ANALYTICAL SOLUTION OF THE MOTION OF THE PLANETS OVER SEVERAL THOUSANDS OF YEARS

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1. Introduction

The results of a planetary theory built by an iterative method are given here in order to show the relation with the secular variation theories and the meaning of the mean elements in these latter theories. The general theories have a validity span of several millions years but a weak precision; on the contrary, the secular variation theories reach a great precision over several thousand years. Two applications of the analytical planetary theories are presented : the relation between the barycentric coordinates and the geocentric ones; the determination of the terms of precession and nutation for the rigid Earth.

2. General theories

2.1. FORM OF THE SOLUTION

The general theories represent the motion of the planets in Fourier series :

$$\begin{aligned} a_i &= a_{i0} + \sum_{\Phi^*} A_{i,\Phi^*} \cos \Phi^* + \sum_{\Phi} A_{i,\Phi} \cos \Phi \\ \lambda_i &= \lambda_{i0} + n_{i0}t + \delta n_i t + \sum_{\Phi^*} B_{i,\Phi^*} \sin \Phi^* + \sum_{\Phi} B_{i,\Phi} \sin \Phi \\ k_i &= \sum_{k=1}^{8} \lambda_{ik} M_k \cos \psi_k + \sum_{k=1}^{8} M_{i,\psi_k} \cos \psi_k \\ &\quad + \sum_{\Phi^*} \epsilon_{\Phi^*} M_{i,\Phi^*} \cos \Phi^* + \sum_{\Phi} \epsilon_{\Phi} M_{i,\Phi} \cos \Phi \end{aligned}$$

S. Ferraz-Mello et al. (eds.), Dynamics, Ephemerides and Astrometry of the Solar System, 17–28.

$$
\begin{aligned}
h_i = & \sum_{k=1}^{8} \lambda_{ik} M_k \sin \psi_k + \sum_{k=1}^{8} M_{i,\psi_k} \sin \psi_k \\
& + \sum_{\Phi^*} M_{i,\Phi^*} \sin \Phi^* + \sum_{\Phi} M_{i,\Phi} \sin \Phi \\
q_i = & \sum_{k=1}^{8} \mu_{ik} N_k \cos \theta_k + \sum_{k=1}^{8} N_{i,\theta_k} \cos \theta_k \\
& + \sum_{\Phi^*} \epsilon_{\Phi^*} N_{i,\Phi^*} \cos \Phi^* + \sum_{\Phi} \epsilon_{\Phi} N_{i,\Phi} \cos \Phi \\
p_i = & \sum_{k=1}^{8} \mu_{ik} N_k \sin \theta_k + \sum_{k=1}^{8} N_{i,\theta_k} \sin \theta_k \\
& + \sum_{\Phi^*} N_{i,\Phi^*} \sin \Phi^* + \sum_{\Phi} N_{i,\Phi} \sin \Phi
\end{aligned}
\tag{1}
$$

with :

$$
\Phi = \sum_{j=1}^{8} r_j \bar{\lambda}_j + \sum_{j=1}^{8} l_j \psi_j + \sum_{j=1}^{8} m_j \theta_j \tag{2}
$$

and

$$
\epsilon_{\Phi} = \text{sign}\left(\sum_{j=1}^{8} r_j + \sum_{j=1}^{8} l_j + \sum_{j=1}^{8} m_j\right)
$$

$\bar{\lambda}_j$, ψ_j, θ_j represent, respectively, the argument of the longitude, the argument of the Lagrange solution in eccentricity, the argument of the Lagrange solution in inclination, connected to the planet j. The matrices (λ_{ik}) and (μ_{ik}) are the matrices of the eigenvectors of the Lagrange solution. In the expressions (1), Φ^* arguments correspond to the case in which all the r_j of the formula (2) are zeros, that is to say to the long period terms.

