ON THE ANNIHILATORS OF THE INJECTIVE HULL OF A MODULE

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In [2, page 151], J. Lambek proposes the following exercise: With any maximal right ideal M of a ring R with 1 associate the ideal $O_M = \{r \varepsilon R : \forall x \varepsilon R \;\; \exists t \notin M, \;\; r x t = 0\}$. Show that O_M is the right annihilator of the injective hull of the right R-module R/M. The purpose of this note is to show that the above statement is true for a much larger class of right ideals than that of maximal regular right ideals of a ring. If R is a ring, let C(R) be a class of right ideals in the ring R such that M ε C(R) if and only if

(i) $R^2 \not\subseteq M$ and there exists $a \varepsilon R$, $a \notin M$, such that $aM \subseteq M$;

(ii) $\mathrm{Hom}_R(\widetilde{R/M}, \widetilde{R/M})$ is a division ring where $\widetilde{R/M}$ is the quasi-injective hull of the right R-module R/M;

(iii) if N is a non-zero submodule of R/M, then there is a non-zero $f \varepsilon \mathrm{Hom}_R(R/M, R/M)$ such that $f(R/M) \subseteq N$.

Clearly, any maximal right ideal M of a ring R with 1, belongs to C(R). However, a member of C(R) is not necessarily a maximal right ideal of the ring R. For example, if R is a commutative ring and P is a prime ideal of R such that $P \neq R$, then R/P satisfies (i) and (iii). By [1, Theorem 3.2, Lemma 3.3], one can also see that R/P satisfies (ii). Hence P ε C(R). In fact if R is a semi-prime ring with a uniform right ideal U such that the (right) singular ideal of R is zero then the right annihilator of the set {u} for $u \varepsilon U$, $u^2 \neq 0$, is a member of C(R) (see [1, Theorem 2.2]).

THEOREM. _Let_ R _be an arbitrary ring with a regular element. If_ M ε C(R) _then_ O_M _is the right annihilator of the injective hull of the right R-module_ R/M.

LEMMA 1. $\{y \varepsilon R/M : yR = 0\} = \{0\}$.

Proof. Let $\Gamma = \{y \varepsilon R/M : yR = 0\}$. If Γ is a non-zero submodule of R/M then by (iii) one can find a non-zero endomorphism f of R/M such that $f(R/M)R \subseteq \Gamma R = 0$ and $R^2 \subseteq M$ since $\mathrm{Ker}\, f = 0$. This of course violates (i).

COROLLARY. _If_ $a \varepsilon R$ _such that_ $aR \subseteq M$ _then_ $a \varepsilon M$.

Proof. If $a \notin M$ then $a + M \varepsilon \Gamma$ in Lemma 1 and Γ would be a non-zero submodule of R/M, which is absurd in view of Lemma 1.

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LEMMA 2. $O_M \subseteq M$.

Proof. For if $O_M \nsubseteq M$, then $O_M + M/M$ is a non-zero submodule of R/M and hence, by (iii), there is a non-zero endomorphism f of R/M such that $f(R/M) \subseteq O_M + M/M$. Let $a \in R$ such that $a \notin M$ and $aM \subseteq M$. Then $f(a + M) = b + M$ for some $b \in O_M$, such that $b \notin M$ since the Ker f is zero by (ii). Since $aM \subseteq M$, $a \notin M$, a induces an endomorphism of R/M, say $g_a : r + M \to ar + M$ for all $r \in R$. g_a is a non-zero endomorphism by the Corollary. Let $t \in R$, $t \notin M$ such that $b\ a\ t = 0$. Then $fg_a(at + M) = b\ a\ t + M = 0$ and $at \notin M$. This means that Ker g_a is not zero. This is a contradiction since the Ker g_a is zero.

Proof of the Theorem. Let $\widehat{R/M}$ be the injective hull of the right R-module R/M and let $(\widehat{R/M})^\gamma = \{r \in R : (\widehat{R/M})r = 0\}$. If there is $r_0 \in (\widehat{R/M})^\gamma$ such that $r_0 \notin O_M$, then there is $a \in R$ such that $r_0at \neq 0$ for any $t \notin M$. That is $(r_0a)^\gamma \subseteq M$. Let c be a regular element in R. Then $(cr_0a)^\gamma \subseteq M$. Let $T = \{x \in \widehat{R/M} : x[(cr_0a)^\gamma] = 0\}$. If $y \in T$, define $f : (cr_0a)r \to yr$. Then f is an R-homorphism from a right ideal cr_0aR into $\widehat{R/M}$. Let $\bar{f}$ be an extension of f to R. Then $y = \bar{f}(cr_0a) = \bar{f}(c)r_0a$ and $y \in (R/M)r_0a$. That is $T \subseteq (\widehat{R/M})r_0a = 0$. By (i) there is $b \in R$, $b \notin M$, such that $bM \subseteq M$. Let $x = b + M$. Then $x[(cr_0a)^\gamma] = 0$ since $(cr_0a)^\gamma \subseteq M$ and $xM \subseteq M$. Therefore $x \in T$ and $b \in M$. This is impossible. Thus $(\widehat{R/M})^\gamma \subseteq O_M$. Now note that for any $r \in R$ and any $b \in O_M$, rb, br are members of O_M. Since $O_M \subseteq M$ by Lemma 2, $RO_M \subseteq M$ and $(R/M)O_M = 0$. If $(\widehat{R/M})O_M \neq 0$ then there exist elements $x \in (\widehat{R/M})$ and $b \in O_M$ such that $xb \neq 0$. Since $(\widehat{R/M})$ is an essential extension of R/M, there is $r_0 \in R$ such that $xbr_0 \neq 0$ and $xbr_0 \in R/M$. Now xbr_0R is a non-zero submodule of R/M by Lemma 1. Let h be a non-zero endomorphism of R/M such that $h(R/M) \subseteq xbr_0R$. Let $a \in R$ such that $a \notin M$ and $aM \subseteq M$. Then $h(a + M) = xbr_0r'$ for some $r' \in R$. Let $t \in R$, $t \notin M$ such that $br_0r't = 0$. Then $h(at + M) = xbr_0r't = 0$ and $at \in M$. This is impossible since the induced endomorphism $g_a : r + M \to ar + M$ is non-zero and Ker $g_a = 0$ by (ii). Thus $(\widehat{R/M})^\gamma = O_M$.

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