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NEW CHARACTERIZATIONS OF THE MEROMORPHIC BESOV, Q_p AND RELATED CLASSES

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Abstract

Functions in the meromorphic Besov, Q_p and related classes are characterized in terms of double integrals of certain oscillation quantities involving chordal distances. Some of the results are analogous to the corresponding results in the analytic case.

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1. Introduction and results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \, dA(z) < \infty,$$

where dA denotes the element of the Euclidean area measure on $\mathbb{D}$. It is well known that

$$\|f - f(0)\|^p_{A^p_\alpha} \simeq \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p+\alpha} \, dA(z) \tag{1.1}$$

for all $f \in \mathcal{H}(\mathbb{D})$ [5, 9]. As usual, the symbol $\simeq$ refers to the asymptotic equality which means that the quantities in other sides of the symbol are comparable, that is, their quotient is bounded and bounded away from zero.

Let $\mathcal{M}(\mathbb{D})$ denote the class of all meromorphic functions in $\mathbb{D}$. For $f \in \mathcal{M}(\mathbb{D})$, the spherical derivative of f at z is defined as $f^\#(z) := |f'(z)|/(1 + |f(z)|^2)$. The chordal

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distance between the points z and w in the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is

$$\chi(z, w) := \begin{cases} \dfrac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} & \text{if } z, w \neq \infty, \\ \dfrac{1}{\sqrt{1+|z|^2}} & \text{if } w = \infty. \end{cases}$$

The following result establishes a partial analogue of the asymptotic equality (1.1) for functions in $\mathcal{M}(\mathbb{D})$.

THEOREM 1.1. *Let $1 \le p < \infty$ and $-1 < \alpha < \infty$, and let $f \in \mathcal{M}(\mathbb{D})$. Then there exists a positive constant C, depending only on p and α, such that*

$$\int_{\mathbb{D}} \chi(f(z), f(0))^p (1-|z|^2)^\alpha \, dA(z) \le C \int_{\mathbb{D}} (f^\#(z))^p (1-|z|^2)^{p+\alpha} \frac{dA(z)}{|z|}.$$

Analogously to the analytic case, the meromorphic Bergman class M^p_α is defined as the set of those $f \in \mathcal{M}(\mathbb{D})$ for which

$$\|f\|^p_{M^p_\alpha} := \int_{\mathbb{D}} \chi(f(z), 0)^p (1-|z|^2)^\alpha \, dA(z) < \infty.$$

Theorem 1.1 implies that

$$\|f\|^p_{M^p_\alpha} \lesssim \int_{\mathbb{D}} (f^\#(z))^p (1-|z|^2)^{p+\alpha} \frac{dA(z)}{|z|} + \chi(f(0), 0)^p \tag{1.2}$$

for all $1 \le p < \infty$, $-1 < \alpha < \infty$ and $f \in \mathcal{M}(\mathbb{D})$. The notation $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. The symbol $\gtrsim$ is understood in a similar fashion. We do not know whether the converse of the asymptotic inequality (1.2) holds or not.

Theorem 1.1 has several consequences. The first one shows that functions in the Besov classes satisfy a certain double integral condition. For $0 < p < \infty$, the Besov class $B^\#_p$ consists of those $f \in \mathcal{M}(\mathbb{D})$ for which

$$\|f\|^p_{B^\#_p} := \int_{\mathbb{D}} (f^\#(z))^p (1-|z|^2)^{p-2} \, dA(z) < \infty.$$

For studies on meromorphic Besov classes, see [1, 2, 7, 12, 16]. For $a \in \mathbb{D}$, we define $\varphi_a(z) := (a-z)/(1-\overline{a}z)$. Then φ_a is the automorphism of $\mathbb{D}$ which interchanges the origin and the point a.

THEOREM 1.2. *Let $1 \le p < \infty$ and $-1 < \alpha < \infty$, and let $f \in \mathcal{M}(\mathbb{D})$. Then there exists a positive constant C, depending only on p and α, such that*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\chi(f(z), f(w))^p}{|1-\overline{w}z|^4} (1-|\varphi_w(z)|^2)^\alpha \, dA(z) \, dA(w) \le C \|f\|^p_{B^\#_p}.$$

A result by Stroethoff [15] states that $f \in \mathcal{H}(\mathbb{D})$ belongs to the analytic Besov space B_p, $2 < p < \infty$, if and only if

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \left| \frac{f(z) - f(w)}{z - w} \right|^p (1 - |z|^2)^{(p/2)-2} (1 - |w|^2)^{(p/2)-2} \, dA(z) \, dA(w) < \infty.$$

The case $\alpha = (p/2) - 2 > -1$ of Theorem 1.2 establishes the following partial meromorphic analogue of this fact. The inequality of the opposite direction for normal meromorphic functions is discussed in Corollary 1.9.