2.2. LONG PERIOD VARIATIONS OF THE SEMI MAJOR AXIS AND OF THE LONGITUDE

General theories are usually limited to the second order with respect to the disturbing masses and the quantities A_{i,Φ^*} of the formula (1) are zeros. In the study of the motion of the four outer planets Jupiter, Saturn, Uranus and Neptune, (Bretagnon and Simon, 1990), (Bretagnon and Francou, 1992) have determined, by an iterative method, perturbations of orders greater than 2 and thus obtained long period terms in the semi major axes. We find, for instance in the semi major axis of Saturn :

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$$
\begin{aligned}
a_{Saturn} = \;& 9.554\,858\,819 & \\
& +0.000\,188\,232\cos(\psi_5-\psi_6) & 54\,017\\
& +0.000\,022\,210\cos(2\psi_5-2\psi_6) & 27\,009\\
& +0.000\,015\,287\cos(\psi_6-\psi_7) & 51\,540\\
& +0.000\,004\,720\cos(2\psi_6-2\theta_6) & 11\,873\\
& +0.000\,003\,587\cos(\psi_5-\psi_7) & 1\,124\,076\\
& +0.000\,003\,309\cos(\psi_5-2\psi_6+\psi_7) & 26\,375\\
& -0.000\,002\,358\cos(2\psi_5-\psi_6-\psi_7) & 56\,744\\
& +\cdots & \qquad (3)
\end{aligned}
$$

In the formula (3), we have given the period of each term in years. By the third Kepler law, these terms produce long period terms with very large amplitude in the longitude. The formula (4) gives the most important long period perturbations of the longitude of Saturn as well as the most important short period terms.

$$
\begin{aligned}
\lambda_{Saturn} = \;& 0.927\,745 + 213.298\,190t\\
& +0.055\,714\sin(\psi_5-\psi_6)\\
& -0.012\,999\sin(\psi_5-\psi_7)\\
& -0.011\,780\sin(\psi_5-\psi_6-\theta_6+\theta_7)\\
& -0.004\,036\sin(\psi_6-\psi_7)\\
& +0.003\,238\sin(2\psi_5-2\psi_6)\\
& -0.000\,744\sin(2\psi_5-\psi_6-\psi_7)\\
& -0.000\,477\sin(2\psi_5-2\psi_6-\theta_6+\theta_7)\\
& +0.000\,462\sin(\psi_5-2\psi_6+\psi_7)\\
& -0.000\,399\sin(\theta_6-\theta_7)\\
& +\cdots\\
& +0.014\,273\sin(2\lambda_5-5\lambda_6+3\psi_6)\\
& +0.003\,917\sin(2\lambda_5-5\lambda_6-\psi_5+4\psi_6)\\
& +0.002\,594\sin(\lambda_5-\lambda_6)\\
& +0.002\,199\sin(2\lambda_5-5\lambda_6+\psi_5+2\psi_6)\\
& +0.001\,172\sin(2\lambda_5-5\lambda_6+\psi_5+3\psi_6-\psi_7)\\
& -0.001\,108\sin(2\lambda_5-5\lambda_6-\psi_5+3\psi_6+\psi_7)\\
& -0.001\,070\sin(\lambda_5-2\lambda_6+\psi_5)\\
& +0.000\,934\sin(\lambda_5-2\lambda_6+\psi_6)\\
& +\cdots \qquad (4)
\end{aligned}
$$

The amplitudes are in radians, time in thousands Julian years from J2000.

These long period perturbations of the longitudes with very large amplitudes are therefore essentially of the third order with respect to the masses. We give the more important ones, expressed in arcseconds, in table 1 for the longitudes of Jupiter and of Saturn, in table 2 for the ones of Uranus and of Neptune.

TABLE 1. Amplitude of the long period terms in the longitudes of Jupiter and Saturn. The unit is the arcsecond.

Argument	Period	B_{J,Φ^*}	B_{S,Φ^*}
$\psi_5 - \psi_6$	54 017	−4 653	11 492
$\psi_5 - \psi_7$	1 124 076	1 606	−2 681
$\psi_5 - \psi_6 - \theta_6 + \theta_7$	1 996 218	986	−2 430
$\psi_6 - \psi_7$	51 540	345	−832
$2\psi_5 - 2\psi_6$	27 009	−272	668
$2\psi_5 - \psi_6 - \psi_7$	56 744	62	−154
$2\psi_5 - 2\psi_6 - \theta_6 + \theta_7$	52 594	40	−98
$\psi_5 - 2\psi_6 + \psi_7$	26 375	−39	95

TABLE 2. Amplitude of the long period terms in the longitudes of Uranus and Neptune. The unit is the arcsecond.

