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UPPER BOUNDS ON THE SEMITOTAL FORCING NUMBER OF GRAPHS

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Abstract

Let G be a graph with no isolated vertex. A semitotal forcing set of G is a (zero) forcing set S such that every vertex in S is within distance 2 of another vertex of S. The semitotal forcing number $F_{t2}(G)$ is the minimum cardinality of a semitotal forcing set in G. In this paper, we prove that it is NP-complete to determine the semitotal forcing number of a graph. We also prove that if $G \neq K_n$ is a connected graph of order $n \geq 4$ with maximum degree $\Delta \geq 2$, then $F_{t2}(G) \leq (\Delta - 1)n/\Delta$, with equality if and only if either $G = C_4$ or $G = P_4$ or $G = K_{\Delta,\Delta}$.

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1. Introduction

Forcing and its variations in graphs are now well studied. The (zero) forcing number of a graph was first introduced by the AIM Minimum Rank–Special Graphs Work Group [2] to bound the maximum nullity/minimum rank of the family of symmetric matrices associated with a graph. Total forcing and semitotal forcing are two variations of forcing, which were first introduced and studied by Davila and Kenter [8] and Chen [6]. The definitions are as follows.

For any two-colouring of the vertex set V of a graph G, say black and white for the two colours, define the *colour-change rule*: a white vertex v is converted to black if it is the only white neighbour of some black vertex u. We say u forces v, written $u \to v$, and also that u is a *forcing vertex*. Let S be a subset of V. Define a two-colouring of G by colouring S black and all other vertices white. The *derived set* $D(S)$ of S is the set of black vertices obtained by iteratively applying the colour-change rule until no more changes are possible. If $D(S) = V$, then we say S is a *forcing set* (also called a *zero forcing set*) of G. The procedure of colouring a graph using the colour-change rule applied to S is called a *forcing process* with respect to S. A *minimum forcing set*

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of G is a forcing set of G of minimum cardinality and the *forcing number*, denoted by $F(G)$, is the cardinality of a minimum forcing set. If S is a forcing set of G and $G[S]$ contains no isolated vertex, then S is a *total forcing set* of G; if S is a forcing set of G and every vertex in S is within distance 2 of another vertex of S, then S is a *semitotal forcing set* of G. The *total forcing number* (respectively, *semitotal forcing number*) of G is the cardinality of a minimum total forcing set (respectively, semitotal forcing set) in G and denoted by $F_t(G)$ (respectively, $F_{t2}(G)$).

Determining the forcing number and the total forcing number for a graph are NP-complete (see [1, 5] and [7], respectively). Therefore, it is difficult to compute the forcing number or the total forcing number of a graph accurately and it is interesting to establish some bounds on these two parameters. Amos *et al.* [3] proved $F(G) \leq ((\Delta-2)n+2)/(\Delta-1)$ for a connected graph G of order n and maximum degree $\Delta \geq 2$, with equality if and only if G is either C_n, K_n or $K_{\Delta,\Delta}$ (see Gentner *et al.* [9] and Lu *et al.* [11]). Caro and Pepper [4] used a greedy algorithm to obtain an improved bound $F(G) \leq ((\Delta-2)n-(\Delta-\delta)+2)/(\Delta-1)$, where δ is the minimum degree of G. We gave a complete characterisation of the extremal graphs for this bound in [10]. For the total forcing number, Davila and Henning [7] showed that if G is a connected graph of order $n \geq 3$ with maximum degree $\Delta \geq 2$, then $F_t(G) \leq \Delta n/(\Delta+1)$, with equality if and only if $G = K_{\Delta+1}$ or $K_{1,\Delta}$.

In this paper, we study the semitotal forcing number of a graph. In Section 2, we give some basic definitions as preliminaries. In Section 3, we prove that it is NP-complete to determine the semitotal forcing number of a graph. In Section 4, we provide some upper bounds on the semitotal forcing number of a graph in terms of its order and maximum degree.

