REPRESENTATIONS SUBDUCED ON AN IDEAL OF A LIE ALGEBRA

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Introduction. This paper considers the properties of the representation of a Lie algebra when restricted to an ideal, the subduced* representation of the ideal. This point of view leads to new forms for irreducible representations of Lie algebras, once the concept of matrices of invariance is developed. This concept permits us to show that irreducible representations of a Lie algebra, over an algebraically closed field, can be expressed as a Lie-Kronecker product whose factors are associated with the representation subduced on an ideal. Conversely, if one has such factors, it is shown that they can be put together to give an irreducible representation of the Lie algebra. A valuable guide to this work was supplied by a paper of Clifford **(1)**.

1. Matrices of invariance. Let an ideal T, of a Lie algebra L over a field F, have the representation Q, that is, $t \to Q(t)$ is a representation of T by matrices with elements in F. If there is a matrix $C(a)$, corresponding to an element a of L, such that

$$[C(a), Q(t)] = Q(a \circ t),\dagger$$

for all elements t in T, then we shall call $C(a)$ a *matrix of invariance* relative to Q. If a matrix of invariance relative to Q exists for each $a \in L$, a matrix of invariance $C(a)$ can be constructed so that

$$\begin{aligned} C(a + b) &= C(a) + C(b) && (b \in L) \\ C(ka) &= kC(a) && (k \in F) \qquad (1.0)\\ C(t) &= Q(t) && (t \in T). \end{aligned}$$

Let $e_1, e_2, \ldots, e_r$ be a basis of T, which extended by $e_{r+1}, e_{r+2}, \ldots, e_n$ gives a basis of L over the field F. From the matrices of invariance of $e_{r+1}, e_{r+2}, \ldots, e_n$ select any particular set $C(e_{r+1}), C(e_{r+2}), \ldots, C(e_m)$ and define $C(e_i) = Q(e_i)$, $i = 1, 2, \ldots, r$. For any $a \in L$ we have

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*This term used in **2**, was drawn to my attention by the referee.

†$[C(a),Q(t)]$ designates the commutator of the matrices $C(a)$ and $Q(t)$ and $a \circ t$ the product of a and t in the Lie algebra L.

$$a = \sum_{i=1}^{n} k_i e_i \qquad (k_i \in F)$$

and setting

$$C(a) = \sum_{i=1}^{n} k_i C(e_i)$$

gives the required construction.

THEOREM 1.1. *Let T be an ideal of a Lie algebra L. Let Q be an irreducible representation of T over an algebraically closed field F. Let $C(a)$, $a \in L$, be a matrix of invariance relative to Q, then*

$$C(a \circ b) = [C(a), C(b)] + c(a, b)I, \qquad (b \in L)$$

where $c(a, b) \in F$, and I is the unit matrix of the dimensions of $C(a)$.

Proof. By the Jacobi identity,

$$(a \circ b) \circ t = a \circ (b \circ t) - b \circ (a \circ t), \qquad (t \in T).$$

Since T is an ideal, $(a \circ b) \circ t$, $a \circ t$, $b \circ t$, are elements of T; we have therefore

$$\begin{aligned} Q((a \circ b) \circ t) &= Q(a \circ (b \circ t)) - Q(b \circ (a \circ t)), \\ [C(a \circ b), Q(t)] &= [C(a), Q(b \circ t)] - [C(b), Q(a \circ t)], \\ &= [C(a), [C(b), Q(t)]] - [C(b), [C(a), Q(t)]]\,, \\ &= [[C(a), C(b)], Q(t)]\,, \end{aligned}$$

since the commutators of matrices satisfy the Jacobi identity. Thus

$$[(C(a \circ b) - [C(a), C(b)]), Q(t)] = 0\,,$$

or

$$(C(a \circ b) - [C(a), C(b)])Q(t) = Q(t)(C(a \circ b) - [C(a), C(b)]).$$

By Schur's lemma* **(5)**,

$$C(a \circ b) = [C(a), C(b)] + c(a, b)I,$$

where $c(a, b) \in F$, and I is the unit matrix of the dimensions of $C(a)$.