Argument	Period	B_{U,Φ^*}	B_{N,Φ^*}
$\psi_5 - \psi_7$	1 124 076	−20 583	9 996
$\psi_7 - \psi_8$	535 721	−431	263
$\psi_5 - \psi_6$	54 017	−248	13
$\psi_5 - \psi_8$	362 810	220	−250
$\theta_7 - \theta_8$	562 640	159	−43
$\psi_6 - \psi_7$	51 540	−97	22
$2\psi_5 - 2\psi_7$	562 038	−13	−34

2.3. LONG PERIOD VARIATIONS OF THE PLANETARY ORBITS

The general theories are developed at the Bureau des Longitudes for all the planets since 1970. They take into account the mutual perturbations of all the planets one another, the relativistic effects, the lunar perturbations. The long period terms of the variables k, h, q, p give the variations of the planetary orbits over several millions years. These solutions bear a fundamental part in paleoclimatology because they allow to date with a great accuracy the glaciations of the Earth of the quaternary period (Berger, 1973), the climatic variations of Mars (Ward, 1979), (Borderies, 1980).

3. Secular variation theories

The general theories represent the motion of the planets over very large time spans but with a weak precision. The secular variation theories attempt to reach a great precision over time spans restricted to a few thousands of years. Therefore it is useless to keep the long period terms in a periodic form. By construction, we directly determine the time polynomial corresponding to the long period part of the formulas (1) and the Poisson series corresponding to the short period terms. Thus, to the long period terms of the formula (3) corresponds the time polynomial of the semi major axis of Saturn :

$$a_{Saturn} = 9.554\,909\,1915 - 21.3896 \times 10^{-6}t + 444 \times 10^{-10}t^2 + 670 \times 10^{-10}t^3 + 110 \times 10^{-10}t^4 \quad (5)$$

where t is reckoned in thousands Julian years from J2000.

The first determinations of these secular terms of the semi major axes were obtained by (Simon and Chapront, 1974). Duriez (1978) has given a proof of the Poisson theorem and established that these terms are of order equal or greater than 3 with respect to the planetary masses.

The expansion of the long period terms with respect to time reduces considerably the size of the series and allows to reach a great precision. Thus, the longitude of Saturn of the formula (4) becomes :

$$\begin{aligned}\lambda_{Saturn} = {} & 0.874\,016\,284 + 213.299\,104\,960t + 0.000\,366\,597t^2 \\ & -806 \times 10^{-9}t^3 - 557 \times 10^{-9}t^4 \\ & +0.013\,944\,575 \sin(2\lambda_5 - 5\lambda_6) + 0.002\,196\,781 \cos(2\lambda_5 - 5\lambda_6) \\ & -0.001\,590\,423t \sin(2\lambda_5 - 5\lambda_6) + 0.005\,404\,368t \cos(2\lambda_5 - 5\lambda_6) \\ & -0.001\,061\,109t^2 \sin(2\lambda_5 - 5\lambda_6) - 0.000\,474\,470t^2 \cos(2\lambda_5 - 5\lambda_6) \\ & + \cdots \end{aligned} \quad (6)$$

In the formula (6), the coefficients are in radians.

The polynomial part of a variable is the mean element of this variable. It represents the development with respect to time of the long period part of the general theories. The mean element contains the most important variations but it does not represent a good approximation of the solution, particularly for the outer planets which include very large short period perturbations as we see in the expression (6).

The present analytical theories VSOP82 (Bretagnon, 1982), TOP82 (Simon, 1983), VSOP87 (Bretagnon and Francou, 1988) have, over one century around J2000, an accuracy of about 10^{-7} for Jupiter and Saturn and of about 10^{-8} for the other planets. They include the mutual perturbations of all the planets one another up to an order with respect to the masses equal or greater than 3, the relativistic contributions of the Schwarzschild problem, the perturbations by the Moon and some asteroids. They are computed using the planetary mass values of IAU (Grenoble, 1976); the integration constants are determined by comparison to DE200 numerical integration of JPL (Standish, 1982).

The precisions of these solutions seem good enough for the two following applications : the determination of the difference between the barycentric time TCB and the geocentric time TCG and the computation of the terms of precession and of nutation for the rigid Earth.

