Appendix A
Long-wavelength reduction

This appendix is concerned with the long-wavelength reduction of the electromagnetic multipole operators. The analysis follows closely the arguments developed in [Bl52] (see also [de66]). Consider first the transverse electric and magnetic multipoles, which govern real photon transitions.[1]

The use of the relations $1/\hbar c = 5.07 \times 10^{10}\,\text{cm}^{-1}\,\text{MeV}^{-1}$ and $R \approx 1.2\,A^{1/3} \times 10^{-13}\,\text{cm}$ allows one to write for real photons

$$kR \approx 6.1 \times 10^{-3}[E_\gamma(\text{MeV})A^{1/3}] \tag{A.1}$$

Evidently $kR \ll 1$ for photons of a few MeV. In this case, the spherical Bessel functions can be expanded as[2]

$$j_J(kx) \rightarrow \frac{(kx)^J}{(2J+1)!!} \qquad ; \; kx \rightarrow 0 \tag{A.2}$$

One also needs from [Ed74]

$$\mathbf{L}Y_{lm} \quad = \quad \frac{1}{i}(\mathbf{r} \times \nabla)Y_{lm} \quad = \quad \sqrt{l(l+1)}\mathscr{Y}^m_{ll1} \tag{A.3}$$

With this relation, the multipole operators in Eqs. (9.16) take the form

$$\begin{aligned}
\hat{T}^{\text{el}}_{JM} \quad &= \quad \frac{1}{k\sqrt{J(J+1)}}\int d^3x \left\{[\nabla \times \mathbf{L}j_J(kx)Y_{JM}]\cdot\hat{\mathbf{J}}_c(\mathbf{x})\right. \\
&\qquad\qquad \left. + k^2[\mathbf{L}j_J(kx)Y_{JM}] \cdot \hat{\boldsymbol{\mu}}(\mathbf{x})\right\} \\
\hat{T}^{\text{mag}}_{JM} \quad &= \quad \frac{1}{\sqrt{J(J+1)}}\int d^3x \left\{[\nabla \times \mathbf{L}j_J(kx)Y_{JM}] \cdot \hat{\boldsymbol{\mu}}(\mathbf{x})\right. \\
&\qquad\qquad \left. + [\mathbf{L}j_J(kx)Y_{JM}]\cdot\hat{\mathbf{J}}_c(\mathbf{x})\right\}
\end{aligned} \tag{A.4}$$

[1] Recall $\mathbf{x} \equiv \mathbf{r}$ and $x \equiv |\mathbf{x}| \equiv r$ in all these discussions.

[2] One has to get all the derivatives off the Bessel functions before they can be expanded — that is the point of the following exercise.

These expressions can now be manipulated in the following manner:

1. The differential orbital angular momentum operator $\mathbf{L}$ in Eq. (A.3) commutes with any function of the radial coordinate $[\mathbf{L}, f(r)] = 0$, and it is hermitian; thus it can be partially integrated in the last two terms on the r.h.s. in the above to get it over to the right [with a sign (-1)].

2. The divergence theorem in Eqs. (9.13) and (9.14) can be used on the first two terms on the r.h.s. of the above to get the curl to the right.

3. One can then get $\mathbf{L}$ to the right in these terms using the first argument [again with a (-1)]. This leads to two types of terms: first

$$\mathbf{L}\cdot\mathbf{v} \;=\; \frac{1}{i}(\mathbf{r}\times\nabla)\cdot\mathbf{v} \;=\; \frac{1}{i}(\nabla\times\mathbf{v})\cdot\mathbf{r} \;=\; -\frac{1}{i}\nabla\cdot(\mathbf{r}\times\mathbf{v}) \tag{A.5}$$

and second

$$\begin{aligned}\mathbf{L}\cdot(\nabla\times\mathbf{v}) \;&=\; \frac{1}{i}(\mathbf{r}\times\nabla)\cdot(\nabla\times\mathbf{v}) \;=\; \frac{1}{i}[\nabla\times(\nabla\times\mathbf{v})]\cdot\mathbf{r} \\ &=\; -\frac{1}{i}\nabla\cdot[\mathbf{r}\times(\nabla\times\mathbf{v})]\end{aligned} \tag{A.6}$$

Here the relation $\nabla\times\mathbf{r} = 0$ has been used in obtaining these equations.