COROLLARY. *If the matrices of invariance are chosen to satisfy* (1.0) *the scalars $c(a, b)$ of the theorem have the properties*

$$\begin{array}{lcl} \text{(i)} & c(a, t) = 0, & (a \in L, t \in T) \\ \text{(ii)} & c(a, a) = 0, & \\ \text{(iii)} & c(a, b) = -c(b, a), & (b \in L) \\ \text{(iv)} & c(a, b \circ d) + c(b, d \circ a) + c(d, a \circ b) = 0 & (d \in L) \end{array}$$

*In an algebraically closed field, the only matrices commuting with an irreducible set of matrices are scalar multiples of the unit matrix.

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$$\text{(v)} \qquad c(a+s, b+t) = c(a, b) \qquad (s \in T)$$
$$\text{(vi)} \qquad c(a+b, d) = c(a, d) + c(b, d)$$
$$\text{(vii)} \qquad c(a, b+d) = c(a, b) + c(a, d),$$
$$\text{(viii)} \qquad c(ka, b) = c(a, kb) = kc(a, b).$$

Proof. The proofs of these properties are trivial with the exception of that for property (iv), which follows. We have

$$\begin{aligned} C(a \circ (b \circ d)) &= [C(a), C(b \circ d)] + c(a, b \circ d)I, \\ &= [C(a), [C(b), C(d)]] + c(b, d)\,[C(a), I] + c(a, b \circ d)I, \\ &= [C(a), [C(b), C(d)]] + c(a, b \circ d)I. \end{aligned}$$

Permuting a, b, and d, cyclically, adding the corresponding equations, then applying the Jacobi identity, we have

$$Q(0) = 0 + (c(a, b\circ d) + c(b, d\circ a) + c(d, a\circ b))I,$$
$$0 = c(a, b\circ d) + c(b, d\circ a) + c(d, a\circ b), \text{ giving (iv).}$$

The elements $c(a, b)$ of F, satisfying the properties (i) to (viii), form a factor set.

It follows from Theorem 1.1 and its corollary that if Q is an irreducible representation of an ideal T of L over an algebraically closed field F and there exist matrices of invariance then we can construct matrices of invariance $C(a)$ satisfying (1.0), $a \in L$, and the correspondence

$$a \rightarrow C(a)$$

is almost a representation of L. Let us call such a correspondence C an *L-projective representation* (L for Lie, and projective because of the analogy with group theory). If we have such an L-projective representation C of L, given by

$$a \rightarrow C(a) = (c_{ij}(a)), \qquad (i, j = 1, 2, \ldots, n)$$

where $c_{ij}(a) \in F$, we can define

$$au_j = C(a)u_j = \sum_{i=1}^{n} c_{ij}(a)u_i, \qquad (j = 1, 2, \ldots, n)$$

for an F-module with the basic elements $u_1, u_2, \ldots, u_n$ to form an L-projective representation module. It is easily verified that

$$\text{(i)} \quad a(u+v) = au + av, \qquad (a \in L)$$
$$\text{(ii)} \quad (a+b)u = au + bu, \qquad (b \in L)$$
$$\text{(iii)} \quad (ka)u = a(ku) = k(au), \qquad (k \in F)$$
$$\begin{aligned} \text{(iv)} \quad (a \circ b)u &= a(bu) - b(au) + c(a, b)u \\ &= ([a, b] + c(a, b))u. \end{aligned}$$

Conversely, if there is an F-module M for which there is defined a unique product au in M for $a \in L$, a Lie algebra, $u \in M$, such that properties (i) to

(iv) are satisfied, then M assigns an L-projective representation to L. We can define irreducibility and reducibility in the usual way.

If $C'(a)$ and $C(a)$, $a \in L$, are two matrices of invariance relative to an irreducible representation Q of an ideal T of L over an algebraically closed field F, then Schur's lemma gives

$$C'(a) = C(a) + c(a)I. \qquad (c(a) \in F).$$

If $C'(a)$ and $C(a)$ satisfy (1.0) then

$$\begin{aligned} c(a+b) &= c(a) + c(b) \\ c(ka) &= kc(a) \\ c(t) &= 0. \end{aligned}$$

Furthermore, if $c'(a, b)$ and $c(a, b)$ are the factor sets corresponding to the L-projective representations C' and C, respectively, then it is easily shown that

$$c'(a, b) = c(a, b) + c(a \circ b).$$

A sufficient condition for the existence of matrices of invariance.