4. Relation between TCB and TCG

The analytical solutions of the motion of the planets and of the Moon ELP2000/82 (Chapront-Touzé and Chapront, 1983) were used in the computation of the relation between the Barycentric Coordinate Time (TCB) and the Geocentric Coordinate Time (TCG) par (Hiramaya *et al* 1987), (Fairhead and Bretagnon, 1990).

Restricted to the terms proportional to c^{-2}, the relation between TCB and TCG is written :

$$\begin{aligned} \mathrm{TCB} &= \mathrm{TCG} + c^{-2}\int_{TCB_0}^{TCB} (U_E + \tfrac{1}{2}v_E^2)\,\mathrm{d}(TCB') \\ &= \mathrm{TCG} + L_C\,TCB \\ &\qquad + 1\,656.674\,564\,\mu\mathrm{s}\,\sin(6\,283.075\,850\,TCB + 6.240\,054) + \cdots \end{aligned}$$

with $L_C = 1.480\,826\,8475 \times 10^{-8}$ and TCB in thousands Julian years.

U_E represents the external mass force function evaluated at the geocentre and has been computed taking into account the Sun, the Moon and the planets from Mercury to Neptune.

By comparison to numerical integrations, T. Fukushima and A. Irwin have shown that this solution has an accuracy of 1.8 ns over (1980–2000). At the beginning, the solution of (Fairhead and Bretagnon, 1990) retained only the periodic terms greater than 0.1 ns. To obtain the precision of 1.8 ns, we have taken into account the 971 periodic terms greater than 0.01 ns.

To improve the relation between TCB and TCG, we have :
- to compute U_E and v_E with planetary motion solutions using recent values of the planetary masses, for instance the ones of the IERS Standards 1992 (McCarthy, 1992);
- to take into account Pluto : $\Delta L_C^{Pluto} \sim 2 \times 10^{-18}$;
- to take into account asteroids (Fukushima, 1995) : $\Delta L_C^{A} \sim 4.5 \times 10^{-18}$;
- to determine the terms proportional to c^{-4}. For these terms, (Moisson, 1995) finds :

$$\begin{aligned} \Delta(TCB - TCG) = &\quad c^{-4}\int_{TCB_0}^{TCB} (\frac{1}{8}v_E^4 + \frac{3}{2}U_E v_E^2 - \frac{1}{2}U_E^2)\,\mathrm{d}(TCB') \\ = &\quad 1.0965 \times 10^{-16}\,TCB - 0.10 \times 10^{-20}\,TCB^2 \\ &\quad -7^s\!.3 \times 10^{-12}\sin(\lambda_3) - 31^s\!.9 \times 10^{-12}\cos(\lambda_3) + \cdots \end{aligned}$$

with TCB in thousands Julian years.

5. Precession and nutation for the rigid Earth

5.1. EQUATIONS OF THE MOTION

We have established the motion equations for the rigid Earth with the Euler angles ψ, ω, φ reckoned in the positive direction and ω being the rotation angle from ecliptic J2000 to the equator of date. We have therefore :

$$\psi = -\psi_A$$
$$\omega = -\omega_A$$

where ψ_A and ω_A represent the luni-solar precession and the obliquity with the notations of (Lieske *et al* 1977). The sidereal time φ is given by :

$$\varphi = \varphi_0 + \varphi_1 t + \Delta\varphi$$

with :

$$\varphi_0 = 4.903\,562\,579\,35$$
$$\varphi_1 = 2\,301\,216.753\,1542 \ \text{rd/thousand Julian years}$$

We also defined :

$$\tilde{\varphi} = \varphi + \alpha$$

with :

$$\alpha = -14°.95$$

longitude of major axis of equatorial ellipse (Bursa, 1992).

Then, the equations are written :

$$\begin{aligned} \ddot{\omega} + \frac{C}{A} \sin\omega_0\, \varphi_1\, \dot{\psi} &= \frac{L}{A} + F_2 + \frac{B-A}{A} F_1 \\ \sin\omega_0\, \ddot{\psi} - \frac{C}{A}\, \varphi_1\, \dot{\omega} &= \frac{M}{A} + G_2 + \frac{B-A}{A} G_1 \\ \ddot{\varphi} &= \frac{N}{C} + H_2 + \frac{B-A}{C} H_1 \end{aligned} \tag{7}$$

with :