4. Since all derivatives are now off the spherical Bessel functions and on the source terms, the Bessel functions may be expanded in the long-wavelength limit according to Eq. (A.2).

5. One next invokes the general vector identity

$$\int x^J Y_{JM}\nabla\cdot[\mathbf{r}\times(\nabla\times\mathbf{v})]\, d^3x = (J+1)\int x^J Y_{JM}\nabla\cdot\mathbf{v}\, d^3x \tag{A.7}$$

This identity holds as long as the source terms $\mathbf{v}(\mathbf{x})$ vanish outside the nucleus.

With these steps the magnetic multipoles take the form

$$\hat{T}^{\text{mag}}_{JM} \approx \frac{1}{i}\frac{k^J}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x\, x^J Y_{JM}\left\{\nabla\cdot\hat{\boldsymbol{\mu}}(\mathbf{x}) + \frac{1}{J+1}\nabla\cdot[\mathbf{r}\times\hat{\mathbf{J}}_c(\mathbf{x})]\right\} \tag{A.8}$$

Partial integration of this result then gives for the long-wavelength limit of the transverse magnetic multipoles

$$\hat{T}^{\text{mag}}_{JM} \approx -\frac{1}{i}\frac{k^J}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x\,\left[\hat{\boldsymbol{\mu}}(\mathbf{x}) + \frac{1}{J+1}\mathbf{r}\times\hat{\mathbf{J}}_c(\mathbf{x})\right]\cdot\nabla x^J Y_{JM} \tag{A.9}$$

Similarly, the electric multipole operators take the form

$$\hat{T}^{\text{el}}_{JM} \approx \frac{1}{i}\frac{k^{J-1}}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x\left\{\nabla\cdot\hat{\mathbf{J}}_c(\mathbf{x}) + \frac{k^2}{J+1}\nabla\cdot[\mathbf{r}\times\hat{\boldsymbol{\mu}}(\mathbf{x})]\right\} x^J Y_{JM} \tag{A.10}$$

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Now use the *continuity equation* on the first term

$$\nabla\cdot\hat{\mathbf{J}}_c(\mathbf{x}) \;=\; \nabla\cdot\hat{\mathbf{J}}(\mathbf{x}) \;=\; -\frac{1}{c}\frac{\partial\hat{\rho}}{\partial t} \;=\; -\frac{i}{\hbar c}\,[\hat{H},\hat{\rho}] \tag{A.11}$$

The matrix element of this relation yields

$$\langle f|[\hat{H},\hat{\rho}]|i\rangle \;=\; (E_f-E_i)\langle f|\hat{\rho}|i\rangle \;=\; -\hbar kc\,\langle f|\hat{\rho}|i\rangle \tag{A.12}$$

Thus, in the matrix element, one can replace[3] $\nabla\cdot\hat{\mathbf{J}}_c(\mathbf{x})\to ik\hat{\rho}(\mathbf{x})$. Thus, for photon emission the long-wavelength limit of the transverse electric multipoles takes the form

$$\hat{T}^{\rm el}_{JM}\approx\frac{k^J}{(2J+1)!!}\sqrt{\frac{J+1}{J}}\int d^3x\left\{x^J Y_{JM}\hat{\rho}(\mathbf{x})-\frac{ik}{J+1}\hat{\boldsymbol{\mu}}(\mathbf{x})\cdot[\mathbf{r}\times\nabla x^J Y_{JM}]\right\} \tag{A.13}$$

The first term in Eq. (A.13) is just the JMth multipole of the charge density. The second term goes as $\hbar kc/mc^2\ll 1$ and hence the contribution of this term is very small compared to that of the first term for real photons.[4]

Make a model where the nucleus is composed of individual nucleons, and where only the leading terms to order $1/m$ are retained in the current, that is, the terms in $\mathbf{p}(i)$ and $\boldsymbol{\sigma}(i)$ [see Eqs. (9.17) and (9.20)]. The $J=1$ transverse magnetic dipole operator for $k\to 0$ then takes the form

$$\hat{T}^{\rm mag}_{1M}\approx\frac{i\sqrt{2}}{3}\frac{\hbar k}{2mc}\sqrt{\frac{3}{4\pi}}\left\{\sum_{i=1}^{Z}\mathbf{l}(i)+\sum_{i=1}^{A}\lambda_i\boldsymbol{\sigma}(i)\right\}_{1M} \tag{A.14}$$

This is the familiar magnetic dipole operator to within a numerical factor and power of k. Here the nucleon magnetic moments in nuclear magnetons are given by $\lambda_p=2.793$ for the proton and $\lambda_n=-1.913$ for the neutron.