THEOREM 1.2. *Let T be an ideal of a Lie algebra L over an arbitrary field F. Let M be an L-F-module and M_1 and M_2, T-F- submodules of M. If $M = M_1 \dot{+} M_2$, then each $a \in L$ can be assigned a matrix of invariance $C(a)$, such that*

$$[C(a), Q(t)] = Q(a \circ t), \qquad (t \in T)$$

where Q is the representation of T assigned by M_1.

Proof. For any $u \in M$, we have $u = u_1 + u_2$, where $u_1 \in M_1$ and $u_2 \in M_2$. The components u_1 and u_2 are unique since the sum of M_1 and M_2 is direct. Thus the correspondences

$$\begin{aligned} \alpha_1 &: u \to u_1 = \alpha_1 u, \\ \alpha_2 &: u \to u_2 = \alpha_2 u, \end{aligned}$$

are homomorphisms of M onto M_1 and M_2 respectively. We can then write

$$u = \alpha_1 u + \alpha_2 u.$$

In particular, for $v \in M_1$

$$av = \alpha_1 av + \alpha_2 av, \qquad (a \in L)$$

then the operator $\alpha_1 a$ is clearly a linear transformation of M_1. For $t \in T$,

$$\begin{aligned} (a \circ t)v &= a(tv) - t(av), \\ &= \alpha_1 a(tv) + \alpha_2 a(tv) - t(\alpha_1 av + \alpha_2 av). \end{aligned}$$

Equating components,

$$(a \circ t)v = \alpha_1 a(tv) - t(\alpha_1 av).$$

Setting the linear transformation $\alpha_1 a = \mathfrak{C}(a)$ and replacing t by its linear transformation $\mathfrak{Q}(t)$ of M_1, we have

$$\mathfrak{Q}(a \circ t)v = \mathfrak{C}(a)(\mathfrak{Q}(t)v) - \mathfrak{Q}(t)(\mathfrak{C}(a)v),$$

or for the corresponding matrices of these linear transformations

$$Q(a \circ t) = C(a)Q(t) - Q(t)C(a) = [C(a), Q(t)].$$

DEFINITION. *If to every element a of L there corresponds a matrix of invariance relative to a representation Q of an ideal T of L then Q will be called invariant under L.*

2. Subduced representations

THEOREM 2.1. *Let A be an irreducible representation of a Lie algebra L. If the irreducible components of the representation subduced by A on an ideal T of L are invariant under L, then the subduced representation is fully reducible to equivalent irreducible components and conversely.*

Proof. Let M be an irreducible L-F-module leading to the representation A of L by matrices. Select any irreducible T-F-module $M_1 \subseteq M$. Let M_1 assign to T the representation Q, invariant under L.

If $M_1 = M$ the theorem follows, so take $M_1 \neq M$, then there is an $a \in L$, such that $aM_1 \not\subseteq M_1$, otherwise M is reducible. Since Q is invariant under L, there is a matrix of invariance $C(a)$ corresponding to a and, consequently, a corresponding linear transformation $\mathfrak{C}(a)$ of M_1. From $aM_1 + M_1$, form the set M_2 of the elements

$$au - \mathfrak{C}(a)u, \qquad (u \in M_1).$$

It is easily verified that M_2 is an F-module. Further

$$\begin{aligned} t(au - \mathfrak{C}(a)u) &= \mathfrak{Q}(t \circ a)u + atu - \mathfrak{Q}(t \circ a)u - \mathfrak{C}(a)(tu) \\ &= a(tu) - \mathfrak{C}(a)(tu) \in M_2. \end{aligned}$$

Thus M_2 is a T-F-module. The correspondence

$$u \to au - \mathfrak{C}(a)u$$

is then an operator homomorphism over F and T of M_1 onto M_2. But M_1 is irreducible, hence the homomorphism is an isomorphism. Since $M_1 \neq M_2$, we have

$$M_1 + aM_1 = M_1 + M_2 = M_1 \dotplus M_2.$$

If $M_1 \dotplus M_2 = M$, the theorem is proved. If $M_1 \dotplus M_2 \neq M$, there exists $b \in L$ such that either

$$bM_1 + M_1 \not\subseteq M_1 \dotplus M_2,$$

or

$$bM_2 + M_2 \not\subseteq M_1 \dotplus M_2.$$

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Otherwise M is reducible. Hence we can continue our construction of isomorphic T-F-modules.