2. Preliminaries

Throughout this paper, we only consider simple, undirected and finite graphs.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let u, v be two vertices of G. If $uv \in E(G)$, then we say u, v are *adjacent*, u is a *neighbour* of v and *vice versa*. The *open neighbourhood* of v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighbourhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. Similarly, for any set $X \subseteq V(G)$, $N_G(X) = \cup_{v\in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$. The *degree* $d_G(v)$ of v is the number of vertices in $N_G(v)$. The *minimum degree* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We call a path connecting u and v a (u, v)*-path*. The *distance* between u and v is the length of a shortest (u, v)-path in G, denoted by $d_G(u, v)$. For a vertex v and a vertex set X, let $d_G(v, X) = \min\{d_G(u, v) \mid u \in X\}$. If the graph G is clear from the context, we write V, E, $N(v)$, $N[v]$, $N(X)$, $N[X]$, $d(v)$, δ, Δ, $d(u, v)$, $d(v, X)$ for short.

An independent set of a graph is a set of pairwise nonadjacent vertices, whereas a clique of a graph is a set of pairwise adjacent vertices. A dominating set in a graph G is a set D of vertices of G such that every vertex not in D is adjacent to at least one vertex in D. For a set of vertices $X \subseteq V(G)$, the *induced subgraph* by X, denoted by

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$G[X]$, is the graph with vertex set X, in which two vertices are adjacent if and only if they are adjacent in G. We denote by $G - X$ the induced subgraph $G[V \setminus X]$; if $X = \{x\}$, we write $G - x$ for short.

Denote a path, a cycle and a complete graph on n vertices by P_n, C_n and K_n, respectively. A complete bipartite graph with parts of sizes a and b is denoted $K_{a,b}$.

Two vertices u and v in a nontrivial connected graph G are twins if u and v have the same neighbours in $V(G) \setminus \{u, v\}$.

OBSERVATION 2.1. If u and v are twins of a connected graph G, then every forcing set of G contains at least one vertex of $\{u, v\}$.

3. Complexity of semitotal forcing

In this section, we show that the semitotal forcing problem is NP-complete. The decision version of the semitotal forcing problem is as follows.

PROBLEM 3.1 (Semitotal Forcing). Instance: a graph $G = (V, E)$ of order n and a positive integer $k \leq n$. Question: does G have a semitotal forcing set of size at most k?

THEOREM 3.2. *The semitotal forcing problem is NP-complete.*

PROOF. We first show that the semitotal forcing problem is in NP. Given a set S of vertices of G, it can be checked in polynomial time whether there is a vertex in S with exactly one neighbour not in S. Moreover, there cannot be more than $|V|$ steps in a forcing process. Thus, a nondeterministic algorithm can check in polynomial time whether a subset of vertices of V is forcing and further semitotal forcing, and whether it has size at most $k + 1$.

To show the hardness, we give a polynomial reduction from the forcing problem, which has been shown to be NP-complete in [1, 5].

Let $G = (V, E)$ be a graph, where $V = \{v_1, \ldots, v_n\}$. We construct a connected graph $G' = (V', E')$ with vertex set $V' = V \cup \{u, w_1, w_2\}$ and edge set

$$E' = E \cup \{uv_i \mid i \in [n]\} \cup \{uw_1, uw_2\}.$$

We will show that G has a forcing set of size at most k if and only if G' has a semitotal forcing set of size at most k.

Suppose that G has a forcing set S of size at most k. We claim that $S' = S \cup \{w_1\}$ is a semitotal forcing set of G'. First, we colour all vertices in S' black and the other vertices of G' white. Then $w_1 \to u$ and further all vertices of $V(G)$ can be forced by applying the colour-change rule to S. Finally, $u \to w_2$. Hence, S' is a forcing set of G'. Since every vertex in S is within distance 2 of the vertex w_1, S' is a semitotal forcing set of G' of size at most $k + 1$.

Conversely, suppose that G' has a semitotal forcing set S' of size at most $k + 1$. By Observation 2.1, at least one vertex of $\{w_1, w_2\}$ belongs to S'. Renaming vertices if necessary, we may assume that $w_1 \in S'$. We can choose a semitotal forcing set S' such that u does not force any vertex of $V(G)$. This is because if u forces a vertex v of $V(G)$,

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then $w_2 \in S'$, and $S'' = (S' \setminus \{w_2\}) \cup \{v\}$ is also a semitotal forcing set of G'. Thus, each force between vertices of $V(G)$ in G' can also be applied for $S := S' \cap V(G)$ in G, since if $v \in V(G)$ has a single white neighbour in G' at some step of the forcing process, it will have the same white neighbour in G. Moreover, since u does not force any vertex in $V(G)$, all vertices in $V(G)$ must be forced by the vertices of S' which are in $V(G)$. Thus, S is a forcing set of G. Additionally, $|S| = |S' \cap V(G)| \leq k + 1 - 1 = k$, so S has size at most k. □

4. General upper bounds

We emphasise that it is NP-hard to compute the semitotal forcing number for a general graph, so it is particularly interesting to find efficient bounds for the semitotal forcing number. In this section, we give some upper bounds on the semitotal forcing number of a graph in terms of its order and maximum degree. We use the following result.