Static Moments. It is useful to make the connection between these general results for the electromagnetic nuclear moments and the static nuclear moments measured in time-independent electric and magnetic fields.

Consider first the static *electric* moments of the nucleus. Suppose one places a static charge distribution $\rho(\mathbf{r})$ in an *external* electrostatic potential $\Phi_{\rm el}(\mathbf{r})$ where the external electric field is given by $\mathbf{E}=-\nabla\Phi_{\rm el}(\mathbf{r})$ (see Fig. A.1). A relevant example is a nucleus in the field of the atomic electrons. The interaction energy is given by

$$U=e_{\rm p}\int\rho(\mathbf{r})\Phi_{\rm el}(\mathbf{r})\,d^3r \tag{A.15}$$

[3] Note this is for photon *emission*; for photon *absorption* one has the opposite sign for this term.

[4] This term can become large in electron scattering where, as we shall see, the appropriate ratio is $\hbar qc/mc^2$ with q the momentum transfer.

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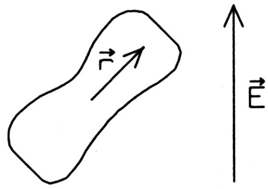

Fig. A.1. Static electric nuclear moments.

The external field satisfies Laplace's equation since it is source-free over the nucleus

$$\nabla^2\Phi_{\rm el}(\mathbf{r}) = 0 \tag{A.16}$$

It is also finite there. Thus the external field in the region of the nucleus can be expanded in terms of the acceptable solutions to Laplace's equation

$$\Phi_{\rm el}(\mathbf{r}) = \sum_{lm} a_{lm} r^l Y_{lm}(\Omega_r) \tag{A.17}$$

The numerical coefficients a_{lm} can be related to various derivatives of the field at the origin. Substitution of Eq. (A.17) into Eq. (A.15) yields

$$U = e_{\rm p} \sum a_{lm} \mathscr{M}^{\rm el}_{lm} \tag{A.18}$$

Here the multipole moments of the charge density are defined by

$$\mathscr{M}^{\rm el}_{lm} = \int x^l Y_{lm}(\Omega_x)\rho(\mathbf{x})\, d^3x \tag{A.19}$$

These are exactly the same expressions, to within a numerical factor and powers of k, as the first term in the transverse electric multipole operators in Eq. (A.13).[5] Note that the nuclear quadrupole moment is conventionally defined by

$$Q = \int (3z^2 - r^2)\rho(\mathbf{x})\, d^3x \tag{A.20}$$

which differs by a numerical constant from $\mathscr{M}^{\rm el}_{20}$.

Consider next the nuclear *magnetic* moments. Take the ground-state expectation value that gives $\langle \partial\hat{\rho}(\mathbf{x})/\partial t\rangle = (i/\hbar)\,\langle[\hat{H},\hat{\rho}]\rangle = 0$. This implies

$$\nabla\cdot\langle\hat{\mathbf{J}}(\mathbf{x})\rangle \quad = \quad \nabla\cdot\langle\hat{\mathbf{J}}_c(\mathbf{x})\rangle = 0 \tag{A.21}$$

Here the general decomposition of current has been invoked

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_c + \nabla \times \hat{\boldsymbol{\mu}} \tag{A.22}$$

[5] The charge multipole *operators* are defined in terms of the charge density *operator*.

