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LETTERS TO THE EDITOR

ON THE WEAK CONVERGENCE OF STOCHASTIC PROCESSES WITH EMBEDDED POINT PROCESSES

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Abstract

We consider a stochastic process in continuous time and two point processes on the real line, all jointly stationary. We show that under a certain mixing condition the values of the process at the points of the second point process converge weakly under the Palm distribution with respect to the first point process, and we identify the limit. This result is a supplement to two other known results which are mentioned below.

STATIONARY POINT PROCESSES; PALM DISTRIBUTIONS; MIXING

1. Statements of the results

Let us consider a stochastic process $\{X_t, t \in \mathbb{R}\}$ with values in a fairly arbitrary space (e.g. a Polish space) and a random point process Φ on $\mathbb{R}$ with points $\cdots < T_{-1} < T_0 \leqq 0 < T_1 < \cdots$. X and Φ are assumed to be jointly stationary. To be more rigorous we can assume that both objects are defined on a common probability space $(\Omega, \mathscr{F}, P)$ and there is a family $\{\theta_t, t \in \mathbb{R}\}$ of measurable bijections of Ω forming a semigroup ($\theta_t \circ \theta_s = \theta_{t+s}$, $\theta_0 =$ identity) such that:
(i) For all $A \in \mathscr{F}$ $\quad P(\theta_{-t}A) = P(A)$.
(ii) For all Borel subsets B of $\mathbb{R}$ and $t, s \in \mathbb{R}$ $\quad X_t \circ \theta_s = X_{t+s}$, $\quad \Phi(B) \circ \theta_t = \Phi(B + t)$.
Let P^0 be the *Palm transformation* of P with respect to Φ (cf. Neveu (1977), Matthes et al. (1978), Franken et al. (1982), Baccelli and Brémaud (1987)). That is, P^0 is another probability measure on $(\Omega, \mathscr{F})$ given by

$$(1) \qquad P^0(A) = \frac{1}{\lambda\,|B|} E \int_B 1_A \circ \theta_t \Phi(dt), \qquad A \in \mathscr{F},$$

where λ is the *rate* of Φ ($\lambda = E\Phi(0, 1]$), which is assumed to be non-zero and finite, E denotes expectation with respect to P and B is any linear Borel set with Lebesgue measure $|B| \neq 0$. It can be proved that the transformation $P \to P^0$ is one-to-one and that P^0 is invariant under the transformations $\{\theta_{T_n}, n \in \mathbb{Z}\}$.

The first theorem can then be stated as follows.

Theorem 1. If

$$(2) \qquad \lim_{t\to\infty} P(A \cap \theta_{-t}B) = P(A)P(B), \qquad A, B \in \mathscr{F},$$

then the P^0-distribution of X_t converges as $t \to \infty$ and the limiting distribution is the P-distribution of X_s (which is the same for all $s \in \mathbb{R}$).

For a proof of this theorem see Matthes et al. (1978), Chapter 9. If condition (2) is satisfied

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then we say that the pair $(P, \{\theta_t\})$ is *mixing*. Another proof has been given by Miyazawa (1977) where (2) has been replaced by the weaker condition

$$\lim_{t\to\infty} P(T_1 \leqq \varepsilon \cap \theta_{-t}B) = P(T_1 \leqq \varepsilon)P(B), \qquad \varepsilon \geqq 0,\ B \in \mathcal{F}. \tag{3}$$

Intuitively, Theorem 1 asserts that, despite the fact that X is not stationary under P^0, it approaches stationarity as $t \to \infty$, provided that the mixing condition (2) or the weaker condition (3) holds.

There is another fact, closely related to Theorem 1, in which the roles of P and P^0 are interchanged. For that reason we can call it the 'dual' of Theorem 1. This is stated as follows.

Theorem 2. If

$$\lim_{n\to\infty} P^0(A \cap \theta_{-T_n}B) = P^0(A)P^0(B), \qquad A, B \in \mathcal{F}, \tag{4}$$

then the P-distribution of X_{T_n} converges as $n \to \infty$ and the limiting distribution is the P^0-distribution of X_{T_k} (which is the same for all $k \in \mathbb{Z}$).