THEOREM 4.1 (Davila and Henning, [7]). *If G is a connected graph of order $n \geq 3$ with maximum degree $\Delta \geq 2$, then*

$$F_t(G) \leq \frac{\Delta}{\Delta + 1} n,$$

with equality if and only if $G = K_n$ or $G = K_{1,n-1}$.

Since every total forcing set is also a semitotal forcing set, we have the consequence.

COROLLARY 4.2. *If G is a connected graph of order $n \geq 3$ with maximum degree $\Delta \geq 2$, then*

$$F_{t2}(G) \leq \frac{\Delta}{\Delta + 1} n,$$

with equality if and only if $G = K_n$ or $G = P_3$.

We will give two improved upper bounds for the semitotal forcing number.

We define a weak partition $(V_1, \ldots, V_k)$ of the set V as a partition where some of the sets may be empty. Algorithm 1 outputs a weak partition of the vertex set V of G. According to Algorithm 1, lines 3–8 iteratively find a pair of vertices u and v with distance 2 in the current graph G^{k-1}, set $v_k = v$ and delete all vertices in $N_{G^{k-1}}[N_{G^{k-1}}[v_k]]$ until the remaining connected components are complete graphs. Again, lines 10–14 iteratively delete the connected components whose order is greater than 2 in the remaining graph. Hence, $G[R]$ is a null graph or every component of $G[R]$ is either an edge or an isolated vertex. For each vertex in $A \cup A'$, its neighbours are in $B \cup B' \cup C$. Thus, the set $A \cup A'$ is independent. Similarly, for $1 \leq i < j \leq r$, there is no edge between B_i and B_j; and there is no edge between R and $A \cup A' \cup B \cup B'$.

We now restrict to $G \neq K_n$. By using Algorithm 1, we present another upper bound on the semitotal forcing number of G in terms of its order and maximum degree.

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Algorithm 1 Weak partition.

Input: A graph $G = (V, E)$ on n vertices
Output: A partition (A, B, C, A', B', R) of V

$k := 0$ and $G^k := G$
$A := \emptyset$, $B := \emptyset$, $C := \emptyset$, $A' := \emptyset$ and $B' := \emptyset$
while $u, v \in V(G^k)$ and $d_{G^k}(u, v) = 2$ **do**
    $k := k + 1$
    $v_k := v$ and *add* v_k to A
    $B_k := N_{G^{k-1}}(v_k)$ and *add* B_k to B
    $C_k := N_{G^{k-1}}(B_k) \setminus N_{G^{k-1}}[v_k]$ and *add* C_k to C
    $G^k := G^{k-1} - v_k - B_k - C_k$
$r := k$ and $G^r := G^k$
while $v \in V(G^r)$ and $d_{G^r}(v) \geq 2$, **do**
    $r := r + 1$
    $v_r := v$ and *add* v_r to A'
    $B_r := N_{G^{r-1}}(v_r)$ and *add* B_r to B'
    $G^r := G^{r-1} - v_r - B_r$
$R := V(G^r)$

THEOREM 4.3. *If $G \neq K_n$ is a connected graph of order $n \geq 4$ with maximum degree $\Delta \geq 2$, then*

$$F_{t2}(G) \leq \frac{\Delta - 1}{\Delta} n,$$

with equality if and only if $G = C_4$ or $G = P_4$ or $G = K_{\Delta,\Delta}$.