COROLLARY 1.3. *Let $2 < p < \infty$ and $f \in \mathcal{M}(\mathbb{D})$. Then there exists a positive constant C, depending only on p, such that*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \left(\frac{\chi(f(z), f(w))}{|1 - \overline{w}z|} \right)^p (1 - |z|^2)^{(p/2)-2} (1 - |w|^2)^{(p/2)-2} \, dA(z) \, dA(w) \\ \leq C \|f\|^p_{B^\#_p}.$$

An application of Theorem 1.1 with $\alpha = 0$ to the function $(f \circ \varphi_w)(rz)$ yields

$$\int_{D(w,r)} \chi(f(z), f(w))^p \, dA(z) \lesssim \int_{D(w,r)} (f^\#(z))^p (1 - |z|^2)^p \frac{dA(z)}{|\varphi_w(z)|}, \tag{1.3}$$

where $D(w, r) := \{z : |\varphi_w(z)| < r\}$ is the pseudohyperbolic disc of (pseudohyperbolic) center $w \in \mathbb{D}$ and radius $r \in (0, 1)$, and the constant of comparison depends only on r. This fact can be used to prove Theorem 1.4. The class $M^\#(p, q, s)$ [13, 16] consists of those $f \in \mathcal{M}(\mathbb{D})$ for which

$$\|f\|^p_{M^\#(p,q,s)} := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^\#(z))^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s \, dA(z) < \infty.$$

THEOREM 1.4. *Let $1 \leq p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$ and $0 < r < 1$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta = q - p$ and $\gamma + \delta = s$, and let $f \in \mathcal{M}(\mathbb{D})$. Then*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1}{|D(z, r)|} \int_{D(z,r)} \chi(f(z), f(w))^p (1 - |z|^2)^\alpha (1 - |w|^2)^\beta \right. \\ \left. \cdot (1 - |\varphi_a(z)|^2)^\gamma (1 - |\varphi_a(w)|^2)^\delta \, dA(w) \right) dA(z) \lesssim \|f\|^p_{M^\#(p,q,s)}.$$

Here $|D(z, r)|$ denotes the Euclidean area of $D(z, r)$, so

$$|D(a, r)| = \pi r^2 \frac{(1 - |a|^2)^2}{(1 - |a|^2 r^2)^2} \tag{1.4}$$

by [6, p. 3].

The class $\mathcal{N}$ of normal functions consists of those $f \in \mathcal{M}(\mathbb{D})$ for which the family $\{f \circ \psi\}$, where ψ is a Möbius transformation of $\mathbb{D}$, is normal in $\mathbb{D}$ in the sense of Montel. It is known that $f \in \mathcal{M}(\mathbb{D})$ is normal if and only if $\|f\|_{\mathcal{N}} := \sup_{z \in \mathbb{D}} f^\#(z)(1 - |z|^2) < \infty$ [11]. The following result establishes a sufficient condition for normal meromorphic functions to belong to $M^\#(p, q, s)$.

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Theorem 1.5. *Let $1 \le p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ and $0 < r < 1$, and let $f \in \mathcal{N}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta = q - p$ and $\gamma + \delta = s$. Then*

$$\|f\|^p_{M^\#(p,q,s)} \lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigg(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi(f(z), f(w))(1-|w|^2)^{\alpha/p}(1-|z|^2)^{\beta/p} \cdot (1-|\varphi_a(w)|^2)^{\gamma/p}(1-|\varphi_a(z)|^2)^{\delta/p}\, dA(w)\bigg)^p dA(z).$$

Theorems 1.4 and 1.5 together with Hölder's inequality yield the following characterization of functions in $M^\#(p, q, s)$. Analogous results for analytic functions can be found in [14].