$$\begin{aligned} F_2 &= -\frac{C}{A}\dot{\psi}\varphi_1(\sin\omega - \sin\omega_0) - \frac{C}{A}\dot{\psi}\Delta\dot{\varphi}\sin\omega - \frac{C-A}{A}\dot{\psi}^2 \sin\omega\cos\omega \\ G_2 &= -\ddot{\psi}(\sin\omega - \sin\omega_0) + \frac{C}{A}\dot{\omega}\Delta\dot{\varphi} - \frac{A+B-C}{A}\dot{\psi}\dot{\omega}\cos\omega \\ H_2 &= -\ddot{\psi}\cos\omega + \dot{\psi}\dot{\omega}\sin\omega \end{aligned} \tag{8}$$

and :

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$$\begin{aligned}
F_1 = \quad & \ddot{\psi} \sin\tilde{\varphi}\cos\tilde{\varphi}\sin\omega + \dot{\psi}\dot{\varphi}(\cos^2\tilde{\varphi} - \sin^2\tilde{\varphi})\sin\omega - 2\dot{\omega}\dot{\varphi}\sin\tilde{\varphi}\cos\tilde{\varphi} \\
& -\ddot{\omega}\sin^2\tilde{\varphi} + \dot{\psi}^2\cos^2\tilde{\varphi}\sin\omega\cos\omega \\
G_1 = \quad & 2\dot{\psi}\dot{\varphi}\sin\tilde{\varphi}\cos\tilde{\varphi}\sin\omega + \ddot{\omega}\sin\tilde{\varphi}\cos\tilde{\varphi} + \dot{\omega}\dot{\varphi}(\cos^2\tilde{\varphi} - \sin^2\tilde{\varphi}) \\
& +\dot{\psi}^2\sin\tilde{\varphi}\cos\tilde{\varphi}\sin\omega\cos\omega - \ddot{\psi}\cos^2\tilde{\varphi}\sin\omega \\
H_1 = \quad & \dot{\omega}^2\sin\tilde{\varphi}\cos\tilde{\varphi} - \dot{\psi}\dot{\omega}(\cos^2\tilde{\varphi} - \sin^2\tilde{\varphi})\sin\omega - \dot{\psi}^2\sin\tilde{\varphi}\cos\tilde{\varphi}\sin^2\omega
\end{aligned} \tag{9}$$

Let $Oxyz$ be the reference frame ecliptic J2000 and $OXYZ$ the non-rotating equator of date. From $Oxyz$, we define $OXYZ$ with 2 rotations:
- a rotation with ψ angle around z axis;
- a rotation with ω angle around X axis.

The quantities L, M, N of the formula (7) represent thus the components, in $OXYZ$, of the torque of the external forces with respect to the geocenter O. A, B, C are the moments of inertia.

5.2. USED MODEL

In the computation of the quantities L, M, N, we take into account, for the influence by the Moon, the terms of the terrestrial potential depending on $C_{n,0}$ for n from 2 to 5, on $C_{2,2}$, $S_{2,2}$ and $C_{3,k}$, $S_{3,k}$ for k from 1 to 3; for the influence by the Sun : $C_{2,0}$, $C_{3,0}$, $C_{2,2}$, $S_{2,2}$, $C_{3,1}$, $S_{3,1}$; for the influence by the planets from Mercury to Neptune : $C_{2,0}$. The lunar theory used is ELP2000/82 (Chapront-Touzé and Chapront, 1983); the one of the planets and the Sun is VSOP87A (Bretagnon and Francou, 1988). In this study, we take the following choices :

a) We study the variations of the rigid Earth equator with respect to the ecliptic and the equinox J2000. So, we have to take into account the perturbations of the equator by the Moon, the Sun and the planets, the motion of which is expressed in rectangular coordinates with respect to the ecliptic and the equator J2000.

b) The solution is expanded in Fourier and Poisson series, the angles of which are linear combinations of the planet longitudes λ_i reckoned from the equinox J2000 and of the Delaunay angles which do not depend on the origin. In consequence, the 18.6 year period perturbation is represented by the angle $\lambda_3 + D - F$ which differs from the longitude Ω of the node of the Moon referred to the equinox of date :

$$\lambda_3 + D - F = \Omega + 180° - p \times t$$

where p is the constant of the precession in longitude and t the time reckoned from J2000.

