 0 \\ 0 & \rho_g & 0 \\ c_{r,z} & 0 & c_{r,r} \end{bmatrix}}_A \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \end{bmatrix}}_B \underbrace{\begin{bmatrix} c_{r,z}/\rho_z & 0 & c_{r,r}/\rho_r \\ c_{y,z}/\rho_z & 1 & c_{y,r}/\rho_r \\ c_{\pi,z}/\rho_z & 0 & c_{\pi,r}/\rho_r \end{bmatrix}}_{D^{-1}}^{-1} \underbrace{\begin{bmatrix} u_{z,t} \\ u_{g,t} \\ u_{r,t} \end{bmatrix}}_{u_t} \tag{34}$$

$$\underbrace{\begin{bmatrix} r_t \\ y_t \\ \pi_t \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} c_{r,z} & 0 & c_{r,r} \\ c_{y,z} & \rho_g & c_{y,r} \\ c_{\pi,z} & 0 & c_{\pi,r} \end{bmatrix}}_C \underbrace{\begin{bmatrix} z_{t-1} \\ g_{t-1} \\ r_{t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} v_{z,t} \\ v_{g,t} \\ v_{r,t} \end{bmatrix}}_{v_t} \tag{35}$$

These equations correspond to Equations (8) and (9) of the heteroskedastic t -QVAR model. The parameters of the ABCD-matrices are identified, and the IRFs can be estimated according to the nonlinear MA(∞) representation of the heteroskedastic t -QVAR model.

For the heteroskedastic t -ABCD representation, the following parameters are estimated:

$$\Theta = (\rho_z, \rho_g, \rho_r, c_{r,z}, c_{r,r}, c_{y,z}, c_{y,r}, c_{\pi,z}, c_{\pi,r}, X_{0,z}, X_{0,g}, X_{0,r}, \omega_z, \omega_g, \omega_r, \beta_z, \beta_g, \beta_r, \alpha_z, \alpha_g, \alpha_r, \alpha_z^*, \alpha_g^*, \alpha_r^*, \nu)'$$
(36)

The heteroskedastic t -ABCD representation is a generalization of the homoskedastic Gaussian-ABCD representation of Fernández-Villaverde et al. (2007) and the heteroskedastic score-driven Gaussian-ABCD representation of Angelini and Gorgi (2018). By assuming that $\Omega_t = \Omega$ and $\nu \rightarrow \infty$, we get the homoskedastic Gaussian-ABCD representation [Fernández-Villaverde et al. (2007)]. Moreover, by assuming that $\Omega_t = \Omega$, we get the new homoskedastic t -ABCD representation. In addition, we note that we provide the following contributions to the work of Angelini and Gorgi (2018): (i) Angelini and Gorgi (2018) assume that only the scale filter is score-driven. We use score-driven transition equation and scale filters. (ii) Angelini and Gorgi (2018) use a score-driven multivariate Gaussian distribution, which we extend to the score-driven multivariate t -distribution. (iii) We extend the score-driven scale filter of Angelini and Gorgi (2018) by adding asymmetric effects to the updates.

3.3. Data

All quarterly time series data of this paper are from Federal Reserve Economic Data (FRED). We use the following variables for the period of 1954 Q3 to 2022 Q1: (a) not seasonally adjusted Effective Federal Funds Rate (in % points), (b) seasonally adjusted US GDP level \tilde{y}_t , and (c) seasonally adjusted US CPI (consumer price index) for all urban consumers. We use the maximum period for which data are available from FRED for these variables. The sample period includes the recent and on-going crisis periods of the coronavirus disease of 2019 (COVID-19) pandemic and the Russian invasion of Ukraine.

The US inflation rate series is computed as quarterly log percentage change (in % points) of the US CPI. The seasonally adjusted US GDP level is detrended by using the following model:

$$\ln \tilde{y}_t = \ln A_t + \ln y_t^\dagger$$
(37)

$$\ln A_t = \gamma + \ln A_{t-1} + z_t$$
(38)

where log aggregate productivity $\ln A_t$ is the unobservable stochastic trend for US GDP level [Giacomini (2013)], and the model is estimated as a Gaussian linear state-space model by using the Kalman filter.

In this paper, the steady state of each observable dependent variable is estimated by using the sample average [Morris (2014)]. There are some negative observations for the US inflation time series. Hence, we do not use the logarithmic percentage difference between the value of the variables and their steady states [Giacomini (2013)], but the sample average is subtracted from each variable [Morris (2014)]. Hence, the sample averages are subtracted from the US inflation rate and the Effective Federal Funds Rate in the computations of π_t and r_t , respectively (both are measured in % points). For percentage deviation of US GDP from its trend, we use $y_t \equiv 100 \times \ln y_t^\dagger$ (hence y_t is in % points), and $E(y_t) = 0$ due to Equations (37) and (38).

Descriptive statistics are presented in Table 1. The evolution of the observable dependent variables r_t , y_t , and π_t , for the period of 1954 Q3 to 2022 Q1, is presented in Figure 1.

