ON THE RELATIVE WIDTHS OF COVERINGS BY CONVEX BODIES

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The purpose of this note is to give an elementary proof of a special case of a theorem suggested by Th. Bang (2; 3) and proved by Lee et al (5; see also 1; 4; 6; 7; 8).

Let $w(K; \xi)$ denote the width of the convex body K in the direction ξ i.e., $w(K; \xi)$ is the length of the interval obtained by projecting K orthogonally onto a line in the direction ξ.

THEOREM. If K, K_1, K_2 are convex regions in the plane with $K \subset K_1 \cup K_2$, then

$$w(K_1; \xi_1)/w(K; \xi_2) + w(K_2; \xi_2)/w(K; \xi_2) \geqslant 1$$

for arbitrary directions ξ_1 and ξ_2.

Proof. If $\xi_1 = \xi_2 (= \xi)$ then K, K_1, K_2 project (on a line in the direction ξ) into intervals I, I_1, I_2. of lengths $w(K; \xi)$, $w(K_1; \xi)$ and $w(K_2; \xi)$ with $I \subset I_1 \cup I_2$. Clearly, in this case $w(K_1; \xi) + w(K_2; \xi) \geqslant w(K; \xi)$.

If $\xi_1 \neq \xi_2$ we may assume them to be perpendicular. (An affine transformation which makes them so leaves convexity, set inclusion and ratio of lengths in the same direction unchanged.) Also, we may assume $K_1 \subset K$, $K_2 \subset K$ (we replace K_1 by $K_1 \cap K$ and K_2 by $K_2 \cap K$ if necessary).

Now let K and K_1 project on a line a (see Fig. 1), in direction ξ_1 into intervals DC and XT; let K and K_2 project on a line b, in direction ξ_2 (perpendicular to ξ_1), into intervals AD and ZY. K_1 is contained in the rectangle r_1 = PQTX and K_2 is contained in the rectangle r_2 = ZRSY. Since $K = K_1 \cup K_2 \subset r_1 \cup r_2$ and K touches the sides of the rectangle r = ABCD it follows that there are lines through E,F,G,H forming a quadrilateral q whose vertices L,M,N,O are not in the interior of r, and q covers K. We pass through these vertices lines parallel to a and b as in Fig. 2. Here it suffices to prove that

$$\overline{X_1T_1}/\overline{C_1D_1} + \overline{Z_1Y_1}/\overline{A_1D_1} \geqslant 1 \quad \text{or}$$

$$\overline{X_1T_1} \cdot \overline{A_1D_1} + \overline{C_1D_1}\ \overline{Z_1Y_1} \geqslant \overline{A_1D_1}\ \overline{C_1D_1}$$

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or $\alpha + \beta + \gamma + \delta \leq$ area (EFGH) where $\alpha, \beta, \gamma, \delta$ denote the areas of the rectangles containing them. From the similar triangles OO_1E and EL_1L, $\overline{OO_1} \cdot \overline{LL_1} = \overline{EO_1} \cdot \overline{EL_1}$ or $\alpha = \alpha'$. Similarly $\beta = \beta'$, $\gamma = \gamma'$, $\delta = \delta'$ and $\alpha + \beta + \gamma + \delta = \alpha' + \beta' + \gamma' + \delta' \leq$ area (EFGH) as desired. This last inequality follows from the fact that the four rectangles containing $\alpha', \beta', \gamma', \delta'$ do not overlap. This is equivalent to the statement: if $\overline{P_1L} \leq \overline{X_1N}$ then $\overline{R_1M} \leq \overline{Z_1O}$. To establish this, observe that $\overline{P_1L} \leq \overline{X_1N}$ implies $\overline{R_1M} \leq R_1S'$. But

$$\overline{EO_1}/\overline{O_1H} = \overline{LL_1}/\overline{D'N'} = \overline{FM_1}/\overline{M_1G} \text{ and } \overline{EO_1} + \overline{O_1H} = \overline{FM_1} + \overline{M_1G}$$

so that $\overline{Z_1O} = \overline{EO_1} = \overline{FM_1} = \overline{R_1S'} \geq \overline{R_1M}$

and the proof is complete.

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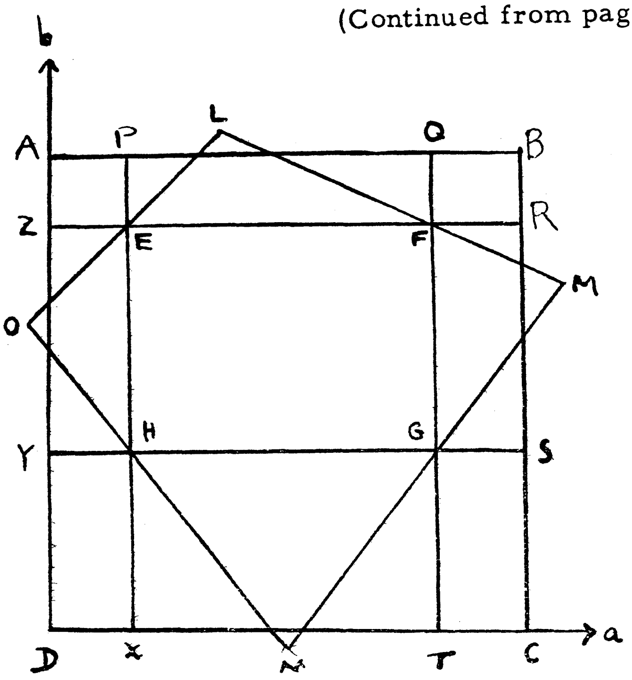

Fig. 1.

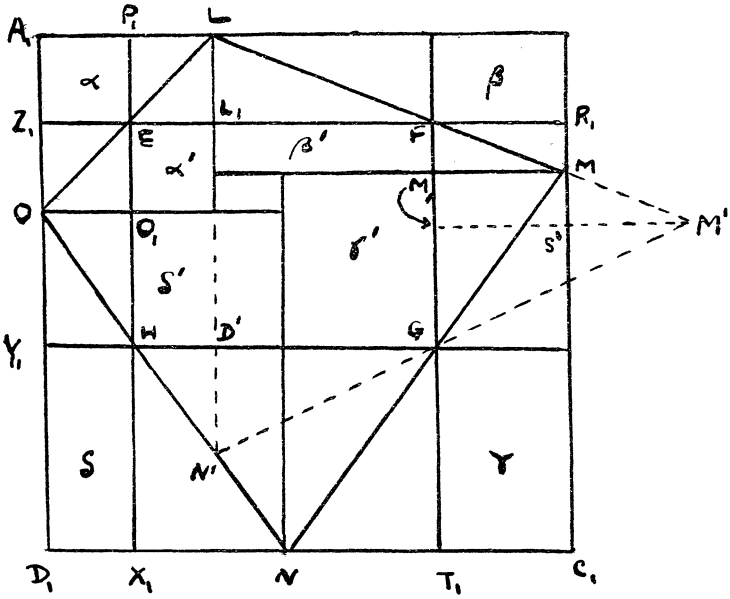

Fig. 2

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