Theorem 1.6. *Let $1 \le p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ and $0 < r < 1$, and let $f \in \mathcal{N}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta = q - p$ and $\gamma + \delta = s$. Then the following conditions are equivalent:*

(1) $f \in M^\#(p, q, s)$;

(2) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigg(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi(f(z), f(w))(1-|w|^2)^{\alpha/p}(1-|z|^2)^{\beta/p} \cdot (1-|\varphi_a(w)|^2)^{\gamma/p}(1-|\varphi_a(z)|^2)^{\delta/p}\, dA(w)\bigg)^p dA(z) < \infty;$$

(3) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigg(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi(f(z), f(w))^p(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \cdot (1-|\varphi_a(z)|^2)^{\gamma}(1-|\varphi_a(w)|^2)^{\delta}\, dA(w)\bigg) dA(z) < \infty.$$

For $0 \le s < \infty$, the meromorphic $Q_s^\#$-class consists of those $f \in \mathcal{M}(\mathbb{D})$ for which

$$\|f\|^2_{Q_s^\#} := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^\#(z))^2 g^s(z,a)\, dA(z) < \infty,$$

where $g(z, a) := -\log|\varphi_a(z)|$ is the Green function of $\mathbb{D}$ [4]. It is known that $Q_s^\# = \mathcal{N} \cap M_s^\#$, where $M_s^\# = M^\#(2, 0, s)$ [16]. Therefore, Theorem 1.6 yields the following result.

Corollary 1.7. *Let $0 < r < 1$ and $0 \le s < \infty$, and let $f \in \mathcal{N}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta = -2$ and $\gamma + \delta = s$. Then the following conditions are equivalent:*

(1) $f \in Q_s^\#$;

(2) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigg(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi(f(z), f(w))(1-|w|^2)^{\alpha/2}(1-|z|^2)^{\beta/2} \cdot (1-|\varphi_a(w)|^2)^{\gamma/2}(1-|\varphi_a(z)|^2)^{\delta/2}\, dA(w)\bigg)^2 dA(z) < \infty;$$

(3) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigg(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi(f(z), f(w))^2(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \cdot (1-|\varphi_a(z)|^2)^{\gamma}(1-|\varphi_a(w)|^2)^{\delta}\, dA(w)\bigg) dA(z) < \infty.$$

Corollary 1.8 is an immediate consequence of Theorem 1.6. It answers partially [13, Question 5].

COROLLARY 1.8. *Let* $1 \le p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ *and* $f \in \mathcal{N}$. *Then*

$$\|f\|^p_{M^\#(p,q,s)} \lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \left(\frac{\chi(f(z), f(w))}{|1-\overline{w}z|}\right)^p (1-|w|^2)^{(q/2)-1}(1-|z|^2)^{(q/2)-1} \cdot (1-|\varphi_a(w)|^2)^{s/2}(1-|\varphi_a(z)|^2)^{s/2}\, dA(w)\, dA(z).$$

Corollaries 1.3 and 1.8 yield the following characterization of functions in the Besov classes.

COROLLARY 1.9. *Let* $2 < p < \infty$ *and* $f \in \mathcal{N}$. *Then* $f \in B^\#_p$ *if and only if*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \left(\frac{\chi(f(z), f(w))}{|1-\overline{w}z|}\right)^p (1-|z|^2)^{(p/2)-2}(1-|w|^2)^{(p/2)-2}\, dA(z)\, dA(w) < \infty. \tag{1.5}$$

It is natural to conjecture that (1.5) is a sufficient condition for $f \in \mathcal{M}(\mathbb{D})$ to be normal. If this is answered in the affirmative, then the assumption $f \in \mathcal{N}$ in Corollary 1.9 may be deleted.

REMARKS. For $0 < p < 2$ the behavior of the Besov class $B^\#_p$ differs a lot from the case $2 \le p < \infty$ when

$$B^\#_p \subsetneq \bigcap_{(p-2)/p<q<1} Q^\#_{q,0} \subset \mathcal{N}_0$$

by [1, Theorem 9], [4, Theorem 3] and [3, Corollary 3]. Here $\mathcal{N}_0$ stands for the class of strongly (little) normal functions which consists of those $f \in \mathcal{M}(\mathbb{D})$ for which $\lim_{|z|\to 1^-} f^\#(z)(1-|z|^2) = 0$. We can distinguish the following three cases.