Table 1. Descriptive statistics

	r_t (Effective Federal Funds Rate)	y_t (detrended US GDP level)	π_t (US inflation rate)
Start date	1954 Q3	1954 Q3	1954 Q3
End date	2022 Q1	2022 Q1	2022 Q1
Frequency of observations	quarterly	quarterly	quarterly
Number of observations T	271	271	271
Mean	0.0000	0.0000	0.0000
Standard deviation	3.6092	1.2557	0.7520
Minimum	-4.5620	-11.1039	-3.1872
Maximum	13.1580	6.8743	2.9994
Skewness	0.9929	-1.8626	0.8236
Excess kurtosis	1.1774	25.4319	2.8473
$\text{Corr}(Y_{i,t}, Y_{i,t-1})$	0.9723	0.1569	0.7520
$\text{Corr}(Y_{i,t} , Y_{i,t-1})$	0.3734	-0.2750	0.4737
Shapiro-Wilk test p -value	0.0000	0.0000	0.0000
ADF-GLS test p -value	0.1520	0.1350	0.1990
ARCH(4) test p -value	0.0000	0.0000	0.0000

Notes: Augmented Dickey-Fuller generalized least squares (ADF-GLS); autoregressive conditional heteroskedasticity (ARCH). For the ADF-GLS test with constant, the modified Bayesian information criterion (BIC) is used for lag-order selection. For the ARCH test, lag-order 4 is used. H_0 for the Shapiro-Wilk test [Shapiro and Wilk (1965)] is the normal distribution (rejected for all variables at all levels of significance). H_0 for the ADF-GLS test is integration of order one (not able to reject for any variable at the 10% level of significance). H_0 for the ARCH test is no ARCH effect is present (rejected for all variables at all levels of significance).

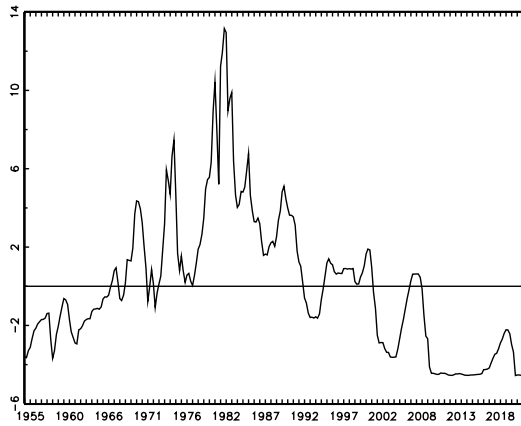
3.4. Parameter estimates and model diagnostics

In the empirical application, we estimate the following models: (i) a homoskedastic Gaussian-QVAR model which is called the homoskedastic Gaussian-ABCD representation; (ii) a heteroskedastic Gaussian-QVAR model which is called the heteroskedastic score-driven Gaussian-ABCD representation; (iii) a homoskedastic t -QVAR model which is called the homoskedastic score-driven t -ABCD representation; and (iv) a heteroskedastic t -QVAR model which is called the heteroskedastic score-driven t -ABCD representation. As aforementioned, models (i), (ii), and (iii) are special cases of model (iv). In Table 2, the ML parameter estimates, likelihood-based model performance metrics, and stationarity and invertibility statistics are presented for all ABCD representations.

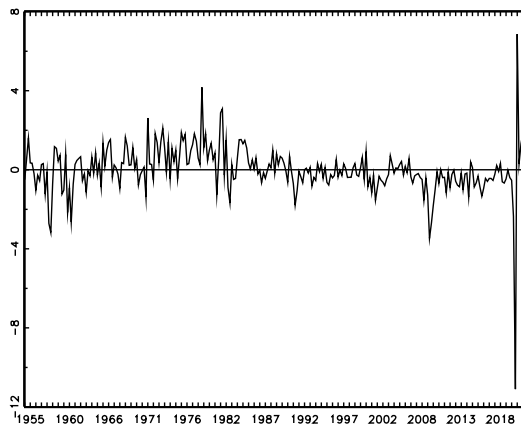
First, almost all parameters of the ABCD-matrices for all representations are significantly different from zero in Table 2. By comparing the ML results for alternative ABCD representations, the estimate of the dynamic parameter ρ_r is lower (though statistically significant) for the heteroskedastic score-driven Gaussian-ABCD and heteroskedastic score-driven t -ABCD representations ($\hat{\rho}_r = 0.0077^{***}$ and $\hat{\rho}_r = 0.0373^{***}$, respectively), than for the homoskedastic Gaussian-ABCD and score-driven homoskedastic t -ABCD representations ($\hat{\rho}_r = 0.7388^{***}$ and $\hat{\rho}_r = 0.3384^{***}$, respectively). As these results may suggest that the dynamic effects for r_t are different for the homoskedastic and heteroskedastic ABCD representations, we return to this point in Section 3.5 where the IRF estimates are presented. Moreover, the parameter estimates for the score-driven heteroskedastic t -ABCD representation show that scale dynamics, that is β_z , β_g , β_r and α_z , α_g , α_r , α_z^* , α_g^* , α_r^* , are significantly different from zero. In addition, the degrees of freedom parameter ν estimates for both t -ABCD representations justify the use of the Student's t -distribution, as their estimates are significantly lower than 30.