For proof see again Matthes et al. (1978). See also Miyazawa (1977) where the mixing condition (4) for the pair $(P^0, \{\theta_{T_n}\})$ has been replaced by the weaker condition

$$\lim_{n\to\infty} P^0(T_1 \leqq \varepsilon \cap \theta_{-T_n}B) = P^0(T_1 \leqq \varepsilon)P^0(B), \qquad \varepsilon \geqq 0,\ B \in \mathcal{F}. \tag{5}$$

We are now in position to state a third case involving two point processes Φ, Ψ with points $\cdots < T_{-1} < T_0 \leqq 0 < T_1 < \cdots$ and $\cdots < S_{-1} < S_0 \leqq 0 < S_1 < \cdots$, respectively. We assume that the three objects X, Φ, Ψ are jointly stationary under measure P and that Φ, Ψ have rates $0 < \lambda,\ \mu < \infty$. Let P^0_Φ, P^0_Ψ be the Palm transformations of P with respect to Φ, Ψ. The theorem can then be stated as follows.

Theorem 3. If

$$\lim_{n\to\infty} P^0_\Psi(\theta_{-S_n}B, \Phi(0, S_1] = k) = P^0_\Psi(B)P^0_\Psi(\Phi(0, S_1] = k), \qquad k \geqq 0,\ B \in \mathcal{F}, \tag{6}$$

then the P^0_Φ-distribution of X_{S_n} converges as $n \to \infty$ and the limiting distribution is the P^0_Ψ-distribution of X_{S_k} (which is the same for all $k \in \mathbb{Z}$).

Observe that (6) is implied by the *mixing* condition for the pair $(P^0_\Psi, \{\theta_{S_n}\})$:

$$\lim_{n\to\infty} P^0_\Psi(A \cap \theta_{-S_n}B) = P^0_\Psi(A)P^0_\Psi(B), \qquad A, B \in \mathcal{F}. \tag{7}$$

2. Proof of Theorem 3

We show that for any bounded and continuous function f from the space of values of the process X into the real line,

$$E^0_\Phi f(X_{S_n}) \xrightarrow[n\to\infty]{} E^0_\Psi f(X_0). \tag{8}$$

We start with the *cycle formula* which relates any two Palm transformations of the same measure P:

$$\lambda P^0_\Phi(A) = \mu E^0_\Psi \int_{(0, S_1]} 1_A \circ \theta_t \Phi(dt), \qquad A \in \mathcal{F}. \tag{9}$$

For a proof of this relation see, for instance, Neveu (1976), Baccelli and Brémaud (1987). An integrated version of (9) for the random variable $f(X_{S_n})$ yields

$$\lambda E^0_\Phi f(X_{S_n}) = \mu E^0_\Psi \int_{(0,S_1]} f(X_{S_n}) \circ \theta_t \Phi(dt) \tag{10}$$
$$= \mu E^0_\Psi \sum_{k=1}^{\infty} f(X_{S_n}) \circ \theta_{T_k} 1(T_k \leqq S_1).$$

Observe that on the set $\{T_k \leqq S_1\}$,

$$S_n \circ \theta_{T_k} = S_n - T_k, \quad \text{and so} \quad f(X_{S_n}) \circ \theta_{T_k} = f(X_{S_n}).$$

Hence (10) can be written as

$$\lambda E^0_\Phi f(X_{S_n}) = \mu E^0_\Psi f(X_{S_n}) \sum_{k=1}^{\infty} 1(T_k \leqq S_1) \tag{11}$$
$$= \mu E^0_\Psi f(X_{S_n}) \Phi(0, S_1]$$
$$\underset{n\to\infty}{\longrightarrow} \mu E^0_\Psi f(X_0) E^0_\Psi \Phi(0, S_1],$$

where we used assumption (6) to get the last limit. Now take A to be the whole space in cycle formula (9). This gives

$$E^0_\Psi \Phi(0, S_1] = \frac{\lambda}{\mu}. \tag{12}$$

Substituting (12) into (11) we get the desired result.

3. Comments

The theorems presented above are generalizations of well-known convergence results for renewal/regenerative processes. In the latter case however, they can be obtained independently by means, for example, of coupling. Lindvall (1982) has shown that a synchronous renewal process couples with its stationary version if the interarrival distribution contains an absolutely continuous component. From this, the mixing condition (7) is immediate. In the general case however, there exist no simple criteria for the mixing condition (to the best of our knowledge).

Furthermore, we note that a theorem similar to Theorem 3 has been proved by König and Schmidt (1986) for the special case of a stationary-ergodic feedback queue, under a different mixing condition.

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