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Since the divergence of the last quantity in Eq. (A.21) vanishes everywhere, it can be expressed as the curl of another vector $\mathbf{M}(\mathbf{x})$

$$\langle \hat{\mathbf{J}}_c(\mathbf{x}) \rangle = \nabla \times \mathbf{M}(\mathbf{x}) \tag{A.23}$$

One can assume that the additional magnetization $\mathbf{M}(\mathbf{x})$ vanishes outside the nucleus, for suppose it does not. Then since its curl vanishes outside the nucleus by Eq. (A.23), it can be written as $\mathbf{M}(\mathbf{x}) = \nabla\chi(\mathbf{x})$ in this region. Now choose a new magnetization $\mathbf{M}'(\mathbf{x}) = \mathbf{M}(\mathbf{x}) - \nabla\chi(\mathbf{x})$. This new magnetization has the same curl everywhere, and now, indeed, vanishes outside the nucleus.

The expectation value of the interaction hamiltonian for the nucleus in an external magnetic field now takes the form

$$\langle \hat{H}_{\text{int}} \rangle = -e_{\text{p}} \int [\nabla \times \mathbf{M}(\mathbf{x})] \cdot \mathbf{A}^{\text{ext}}(\mathbf{x})\, d^3x - e_{\text{p}} \int \boldsymbol{\mu}(\mathbf{x}) \cdot \mathbf{B}^{\text{ext}}(\mathbf{x})\, d^3x \tag{A.24}$$

Here $\boldsymbol{\mu} \equiv \langle \hat{\boldsymbol{\mu}} \rangle$. The use of Eqs. (9.13) and (9.14) permits this expression to be rewritten as

$$\langle \hat{H}_{\text{int}} \rangle = -e_{\text{p}} \int [\mathbf{M}(\mathbf{x}) + \boldsymbol{\mu}(\mathbf{x})] \cdot \mathbf{B}^{\text{ext}}(\mathbf{x})\, d^3x \tag{A.25}$$

Since $\mathbf{B}^{\text{ext}}(\mathbf{x})$ is an external magnetic field with no sources over the nucleus, it satisfies Maxwell's equations there

$$\nabla \cdot \mathbf{B}^{\text{ext}} \quad = \quad \nabla \times \mathbf{B}^{\text{ext}} \quad = \quad 0 \tag{A.26}$$

Thus one can write in the region of interest

$$\begin{aligned} \mathbf{B}^{\text{ext}} &= -\nabla\Phi_{\text{mag}} \\ \nabla^2\Phi_{\text{mag}} &= 0 \end{aligned} \tag{A.27}$$

One can now proceed with exactly the same arguments used on the electric moments. The energy of interaction is given by

$$\begin{aligned} \langle \hat{H}_{\text{int}} \rangle &= e_{\text{p}} \int [\mathbf{M}(\mathbf{x}) + \boldsymbol{\mu}(\mathbf{x})] \cdot \nabla\Phi_{\text{mag}}(\mathbf{x})\, d^3x \\ &= -e_{\text{p}} \int \Phi_{\text{mag}} \nabla \cdot (\mathbf{M} + \boldsymbol{\mu})\, d^3x \end{aligned} \tag{A.28}$$

The divergence in the last equation evidently plays the role of the "magnetic charge." Thus, just as before, when the general solution to Laplace's equation is substituted for the magnetic potential Φ_{mag}, all one needs are the magnetic charge multipoles given by

$$\begin{aligned} \mathscr{M}_{lm}^{\text{mag}} &= -\int x^l Y_{lm}(\Omega_x) \nabla \cdot (\mathbf{M} + \boldsymbol{\mu})\, d^3x \qquad \text{(A.29)} \\ &= -\int x^l Y_{lm}(\Omega_x) \nabla \cdot [\frac{1}{l+1} \mathbf{r} \times (\nabla \times \mathbf{M}) + \boldsymbol{\mu}]\, d^3x \end{aligned}$$

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The second equality follows with the aid of the identity in Eq. (A.7). A partial integration, and the restoration to operator form yields the final result for the relevant static magnetic multipole operators

$$\hat{\mathscr{M}}_{lm}^{\text{mag}} = \int d^3x \left[\hat{\boldsymbol{\mu}}(\mathbf{x}) + \frac{1}{l+1}\mathbf{r} \times \hat{\mathbf{J}}_c(\mathbf{x})\right] \cdot \nabla x^l Y_{lm} \tag{A.30}$$

This is recognized to be, within a numerical factor and powers of k, the long-wavelength limit of the transverse magnetic multipole operator in Eq. (A.9).

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