Second, the statistical performances of the ABCD representations are compared by using the LL, Akaike information criterion, Bayesian information criterion, and Hannan-Quinn criterion metrics in Table 2. We find that the performances of the Gaussian-ABCD representations are

(a) Effective Federal Funds Rate r_t (in percentage points), centered at zero.



(b) US GDP level y_t (in percentage points), centered at zero.



(c) US inflation rate π_t (in percentage points), centered at zero.

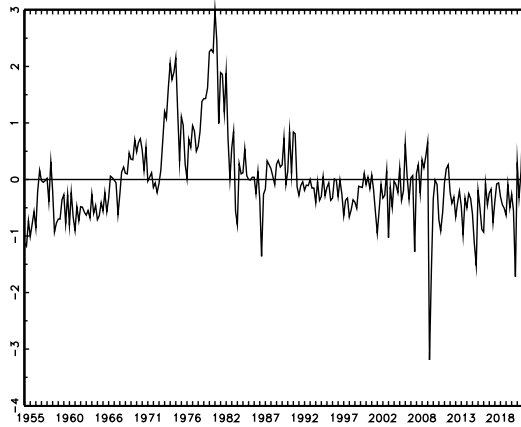


Figure 1. Observable dependent variables for the period of 1954 Q3 to 2022 Q1.

Table 2. Parameter estimates and model diagnostics

Homoskedastic Gaussian-ABCD		Heteroskedastic Score-driven Gaussian-ABCD		Homoskedastic Score-driven t-ABCD		Heteroskedastic Score-driven t-ABCD	
ρ_z	0.6116*** (0.0470)	ρ_z	0.4505*** (0.0175)	ρ_z	0.8745*** (0.0375)	ρ_z	0.7555*** (0.0364)
ρ_g	0.0826* (0.0486)	ρ_g	0.3323*** (0.0121)	ρ_g	0.4858*** (0.0675)	ρ_g	0.4213*** (0.0750)
ρ_r	0.7388*** (0.0261)	ρ_r	0.0077*** (0.0003)	ρ_r	0.3384*** (0.0082)	ρ_r	0.0373* (0.0217)
$c_{r,z}$	1.7089* (0.9705)	$c_{r,z}$	0.1222*** (0.0399)	$c_{r,z}$	3.3188*** (0.9632)	$c_{r,z}$	2.7961+ (1.9136)
$c_{r,r}$	0.9718*** (0.0145)	$c_{r,r}$	0.9474*** (0.0135)	$c_{r,r}$	0.9645*** (0.0129)	$c_{r,r}$	0.9605*** (0.0107)
$c_{y,z}$	7.4279*** (0.9349)	$c_{y,z}$	0.1733*** (0.0365)	$c_{y,z}$	4.9599*** (1.2109)	$c_{y,z}$	5.6437+ (3.6881)
$c_{y,r}$	0.0798*** (0.0269)	$c_{y,r}$	0.0620** (0.0270)	$c_{y,r}$	0.0637** (0.0263)	$c_{y,r}$	0.0514+ (0.0317)
$c_{\pi,z}$	10.9102*** (0.6212)	$c_{\pi,z}$	0.4637*** (0.0593)	$c_{\pi,z}$	12.0257*** (0.4471)	$c_{\pi,z}$	18.2881* (9.8584)
$c_{\pi,r}$	0.1168*** (0.0205)	$c_{\pi,r}$	0.0436** (0.0179)	$c_{\pi,r}$	0.0747*** (0.0247)	$c_{\pi,r}$	0.0234 (0.0213)
$\chi_{0,z}$	-0.0652** (0.0331)	$\chi_{0,z}$	-1.9689*** (0.0363)	$\chi_{0,z}$	-0.0724*** (0.0268)	$\chi_{0,z}$	-0.0563 (0.1163)
$\chi_{0,g}$	3.6299 (13.2357)	$\chi_{0,g}$	2.0111*** (0.0363)	$\chi_{0,g}$	0.2280 (1.3749)	$\chi_{0,g}$	0.9708 (5.2309)
$\chi_{0,r}$	-3.5804*** (0.7812)	$\chi_{0,r}$	-3.4893*** (0.0363)	$\chi_{0,r}$	-3.4731*** (0.8028)	$\chi_{0,r}$	-3.4084 (2.8854)
$\Omega_{1,1}$	0.0263*** (0.0028)	ω_z	-0.1856*** (0.0336)	$\Omega_{1,1}$	0.0276*** (0.0019)	ω_z	-1.2153** (0.5228)
$\Omega_{2,2}$	1.1449*** (0.0425)	ω_g	-0.0442 (0.0350)	$\Omega_{2,2}$	0.7895*** (0.0418)	ω_g	-0.0714** (0.0311)
$\Omega_{3,3}$	0.6355*** (0.0223)	ω_r	-0.3619*** (0.0177)	$\Omega_{3,3}$	0.3091*** (0.0075)	ω_r	-0.3563*** (0.0964)
		β_z	0.8044*** (0.0057)	ν	12.0485*** (1.6929)	β_z	0.7632*** (0.1002)
		β_g	0.7958*** (0.0059)			β_g	0.7051*** (0.0842)
		β_r	0.9334*** (0.0031)			β_r	0.9041*** (0.0196)
		α_z	0.0412*** (0.0031)			α_z	0.0059** (0.0026)
		α_g	0.1095*** (0.0053)			α_g	0.1175*** (0.0198)
		α_r	0.0000** (0.0000)			α_r	-0.0001 (0.0007)
		α_z^*	0.0084*** (0.0018)			α_z^*	0.0012*** (0.0004)
		α_g^*	0.0513*** (0.0078)			α_g^*	0.0704*** (0.0211)
		α_r^*	0.0000*** (0.0000)			α_r^*	-0.0042** (0.0017)
						ν	13.1399*** (1.8043)
LL	-3.4356	LL	-2.7977	LL	-3.1467	LL	-2.6755
AIC	6.9819	AIC	5.7724	AIC	6.4115	AIC	5.5355
BIC	7.1813	BIC	6.0914	BIC	6.6241	BIC	5.8678
HQC	7.0620	HQC	5.9005	HQC	6.4968	HQC	5.6689
Stat $_{\mu}$	0.9718	Stat $_{\mu}$	0.9474	Stat $_{\mu}$	0.9645	Stat $_{\mu}$	0.9605
Inv $_{\mu}$	0.0000	Inv $_{\mu}$	0.0000	Inv $_{\mu}$	-0.4752	Inv $_{\mu}$	-0.4867
		Stat $_{\lambda,z}$	0.8044			Stat $_{\lambda,z}$	0.7632
		Stat $_{\lambda,g}$	0.7958			Stat $_{\lambda,g}$	0.7051
		Stat $_{\lambda,r}$	0.9334			Stat $_{\lambda,r}$	0.9041
		Inv $_{\lambda,z}$	-0.4108			Inv $_{\lambda,z}$	-0.3950
		Inv $_{\lambda,g}$	-0.4608			Inv $_{\lambda,g}$	-0.3066
		Inv $_{\lambda,r}$	-0.1068			Inv $_{\lambda,r}$	-0.0813