Since M is finite and each additional module is non-zero, a finite number of such constructions will exhaust M. Hence A subduces on the ideal T a representation which is completely reducible into equivalent irreducible components.

Proof of the converse. Since A subduces on T a representation which is fully reducible, the corresponding representation module M, considered as a T-F-module, can be written in the form

$$M = M_i \dotplus (M_1 \dotplus \ldots \dotplus M_{i-1} \dotplus M_{i+1} \dotplus \ldots \dotplus M_r).$$

Theorem 1.2 then assures us of the existence of matrices of invariance for the representation assigned to T by M_i.

In order to consider the nature of subduced representations in greater detail, we turn to the Lie-Kronecker product of matrices, namely, if A and B are any square matrices, not necessarily of the same dimensions, their *Lie-Kronecker product*, designated by $A \otimes B$, is defined by the equation

$$A \otimes B = A \times I_B + I_A \times B,$$

where $\times$ is the Kronecker product of matrices, and I_A, I_B are the unit matrices with the dimensions of A and B, respectively. This product is derived in a natural way by a consideration of the product of representation modules (**8**, p. 26).

For our purposes, we extend this concept to L-projective representation modules.

Let M and N be L-projective representation modules of L. We can then define a linear transformation $\mathfrak{A}(a)$ of the product module MN by the equations

$$\mathfrak{A}(a)(uv) = (au)v + u(av), \qquad (a \in L)$$

$$\mathfrak{A}(a) \sum_{i=1}^{h} \pm u_i v_i = \sum_{i=1}^{h} \pm (au_i)v_i + \sum_{i=1}^{h} \pm u_i(av_i).$$

The linear transformation $\mathfrak{A}(a)$ is uniquely determined by a, so we define the product $a(uv)$ by

$$a(uv) = \mathfrak{A}(a)(uv).$$

THEOREM 2.2. *Let M and N assign L-projective representations to the Lie algebra L, whose factor sets are $c(a, b)$ and $d(a, b)$, respectively. Then the product module MN, for which there is defined a left multiplication by elements of L as above, assigns an L-projective representation to L with the factor set $c(a, b) + d(a, b)$.*

Proof.

$$\begin{aligned}(a \circ b)(uv) &= ((a \circ b)u)v + u((a \circ b)v)\\ &= ([a, b]u)v + u([a, b]v) + (c(a, b) + d(a, b))uv\\ &= ([a, b] + c(a, b) + d(a, b))uv.\end{aligned}$$

That MN has the remaining properties of an L-projective representation module is easily verified.

THEOREM 2.3. *Let A be an irreducible representation of a Lie algebra L in an algebraically closed field F. Let A subduce on an ideal T of L a representation completely reducible to r irreducible components equivalent to a representation G. A is then the Lie-Kronecker product of two irreducible L-projective representations C and U of L, where C has the degree of G; U, the degree r, and their factor sets differ only in sign. U is actually an L-projective representation of the residue class algebra L/T.*

Proof. Let M and M_1 be the representation modules assigning the representations A and G to L and T, respectively, then

$$M = M_1 \dotplus M_2 \dotplus \ldots \dotplus M_r,$$

and

$$M_1 \cong M_i,\ i = 2, 3, \ldots, r$$

over T. Let α_i be the operator isomorphism between M_1 and M_i, that is, the isomorphism $M_1 \cong M_i$ is accomplished by the correspondence

$$u \rightarrow \alpha_i u, \qquad (u \in M_1,\ \alpha_i u \in M_i)$$

such that for $t \in T$, $k \in F$, $tu \rightarrow \alpha_i tu = t\alpha_i u$, $ku \rightarrow \alpha_i ku = k\alpha_i u$, then

$$a\alpha_j u = \sum_{i=1}^{r} \mathfrak{A}_{ij}(a)\alpha_i u, \qquad (u \in M_1;\ j = 1, 2, \ldots, r; a \in L)$$

where $\mathfrak{A}_{ij}(a)$ is a linear transformation of M_i. Let its corresponding matrix be $A_{ij}(a)$.