(a) If $0 < p < 1$, then [2, Theorem 5] guarantees that the class $B^\#_p$ contains nonconstant analytic functions. This is a clear distinction to the analytic case where B_p reduces to the space of constant functions for all $0 < p \le 1$ [17].

(b) The class $B^\#_1$ contains only either constant or nonnormal functions by [2, Theorem 4].

(c) If $1 < p < 2$, then $B^\#_p$ contains both some nonconstant normal functions (for example, all polynomials) and nonnormal functions [2, Theorems 4 and 5]. Indeed, by [2, Theorem 5], for each $1 < p < 2$ there exists an $f_p \in \mathcal{H}(\mathbb{D})$ such that

$$\int_{\mathbb{D}} (f^\#_p(z))^p (1-|z|^2)^{p-2}\, dA(z) \le \int_{\mathbb{D}} (f^\#_p(z))^p (1-|z|^2)^{-1}\, dA(z) < \infty.$$

Then $f_p \in B^\#_p$ and (11) in [2] is satisfied. Therefore, f_p is not a normal function and, in fact, f_p satisfies $\limsup_{z\to\zeta} f^\#(z)(1-|z|^2) = \infty$ for all $\zeta \in \partial\mathbb{D}$ [8].

The remaining part of the present paper is devoted to the proofs of Theorems 1.1, 1.2, 1.4 and 1.5.

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2. Proof of Theorem 1.1

First let $p=1$, and let $0<t<1$. Since $\chi(f(z), f(0)) \le \int_0^1 f^{\#}(tz)|z|\,dt$, Fubini's theorem and integration by parts yield

$$
\begin{aligned}
\int_{\mathbb{D}} \chi(f(z), f(0))(1-|z|^2)^{\alpha}\,dA(z) &\lesssim \int_{\mathbb{D}}\int_0^1 f^{\#}(tz)\,dt|z|(1-|z|)^{\alpha}\,dA(z)\\
&= \int_0^1 \int_{D(0,t)} f^{\#}(w)|w|\left(1-\frac{|w|}{t}\right)^{\alpha}\frac{dt}{t^3}\,dA(w)\\
&= \int_{\mathbb{D}} f^{\#}(w)|w| \int_{|w|}^1 \left(1-\frac{|w|}{t}\right)^{\alpha}\frac{dt}{t^3}\,dA(w)\\
&= \int_{\mathbb{D}} f^{\#}(w) \int_{|w|}^1 (1-s)^{\alpha} s\,ds \frac{dA(w)}{|w|}\\
&= \int_{\mathbb{D}} f^{\#}(w)\frac{(1-|w|)^{1+\alpha}}{\alpha+1}\\
&\quad\times\left(|w|+\frac{1-|w|}{\alpha+2}\right)\frac{dA(w)}{|w|}\\
&\lesssim \int_{\mathbb{D}} (f^{\#}(w))(1-|w|^2)^{1+\alpha}\frac{dA(w)}{|w|},
\end{aligned}
$$

which is the desired asymptotic inequality for $p=1$.

If $p>1$, choose $q>((p-1)/p)$ such that $\alpha-pq+p>0$. By Hölder's inequality,

$$
\begin{aligned}
\chi(f(z), f(0)) &\le \int_0^1 f^{\#}(tz)|z|\,dt = \int_0^1 f^{\#}(tz)(1-t|z|)^q\frac{|z|\,dt}{(1-t|z|)^q}\\
&\le \left(\int_0^1 f^{\#}(tz)^p(1-t|z|)^{pq}\,dt\right)^{1/p}\left(\int_0^1 \frac{|z|^{p/(p-1)}\,dt}{(1-t|z|)^{pq/(p-1)}}\right)^{(p-1)/p}\\
&\lesssim \left(\int_0^1 f^{\#}(tz)^p(1-t|z|)^{pq}\,dt|z|(1-|z|)^{p-1-pq}\right)^{1/p}
\end{aligned}
$$

from which Fubini's theorem yields

$$
\begin{aligned}
&\int_{\mathbb{D}} \chi(f(z), f(0))^p(1-|z|^2)^{\alpha}\,dA(z)\\
&\quad\lesssim \int_{\mathbb{D}}\int_0^1 f^{\#}(tz)^p(1-t|z|)^{pq}\,dt|z|(1-|z|)^{\alpha+p-1-pq}\,dA(z)\\
&\quad= \int_0^1\int_{D(0,t)} f^{\#}(w)^p(1-|w|)^{pq}|w|\left(1-\frac{|w|}{t}\right)^{\alpha-pq+p-1}\frac{dt}{t^3}\,dA(w)
\end{aligned}
$$