Notes: AIC, Akaike information criterion; BIC, Bayesian information criterion; HQC, Hannan–Quinn criterion; LL, log-likelihood. Standard errors are reported in parentheses. The bold likelihood-based model performance metrics indicate superior statistical performance. The heteroskedastic score-driven Gaussian-ABCD representation is the DSGE model of Angelini and Gorgi (2018). The Gaussian-ABCD representations have VAR(1) representations, hence $\text{Inv}_{\mu} = 0$. ***, **, *, and + indicate parameter significance at the 1%, 5%, 10%, and 15% levels, respectively.

improved by using the score-driven t -ABCD representations. We also find that the best statistical performance is provided by the heteroskedastic score-driven t -ABCD representation.

Third, the empirical estimates of the covariance stationarity and Lyapunov exponent statistics are also presented in Table 2. For the Gaussian-ABCD representations, covariance stationarity is supported by the Stat_μ statistic. For the t -ABCD representations, covariance stationarity and invertibility are supported by the Stat_μ and Inv_μ statistics, respectively. For both heteroskedastic ABCD representations, covariance stationarity and invertibility are supported by the $\text{Stat}_{\lambda,z}$, $\text{Stat}_{\lambda,g}$, $\text{Stat}_{\lambda,r}$, and $\text{Inv}_{\lambda,z}$, $\text{Inv}_{\lambda,g}$, $\text{Inv}_{\lambda,r}$, respectively. We note that both Gaussian-ABCD representations of the DSGE model of An and Schorfheide (2007) have VAR(1) forms [see Morris (2014)], hence $\text{Inv}_\mu = 0$ for those models.

Fourth, in Figure 2, as a graphical illustration of heteroskedasticity for the DSGE model, the evolution of $\epsilon_{i,t} \pm \sigma_{i,t}$ (i.e., error \pm one standard deviation) is presented for the period of 1954 Q3 to 2022 Q1, by using the estimates for the best-performing heteroskedastic score-driven t -ABCD representation. The figure shows that the standard deviations of the error terms are time-varying for the DSGE model.

3.5. Impulse response analysis

IRFs are identified using sign restrictions, based on 10,000 Monte Carlo simulations of the matrix Ω , in accordance with the procedure of Rubio-Ramírez et al. (2010). First, the ML estimates of Ω are used. Second, an $\mathcal{N} \times \mathcal{N}$ matrix \tilde{K} of i.i.d. $N(0, 1)$ numbers is simulated. Third, the QR decomposition of \tilde{K} is performed, and the resulting matrices are denoted \tilde{Q} and \tilde{R} . Fourth, we define $\tilde{\Omega} \equiv \Omega \times \tilde{Q}'$ for each simulation. The parameter matrices Ω and Ω_t are replaced by $\tilde{\Omega}$ in the IRFs.