Since T is an ideal, $a \circ t \in T$, hence

$$(a \circ t)\alpha_j u = \alpha_j((a \circ t)u), \qquad (u \in M_1)$$

$$a(t\alpha_j u) - t(a\alpha_j u) = \alpha_j((a \circ t)u),$$

$$a(\alpha_j(tu)) - t\sum_{i=1}^{r} \mathfrak{A}_{ij}(a)\alpha_i u = \alpha_j((a \circ t)u),$$

$$\sum_{i=1}^{r} \mathfrak{A}_{ij}(a)\alpha_i(tu) - \sum_{i=1}^{r} t(\mathfrak{A}_{ij}(a)\alpha_i u) = \sum_{i=1}^{r} \delta_{ij}\alpha_i((a \circ t)u)$$

where δ_{ij} is the Kronecker delta. Thus

$$\sum_{i=1}^{r} (\mathfrak{A}_{ij}(a)\alpha_i(tu) - t(\mathfrak{A}_{ij}(a)\alpha_i u)) = \sum_{i=1}^{r} \delta_{ij}\alpha_i((a \circ t)u).$$

Replacing the element t of T by its linear transformation $\mathfrak{G}(t)$, assigned by M_1, we have

$$\sum_{i=1}^{r} [\mathfrak{A}_{ij}(a), \mathfrak{G}(t)]\alpha_i u = \sum_{i=1}^{r} (\delta_{ij}\mathfrak{G}(a \circ t))\alpha_i u.$$

Since $M = M_i \dot{+} (M_2 \dot{+} \ldots \dot{+} M_r)$, Theorem 1.2 assures us of the invariance of G under L. Hence we can construct matrices of invariance $C(a)$, $a \in L$, with properties (1.0) and then the correspondence $a \to C(a)$ is an L-projective representation. Let $\mathfrak{C}(a)$ be the linear transformation corresponding to $C(a)$, then we can write our last equation in the form

$$\sum_{i=1}^{r} ([(\mathfrak{A}_{ij}(a), \mathfrak{G}(t)] - \delta_{ij}[\mathfrak{C}(a), \mathfrak{G}(t)])\alpha_i u = 0.$$

Consequently

$$[A_{ij}(a) - \delta_{ij}C(a)), G(t)] = 0,$$

where $G(t)$ is the matrix corresponding to $\mathfrak{G}(t)$, is a valid matrix equation for all $t \in T$ and $i, j = 1, 2, \ldots, r$. Applying Schur's lemma, we have

$$A_{ij}(a) - \delta_{ij}C(a) = U_{ij}(a)I_G, \quad (U_{ij}(a) \in F)$$

I_G being the unit matrix with the dimensions of G, thus

$$A_{ij}(a) = \delta_{ij}C(a) + U_{ij}(a)I_G,$$

and so

$$\begin{aligned} A(a) = (A_{ij}(a)) &= C(a) \times I_U + I_C \times U(a), \\ &= C(a) \otimes U(a), \\ &= C \otimes U(a), \end{aligned}$$

where $U(a) = (U_{ij}(a))$ and since $I_C = I_G$.

Observing that $C \otimes U$ is a representation and C is an L-projective representation it is easily shown that U is an L-projective representation of L with a factor set differing only in sign from that of C.

For $t \in T$, we have $A(t) = Q(t) = G(t) \times I_r$, where I_r is the unit matrix of degree r. Also

$$\begin{aligned} A(t) &= C(t) \times I_U + \mathrm{I}_C \times U(t), \\ &= G(t) \times I_r + I_C \times U(t), \end{aligned}$$

thus

$$0 = I_C \times U(t),$$

giving

$$U(t) = 0.$$

Hence U gives an L-projective representation of the residue class algebra L / T.

C and U are irreducible L-projective representations of L since otherwise A is reducible.

The imbedding of irreducible representations.

THEOREM 2.4. *Let T be an ideal of a Lie algebra L. Let Q be an irreducible* *representation of T invariant under L. Let $c(a, b)$, $(a, b \in L)$, be a factor set of the corresponding L-projective representation C of L. Then a necessary and sufficient condition that Q can be imbedded in an irreducible representation of L is that the factor set $-c(a, b)$ can be realized by an L-projective representation U^* of L / T.*

Proof. The necessity of the condition is shown by Theorem 2.3. The condition is also sufficient. For, taking an irreducible component U of the representation U^*, we set $A = C \otimes U$. By Theorem 2.2, A is certainly a representation. To show A is irreducible we require the following lemma.