c) In the Fourier and Poisson series, we keep only the linear part of the mean longitudes of the planets and of the Delaunay angles D, F, l_M; the poly-

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nomial parts, the degree of which is equal or greater than 2 are expanded in Poisson series.

d) The perturbations due to the Moon are computed as a whole. We do use a representation of the lunar motion in rectangular coordinates, containing the perturbations of the main problem, the direct and indirect planetary perturbations, the perturbations due to the terrestrial potential, the tidal effects. In the same way, we use a solution of the Sun in rectangular coordinates reckoned with respect to the Earth but not to the Earth-Moon barycenter.

e) The computation was performed with the value of the precession constant given by (Williams *et al* 1991) and used by (Simon *et al* 1994) :

$$p = 50\,288''.200/\text{thousand Julian years.}$$

This value corresponds to :

$$\left(\frac{\mathrm{d}\psi_A}{\mathrm{d}t}\right)_{t=0} = 50\,385''.0672.$$

The value of the geodesic precession p_g determined by (Brumberg *et al* 1991) is :

$$p_g = 19''.1988.$$

We have therefore solved equations (7) fixing the value of the moment of inertia C :

$$C = 1.805\,465\,872 \times 10^{-15} (m_S\,\text{au}^2) \tag{10}$$

in order to obtain :

$$-\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)_{t=0} = 50\,385''.0672 + 19''.1988 = 50\,404''.2660. \tag{11}$$

The value of the obliquity is :

$$\varepsilon_0 = 23^\circ 26' 21''.412.$$

In the relation (10), m_S is the mass of the Sun :

$$m_S = 332\,946.045 m_E$$

At this value of C corresponds the dynamical ellipticity H_d :

$$H_d(50\,288''.2) = 0.003\,273\,800\,45$$

For $p = 50\,287''.7$, one obtains :

$$H_d(50\,287''.7) = 0.003\,273\,767\,98$$

TABLE 3. Most important perturbations of the nutation. Amplitudes are in $10^{-6\prime\prime}$, periods in days.

Origin	Argument	Amplitude in ψ	Amplitude in ω	Period
Moon $C_{2,0}$	$\lambda_3 + D - F$	17 292 345.65	9 227 970.05	6 793.48
$C_{3,0}$	$\lambda_3 + D - l$	104.05	88.95	3 232.61
$C_{4,0}$	$\lambda_3 + D - F$	0.73	6.84	6 793.48
$C_{5,0}$	$\lambda_3 + D - l$	0.01	0.00	3 232.61
$C_{2,2} - S_{2,2}$	$2\lambda_3 + 2D - 2\varphi$	29.44	11.71	0.52
$C_{3,1} - S_{3,1}$	$\lambda_3 + D + \varphi$	38.44	15.25	0.96
$C_{3,2} - S_{3,2}$	$\lambda_3 + D - 2\varphi$	0.39	0.14	0.51
$C_{3,3} - S_{3,3}$	$\lambda_3 + D - 3\varphi$	0.41	0.20	0.34
Sun $C_{2,0}$	$2\lambda_3$	1 276 723.69	552 395.17	182.63
$C_{3,0}$	λ_3	0.26	0.22	365.26
$C_{2,2} - S_{2,2}$	$2\lambda_3 - 2\varphi$	12.32	4.90	0.50
$C_{3,1} - S_{3,1}$	$\lambda_3 + \varphi$	2.79	1.11	1.00
Mercury $C_{2,0}$	$\lambda_1 - 4\lambda_3$	1.03	0.43	2 432.11
Venus $C_{2,0}$	$3\lambda_2 - 5\lambda_3$	216.71	90.76	2 959.21
Mars $C_{2,0}$	$\lambda_3 - 2\lambda_4$	11.55	0.95	5 764.01
Jupiter $C_{2,0}$	$2\lambda_5$	104.41	45.69	2 166.29
Saturn $C_{2,0}$	$2\lambda_6$	12.15	5.16	5 379.61
Uranus $C_{2,0}$	$2\lambda_7$	0.65	0.29	15 344.24
Neptune $C_{2,0}$	$2\lambda_8$	0.40	0.16	30 091.15
Complements	$\lambda_3 + D - F$	15 361.43	0.04	6 793.48