For each simulation of $\tilde{\Omega}$, sign restrictions are used in accordance with Table 3, for which we refer to the works of Leeper et al. (2009) and Cogan et al. (2010). For the simulations of $\tilde{\Omega}$ that satisfy the sign restrictions of Table 3, we report the 5%, 50%, and 95% percentiles of the IRFs up to 20 leads in Figures 3 and 4 for the classical homoskedastic Gaussian-ABCD and the new heteroskedastic score-driven t -ABCD representations, respectively.

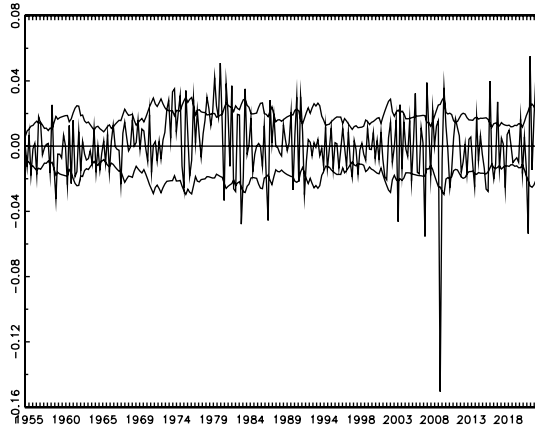
We highlight the following IRF results: First, the relatively low estimate of ρ_r for the heteroskedastic ABCD representations, compared to the homoskedastic ABCD representations (Section 3.4), might be interpreted as a lower effect of an interest rate shock on subsequent interest rates, but, as is shown in Panel (c) of Figures 3 and 4, that is not the case. The persistence of interest rate shocks is similar for all ABCD representations [Panel (c) of Figures 3 and 4].

Second, a difference between the IRFs of the two ABCD representations, which may have policy implications, is in relation to the effects of an aggregate productivity shock on the US inflation rate $\tilde{\epsilon}_{z,t} \rightarrow \pi_{t+j}$. For the homoskedastic Gaussian-ABCD representation, $\tilde{\epsilon}_{z,t} \rightarrow \pi_{t+j}$ is negative, and its contemporaneous value is $-0.41[-0.11, -0.62]$ [Panel (g) of Figure 3]. For the heteroskedastic t -ABCD representation, $\tilde{\epsilon}_{z,t} \rightarrow \pi_{t+j}$ is also negative, but its effect is lower in absolute value, that is, its contemporaneous value is $-0.10[-0.02, -0.14]$ [Panel (g) of Figure 4].

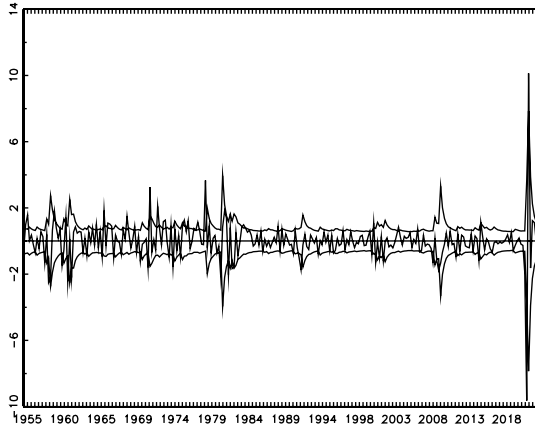
Third, another difference between the IRFs of different ABCD representations is in relation to the effects of government spending shocks on the US inflation rate $\tilde{\epsilon}_{g,t} \rightarrow \pi_{t+j}$ [Panel (h) of Figures 3 and 4]. For the homoskedastic Gaussian-ABCD representation, $\tilde{\epsilon}_{g,t} \rightarrow \pi_{t+j}$ is positive contemporaneously, and its value is $0.28[0.02, 0.61]$ [Panel (h) of Figure 3]. For the heteroskedastic t -ABCD representation, $\tilde{\epsilon}_{g,t} \rightarrow \pi_{t+j}$ is also positive, but its effect is lower in absolute value, that is, its contemporaneous value is $0.07[0.01, 0.14]$ [Panel (h) of Figure 4].

Fourth, another difference between the IRFs of different ABCD representations is in relation to monetary policy shocks, that is, effects of interest rate shocks on the US inflation rate $\tilde{\epsilon}_{r,t} \rightarrow \pi_{t+j}$ [Panel (i) of Figures 3 and 4]. For the homoskedastic Gaussian-ABCD representation, $\tilde{\epsilon}_{r,t} \rightarrow \pi_{t+j}$ is negative, and its contemporaneous value is $-0.25[-0.02, -0.61]$ [Panel (i) of Figure 3].