LEMMA 2.1. *Let L have the equivalent representations $C \otimes U$ and $C' \otimes U'$, formed according to the theorem from an irreducible representation Q of T. If C and C' are equivalent, then U and U' are equivalent.*

Proof. For $a \in L$, we have

$$X(C'(a) \otimes U'(a))X^{-1} = C(a) \otimes U(a),$$
$$Y^{-1}C'(a)Y = C(a).$$

Consequently,

$$X((YC(a)Y^{-1}) \otimes U'(a))X^{-1} = C(a) \otimes U(a),$$
$$X(YC(a)Y^{-1} \times I_{U'} + I_C \times U'(a))X^{-1} = C(a) \otimes U(a),$$
$$X(Y \times I_{U'})(C(a) \otimes U'(a))(Y^{-1} \times I_{U'})X^{-1} = C(a) \otimes U(a) \qquad (1).$$

Setting $X(Y \times I_{U'}) = Z$, and replacing a by $t \in T$, this equation becomes

$$Z(Q(t) \times I_U)Z^{-1} = Q(t) \times I$$

or

$$Z(Q(t) \times I_U) = (Q(t) \times I_U)Z \quad \text{for all} \quad t \in T.$$

Applying Schur's lemma gives Z the form $I_Q \times W$, where W is non-singular. Substituting this form of Z in (1) gives

$$C(a) \times I_{U'} + I_C \times WU'(a)W^{-1} = C(a) \times I_U + I_C \times U(a)$$
$$WU'(a)W^{-1} = U(a),$$

proving the lemma.

Returning to the theorem, we can now prove that the representation $A = C \otimes U$ is irreducible. Let M_c and M_u be the modules assigning the L-projective representations C and U to L. By Theorem 2.2, $M = M_cM_u$ assigns the ordinary representation $C \otimes U$ to L. Further, since

$$A(t) = C(t) \times I_U + I_C \times U(t), \qquad (t \in T).$$
$$= Q(t) \times I_U,$$

M, considered as a T-F-module is the direct sum of irreducible T-F-modules operator isomorphic to M_c, that is,

$$M = M_1 \dotplus M_2 \dotplus \ldots \dotplus M_s, \; M_1 = M_c, \; M_1 \cong M_i \text{ over } T.$$

This form of M is then a Remak decomposition **(9)**.

Let us assume that M is reducible, then M properly contains an L-F-module M'. With suitably chosen subscripts we then have

$$M = M' \dotplus M_{r+1} \dotplus \ldots \dotplus M_s.$$

Since any submodule of a module with a Remak decomposition has a Remak decomposition and, furthermore, since different Remak decompositions of the same module are equal in length, and the components are operator isomorphic in some order, we have

$$M = M_1' \dotplus M_2' \dotplus \ldots \dotplus M_r' \dotplus M_{r+1} \dotplus \ldots \dotplus M_s,$$

where $M_1' \cong M_i' \cong M_j$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$. These operator isomorphisms assure us of the irreducibility of the M'_i. Let α_i be the operator isomorphism of M_1' onto M_i', or M_i if $i \geqslant r + 1$. Then for $u \in M_i$,

$$a\alpha_g u = \sum_{i=1}^{r} \mathfrak{A}_{ig}(a)\alpha_i u, \qquad (a \in L, g \leqslant r)$$

since M' is invariant under L. Also

$$a\alpha_h u = \sum_{i=1}^{s} \mathfrak{A}_{ih}(a)\alpha_i u, \qquad (s \geqslant h > r).$$

As in Theorem 2.3, these equations lead to the matrix equations

$$A'_{ik}(a) = C'(a)\delta_{ik} + I_C U_{ik}(a),$$

but with the further property that

$$I_C U'_{ik}(a) = 0, \quad \text{for} \quad k \leqslant r, i > r,$$

giving

$$U'_{ik}(a) = 0, \quad \text{for} \quad k \leqslant r, i > r.$$

Thus $U'(a) = (U_{ik}'(a))$ is a reducible representation, making $A'(a) = C'(a) \otimes U'(a)$ also reducible. Since there is an operator isomorphism between the two Remak decompositions of M and also between M_1 and M'_1 it follows that the representation $C' \otimes U'$ is equivalent to $C \otimes U$ and C is equivalent to C'. By the lemma U is equivalent to U', contrary to U being irreducible. Thus the assumption that M is reducible is contradicted.

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