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$$
\begin{aligned}
&= \int_{\mathbb{D}} f^{\#}(w)^p (1-|w|)^{pq} |w| \int_{|w|}^{1} \left(1 - \frac{|w|}{t}\right)^{\alpha - pq + p - 1} \frac{dt}{t^3}\, dA(w) \\
&\lesssim \int_{\mathbb{D}} (f^{\#}(w))^p (1-|w|^2)^{p+\alpha} \frac{dA(w)}{|w|}.
\end{aligned}
$$

3. Proof of Theorem 1.2

By the change of variable $z = \varphi_w(u)$, Theorem 1.1 and Fubini's theorem,

$$
\begin{aligned}
I(f) &:= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\chi(f(z), f(w))^p}{|1-\overline{w}z|^4} (1 - |\varphi_w(z)|^2)^{\alpha}\, dA(z)\, dA(w) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \chi((f \circ \varphi_w)(u), (f \circ \varphi_w)(0))^p (1-|u|^2)^{\alpha}\, dA(u) \frac{dA(w)}{(1-|w|^2)^2} \\
&\lesssim \int_{\mathbb{D}} \int_{\mathbb{D}} ((f \circ \varphi_w)^{\#}(u))^p (1-|u|^2)^{p+\alpha} \frac{dA(u)}{|u|} \frac{dA(w)}{(1-|w|^2)^2} \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(\varphi_w(u)))^p (1-|\varphi_w(u)|^2)^p (1-|u|^2)^{\alpha} \frac{dA(u)}{|u|} \frac{dA(w)}{(1-|w|^2)^2} \\
&= \int_{\mathbb{D}} (f^{\#}(z))^p (1-|z|^2)^{p+\alpha} \int_{\mathbb{D}} |\varphi_w'(z)|^{\alpha+2} \frac{dA(w)}{|\varphi_w(z)|(1-|w|^2)^2}\, dA(z).
\end{aligned}
$$

But now

$$
\begin{aligned}
\int_{\mathbb{D}} |\varphi_w'(z)|^{\alpha+2} \frac{dA(w)}{|\varphi_w(z)|(1-|w|^2)^2} &= (1-|z|^2)^{-(\alpha+2)} \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha}}{|u|}\, dA(u) \\
&\simeq (1-|z|^2)^{-(\alpha+2)},
\end{aligned}
$$

and the assertion follows.

4. Proof of Theorem 1.4

Routine calculations and (1.4) show that for $w \in D(z, r)$ and $a \in \mathbb{D}$,

$$
1 - |z|^2 \simeq 1 - |w|^2 \simeq |1 - \overline{w}z| \simeq |D(z, r)|^{1/2} \tag{4.1}
$$

and

$$
1 - |\varphi_a(z)|^2 \simeq 1 - |\varphi_a(w)|^2, \tag{4.2}
$$

where the constants of comparison depend only on r. By (4.1), (4.2) and (1.3),

$$
\begin{aligned}
I &:= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} \chi(f(z), f(w))^p (1-|z|^2)^{\alpha} (1-|w|^2)^{\beta} \right. \\
&\qquad \left. \cdot (1-|\varphi_a(z)|^2)^{\gamma} (1-|\varphi_a(w)|^2)^{\delta}\, dA(w) \right) dA(z) \\
&\lesssim \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\int_{D(z,r)} (f^{\#}(w))^p (1-|w|^2)^p \frac{dA(w)}{|\varphi_z(w)|} \right) (1-|z|^2)^{q-p-2} \\
&\qquad \cdot (1-|\varphi_a(z)|^2)^{s}\, dA(z)
\end{aligned}
$$

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from which (4.1), (4.2) and Fubini's theorem yield

$$
\begin{aligned}
I &\lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{D(z,r)} (f^{\#}(w))^p (1-|w|^2)^{q-2} (1-|\varphi_a(w)|^2)^s \frac{dA(w)}{|\varphi_z(w)|} \right) dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{D(w,r)} \frac{dA(z)}{|\varphi_z(w)|} \right) (f^{\#}(w))^p (1-|w|^2)^{q-2} (1-|\varphi_a(w)|^2)^s \, dA(w) \\
&\simeq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(w))^p (1-|w|^2)^{q} (1-|\varphi_a(w)|^2)^s \, dA(w).
\end{aligned}
$$