(a) $\epsilon_{z,t}$ and $\epsilon_{z,t} \pm \sigma_{z,t}$ (Effective Federal Funds Rate).



(b) $\epsilon_{g,t}$ and $\epsilon_{g,t} \pm \sigma_{g,t}$ (US GDP level).



(c) $\epsilon_{r,t}$ and $\epsilon_{r,t} \pm \sigma_{r,t}$ (US inflation rate).

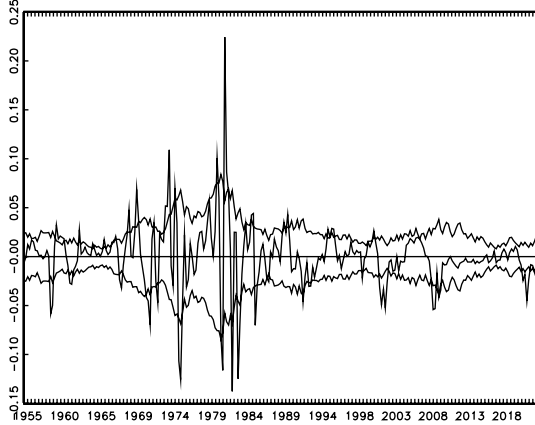


Figure 2. Evolution of ϵ_t and $\epsilon_t \pm \sigma_t$ for score-driven heteroskedastic t -ABCD (1954 Q3 to 2022 Q1).
Notes: $\sigma_{i,t} = \exp(\lambda_{i,t})[\nu/(\nu - 2)]^{1/2}$ for $i = z, g, r$.

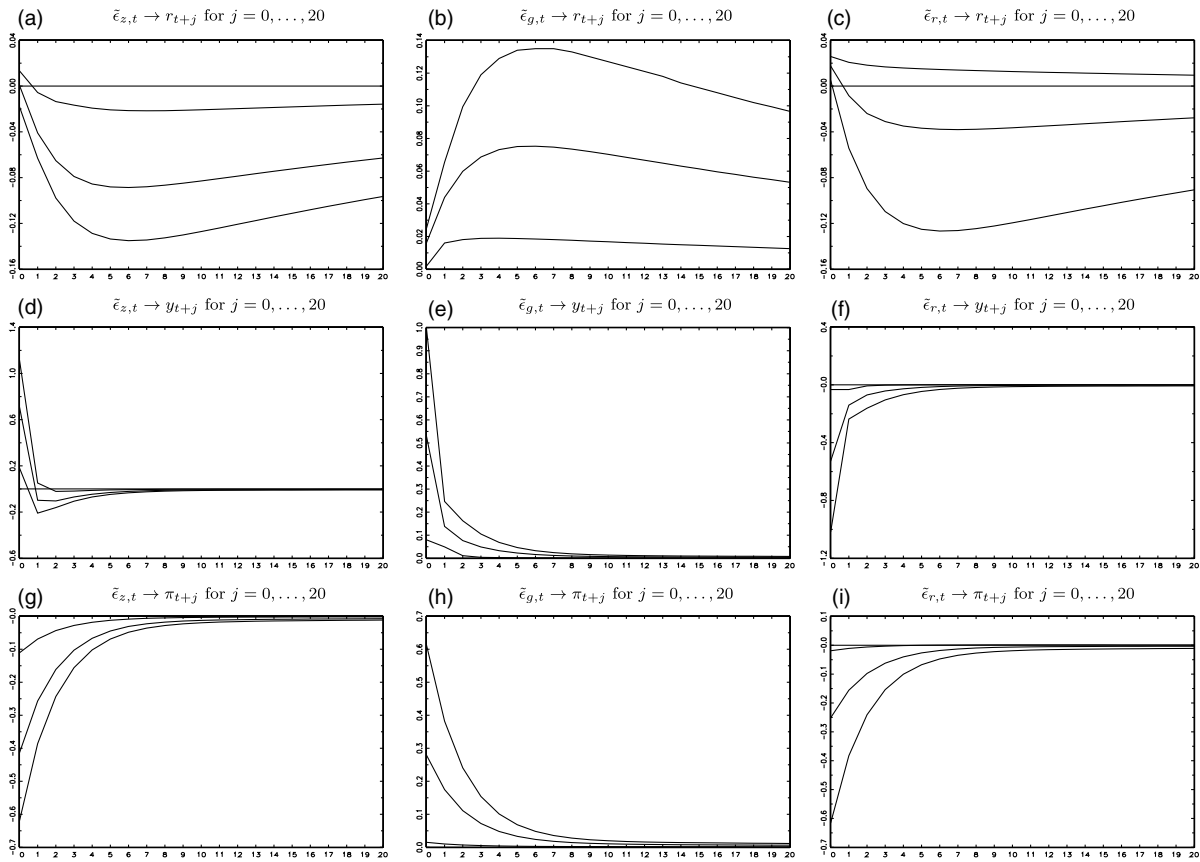


Figure 3. IRFs for the homoskedastic Gaussian-ABCD representation (5%, 50%, and 95% percentiles).