5.3. RESULTS

We give in table 3 the most important perturbation of each component. For the planets, it is the direct influence which is concerned. Complements correspond to the quantities F_1, G_1, H_1, F_2, G_2, H_2 of the equations (7). Table 4 gives the different components to the secular terms of ψ_A and of ω_A. In table (5) we compare the polynomial parts of ψ_A and ω_A to the results of (Simon *et al* 1994) and of (Williams, 1994). The difference with Williams *et al* in $\psi_A(t)$ results from a different choice of precession constant and an insufficient model in Simon *et al* explains the difference in $\omega_A(t)$. Besides, Williams takes into account the secular variation of J_2 that explains the discrepancy in $\psi_A(t^2)$. With the value of $\dot{J}_2$ (Bursa, 1992) :

$$\dot{J}_2 = (-2.8 \pm 0.3) \times 10^{-9}/\text{century}$$

we compute the following perturbation :

$$\begin{aligned}\Delta\psi_A(\dot{J}_2) = \quad & -0''.651\,804t^2 + 0''.001\,849t^3 + 0''.000\,022t^4 + \cdots \\ & -0''.000\,447t\sin(\lambda_3 + D - F) + \cdots\end{aligned}$$

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TABLE 4. Secular term of ψ_A and of ω_A in arcseconds per thousand years.

Origin	ψ_A	ω_A
Moon $C_{2,0}$	34 455.298 798	−0.254 417
$C_{3,0}$	−0.000 057	−0.000 011
$C_{4,0}$	0.025 192	
Sun $C_{2,0}$	15 948.860 274	0.002 923
$C_{3,0}$	−0.000 026	−0.000 005
Mercury $C_{2,0}$	0.003 698	−0.000 088
Venus $C_{2,0}$	0.181 582	−0.016 814
Mars $C_{2,0}$	0.005 999	0.000 357
Jupiter $C_{2,0}$	0.117 060	0.002 804
Saturn $C_{2,0}$	0.005 208	0.000 220
Uranus $C_{2,0}$	0.000 100	0.000 001
Neptune $C_{2,0}$	0.000 029	0.000 001
Complements	−0.231 857	
$-p_g$	−19.198 800	
	50 385.067 200	−0.265 029

$$\Delta\omega_A(\dot{J}_2) = \begin{array}{l} 0''.000\,003t^2 - 0''.000\,088t^3 + 0''.000\,150t^4 + \cdots \\ +0''.000\,239t\cos(\lambda_3 + D - F) + \cdots \end{array}$$

TABLE 5. Secular variations of ψ_A and of ω_A.

	t	t^2	t^3	t^4
ψ_A	50 385.067 200	−107.246 837	−1.144 309	1.329 708
Simon *et al*	50 385.067 200	−107.237 4	−1.142 4	1.327 9
Williams	50 384.565 010	−107.897 7	−1.141	1.33
ω_A	−0.265 029	5.129 643	−7.732 154	−0.004 852
Simon *et al*		5.129 4	−7.727 6	−0.004 8
Williams	−0.244 00	5.126 8	−7.727	

From ψ_A and ω_A, we have computed the variables p_A and ε_A and compared to the KS solution of (Kinoshita and Souchay, 1990). Kinoshita and Souchay use the following value of the precession constant :

$$p = 50\,290''.966.$$

So, we have multiplied their solution by :

$$\text{coef} = \frac{50\,404.266}{50\,407.032}$$

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TABLE 6. Difference p_A−coef KS. Unit is $10^{-6}{}''$.

Argument	sin	cos	Period
$2\lambda_5 - 5\lambda_6$	−499	−660	883 y
$3\lambda_3$	216	888	121.75 d
$\lambda_3 + D - F$	−25	−384	6 793.48 d
$2\lambda_3 + 2D - F - l_M$	20	281	6 167.21 d
$4\lambda_3 - 8\lambda_4 + 3\lambda_5$	−52	166	1 783 y
$\lambda_3 + D - l_M$	−106	96	3 232.61 d
$8\lambda_2 - 13\lambda_3$	79	31	239 y
$2\lambda_3 + 2D$	81	−3	13.66 d
$F - l_M$	−33	33	2 190.35 d

in order to make the solutions comparables. We give in table 6 the most important discrepancies p_A − coef KS.

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