5. Proof of Theorem 1.5

Let $z,\ w \in \widehat{\mathbb{C}}$, and define

$$
F(z, w) := \begin{cases} \dfrac{w-z}{1+\overline{w}z}, & w \in \mathbb{C}, \\[2ex] \dfrac{1}{z}, & w = \infty. \end{cases}
$$

A direct calculation shows that $|F(z, w)|^2 = \chi^2(z, w)/(1-\chi^2(z, w))$ for all z, $w \in \widehat{\mathbb{C}}$. Denote the pseudohyperbolic distance between the points z and w in $\mathbb{D}$ by $\rho(z, w) := |\varphi_z(w)|$. By the uniform (ρ, χ)-continuity of f, there is an $r_1 \in (0, 1)$ such that $\chi(f(z), f(w)) < \frac{1}{2}$ for $\rho(z, w) < r_1$ [10]. It follows that

$$
|F(f(z), f(w))| = \frac{\chi(f(z), f(w))}{\sqrt{1-\chi^2(f(z), f(w))}} < \frac{2}{\sqrt{3}} \chi(f(z), f(w)) \tag{5.1}
$$

for $\rho(z, w) < r_1$. Since $f \in \mathcal{M}(\mathbb{D})$, there is an $r_2 \in (0, 1)$ such that the function $g_z(w) := F((f \circ \varphi_z)(w), f(z))$ is analytic in $D(0, r_2) := \{w : \rho(0, w) = |w| < r_2\}$ for all $z \in \mathbb{D}$, and hence its Maclaurin series is of the form $\sum_{k=1}^{\infty} a_k(z) w^k$ in $D(0, r_2)$. Therefore,

$$
\begin{aligned}
f^{\#}(z)(1-|z|^2) = |a_1| &= \frac{2}{r^4} \left| \int_{D(0,r)} \overline{w} g_z(w) \, dA(w) \right| \\
&\le \frac{2}{r^3} \int_{D(0,r)} \left| F((f\circ\varphi_z)(w), f(z)) \right| dA(w)
\end{aligned} \tag{5.2}
$$

for any $r \in (0, r_2)$. Now let $r < \min\{r_1, r_2\}$. Then (5.2) and (5.1) yield

$$
\begin{aligned}
I(f) &:= \int_{\mathbb{D}} (f^{\#}(z)(1-|z|^2))^p (1-|z|^2)^{q-p} (1-|\varphi_a(z)|^2)^s \, dA(z) \\
&\le \int_{\mathbb{D}} \left(\frac{2}{r^3} \int_{D(0,r)} |F((f\circ\varphi_z)(w), f(z))| \, dA(w) \right)^p \\
&\quad \cdot (1-|z|^2)^{q-p} (1-|\varphi_a(z)|^2)^s \, dA(z)
\end{aligned}
$$

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$$
\begin{aligned}
&= \int_{\mathbb{D}} \left(\frac{2}{r^3} \int_{D(z,r)} |F(f(u), f(z))| |\varphi_z'(u)|^2 \, dA(u) \right)^p \\
&\quad \cdot (1 - |z|^2)^{q-p} (1 - |\varphi_a(z)|^2)^s \, dA(z) \\
&\le \int_{\mathbb{D}} \left(\frac{4}{\sqrt{3} r^3} \int_{D(z,r)} \chi(f(u), f(z)) |\varphi_z'(u)|^2 \, dA(u) \right)^p \\
&\quad \cdot (1 - |z|^2)^{q-p} (1 - |\varphi_a(z)|^2)^s \, dA(z) \qquad (5.3)
\end{aligned}
$$

from which the assertion for $r < \min\{r_1, r_2\}$ follows by (4.1) and (4.2). If $r \geq \min\{r_1, r_2\}$, choose $c > 1$ such that $r^* := r/c < \min\{r_1, r_2\}$. Then (5.3) together with (4.1) and (4.2) give the assertion for r^*. To obtain the assertion for r, it remains to make the set of integration larger by replacing $D(z, r^*)$ by $D(z, r)$ and note that there is a constant C, depending only on c, such that $|D(z, r^*)| \geq C|D(z, r)|$ for all $z \in \mathbb{D}$.

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