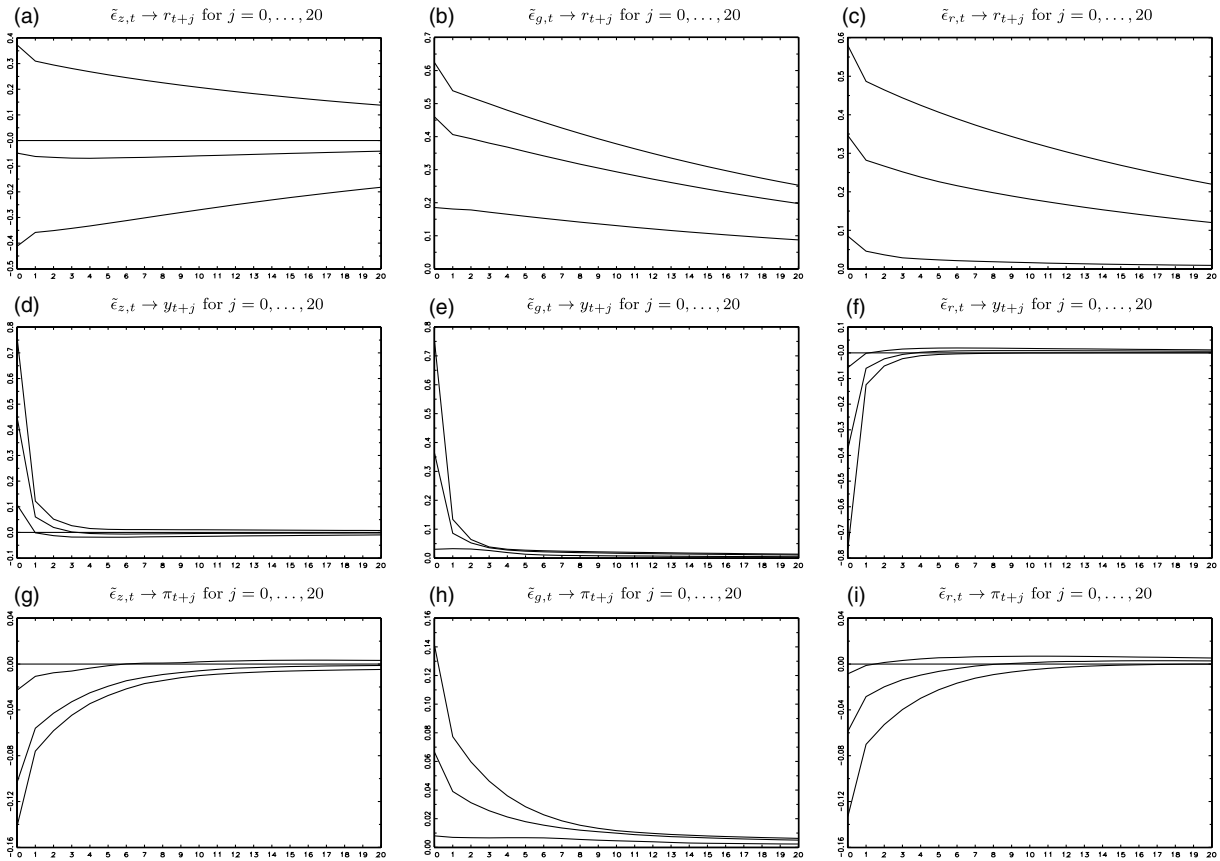


Figure 4. IRFs for the heteroskedastic score-driven t -ABCD representation (5%, 50%, and 95% percentiles).

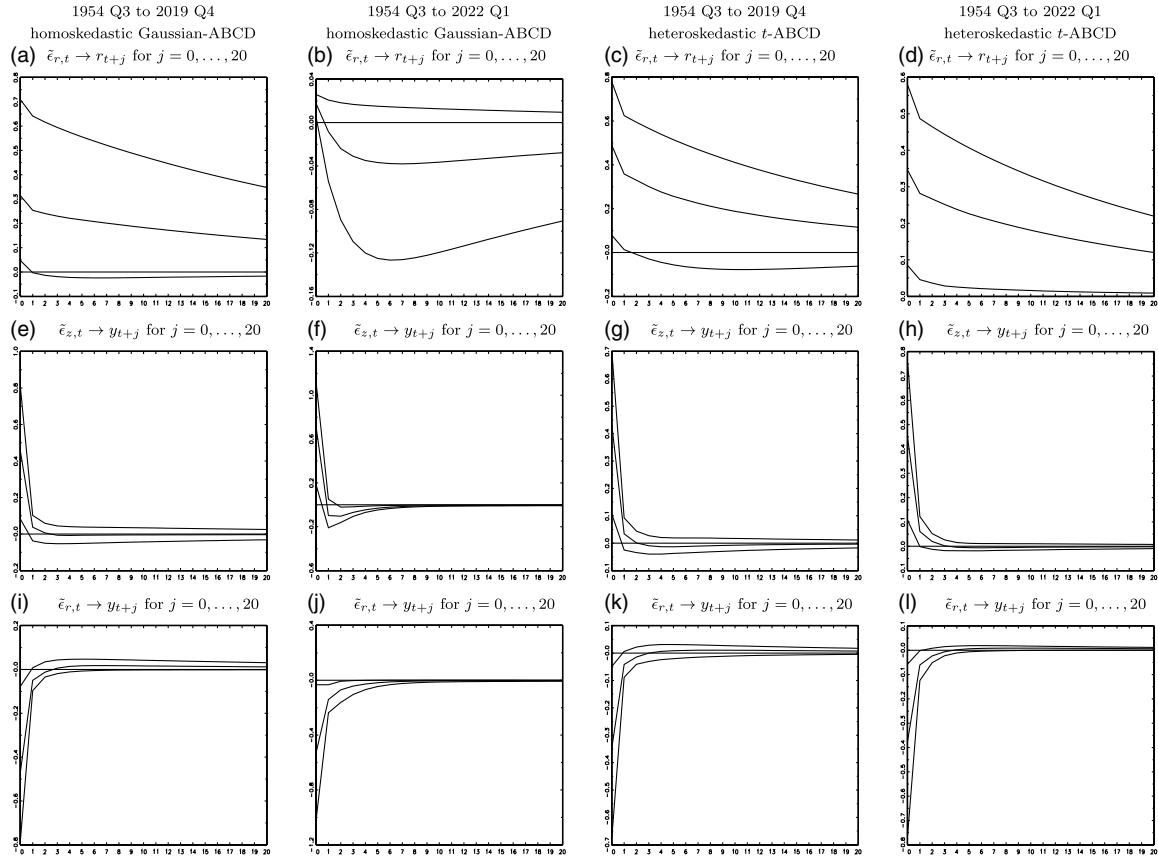


Figure 5. Comparison of IRFs estimates for the periods of 1954 Q3 to 2019 Q4 and 1954 Q3 to 2022 Q1 (5%, 50%, and 95% percentiles).

Table 3. Sign restrictions on impact responses

	Aggregate productivity shock $\epsilon_{z,t}$	Government spending shock $\epsilon_{g,t}$	Monetary policy shock $\epsilon_{r,t}$
r_t	Unrestricted	+	+
y_t	+	+	–
π_t	–	+	–

For the heteroskedastic t -ABCD representation, $\tilde{\epsilon}_{r,t} \rightarrow \pi_{t+j}$ is also negative, but its effect is lower in absolute value, that is, its contemporaneous value is $-0.06[-0.01, -0.13]$ [Panel (i) of Figure 4].

Fifth, as measured by the IRF confidence intervals, the estimation precision of the IRF for US GDP growth [Panels (d)–(f) of Figures 3 and 4] and US inflation rate [Panels (g)–(i) of Figures 3 and 4] is higher for the heteroskedastic score-driven t -ABCD representation than for the homoskedastic Gaussian-ABCD representation. This indicates that the score-driven updates of the dependent variable may improve the statistical inferences of the impulse responses.

Finally, in Figure 5, we compare some of the IRF estimates for the homoskedastic Gaussian-ABCD representation and the heteroskedastic score-driven t -ABCD representation, by using data for the periods of 1954 Q3 to 2019 Q4 and 1954 Q3 to 2022 Q1 (i.e., data without and with the COVID-19 pandemic, respectively). The IRF estimates show that the heteroskedastic score-driven t -ABCD representation is robust to the COVID-19 pandemic, while for the homoskedastic Gaussian-ABCD representation some of the IRF estimates are significantly different for the two samples. These results indicate the robustness of the score-driven t -ABCD representation to extreme observations.

4. Conclusions

In this paper, we have introduced the heteroskedastic score-driven state-space model with Student's t -distributed innovations, and we have applied the new nonlinear model in an empirical macroeconomic way to a well-known DSGE model. In the heteroskedastic score-driven state-space model with Student's t -distributed innovations, the coefficients are time-invariant and the error terms are conditionally heteroskedastic, similar to the best-performing model in Chan and Eisenstat (2018).

We have used the robust empirical macroeconomic model with time-varying conditional moments, to estimate a new heteroskedastic score-driven t -ABCD representation of the DSGE model of An and Schorfheide (2007). We have reported estimation results by using data on economic output, inflation, interest rate, government spending, aggregate productivity, and consumption of the USA for the period of 1954 Q3 to 2022 Q1. We found a superior statistical performance, lower policy-relevant effects, and a higher estimation precision of the IRF for US GDP growth and US inflation rate for the heteroskedastic score-driven t -ABCD representation rather than for the homoskedastic Gaussian-ABCD representation. These results are due to the robustness of the heteroskedastic t -QVAR model, even including the period of the COVID-19 pandemic and the start of the Russian invasion of Ukraine, which may motivate its practical use for empirical macroeconomic analysis.

Supplementary material. To view supplementary material for this article, please visit <https://doi.org/10.1017/S1365100522000712>.

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