PROOF. Let $G \neq K_n$ be a connected graph of order $n \geq 4$ with maximum degree $\Delta \geq 2$. If $\Delta = 2$, then $G = P_n$ or $G = C_n$. In both cases, $F_{t2}(G) = 2 \leq n/2 = (\Delta - 1)n/\Delta$, as desired. Further, if $F_{t2}(G) = (\Delta - 1)n/\Delta$, then $n = 4$. Thus, $G = C_4$ or $G = P_4$. Hence, we assume that $\Delta \geq 3$ in what follows.

Applying Algorithm 1 to $G = (V, E)$, we get a weak partition (A, B, C, A', B', R) of V, and $A = \{v_1, \ldots, v_k\}$, $B = \{B_1, \ldots, B_k\}$, $C = \{C_1, \ldots, C_k\}$, $A' = \{v_{k+1}, \ldots, v_r\}$, $B' = \{B_{k+1}, \ldots, B_r\}$. Since $G \neq K_n$, the sets A, B and C are not empty. For $1 \leq i \leq k$, let G_i be the graph induced by $\{v_i\} \cup B_i \cup C_i$; for $k + 1 \leq i \leq r$, let G_i be the graph induced by $\{v_i\} \cup B_i$. Note that r may be equal to k. Let G_i have order n_i and $|B_i| = b_i$, $|C_i| = c_i$. In what follows, we consider G_i and divide into two cases.

Case 1: $1 \leq i \leq k$. We divide into two subcases.

Subcase 1.1: $b_i = 1$ and $c_i = 1$. In this subcase, $G_i = P_3$ and $n_i = 3$. Let $S_i = \{v_i\} \cup B_i$. Then S_i is a semitotal forcing set of G_i and

$$|S_i| = 2 = \frac{2}{3} \times 3 \leq \frac{\Delta - 1}{\Delta} n_i. \tag{4.1}$$

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Subcase 1.2: $b_i \geq 2$ or $c_i \geq 2$. By Algorithm 1, we note that the set B_i dominates the set C_i. Let D_i be a minimum set of vertices in B_i that dominate C_i and $|D_i| = d_i$. Note that $1 \leq d_i \leq b_i$. Let $D_i = \{x^i_1, \ldots, x^i_{d_i}\}$. By the minimality of the set D_i, each vertex x^i_j in D_i dominates a vertex y^i_j in C_i that is not dominated by the other vertices in D_i, where $j \in [d_i]$. Now, let $D'_i = \{y^i_1, \ldots, y^i_{d_i}\}$ and $L_i = C_i \setminus D'_i$. Let $|L_i| = l_i$. Then $c_i = d_i + l_i$ and $n_i = b_i + c_i + 1 = b_i + d_i + l_i + 1$. Each vertex in D_i is adjacent to v_i and to exactly one vertex in D'_i, and therefore is adjacent to at most $\Delta - 2$ vertices in L_i, implying that $l_i \leq d_i(\Delta - 2)$. Let $S_i = V(G_i) \setminus (D'_i \cup \{x^i_1\})$. By the construction, S_i is semitotal. Further, the set S_i is a forcing set of G_i since $v_i \to x^i_1$ first, and then $x^i_j \to y^i_j$ for $j \in [d_i]$. Thus, the set S_i is a semitotal forcing set of G_i. Moreover,

$$\begin{aligned} |S_i| = b_i + l_i \leq b_i + d_i(\Delta - 2) &= b_i - d_i + d_i(\Delta - 1) \\ &\leq \Delta - 1 + d_i(\Delta - 1) = (d_i + 1)(\Delta - 1), \end{aligned} \tag{4.2}$$

which implies that $d_i + 1 \geq |S_i|/(\Delta - 1)$. Thus, $n_i = b_i + l_i + d_i + 1 = |S_i| + d_i + 1 \geq |S_i| + |S_i|/(\Delta - 1)$ and further $|S_i| \leq (\Delta - 1)n_i/\Delta$.

Case 2: $k + 1 \leq i \leq r$. In this case, $G_i = G[\{v_i\} \cup \{B_i\}]$ is a complete graph. Since $G \neq K_{\Delta+1}$, we have $2 \leq b_i \leq \Delta - 1$. Let $S_i = B_i$. It is clear that S_i is a semitotal forcing set of G_i. Thus, $n_i = b_i + 1$ and

$$|S_i| = b_i \leq \frac{\Delta - 1}{\Delta}(b_i + 1) = \frac{\Delta - 1}{\Delta} n_i.$$

The set S_i constructed for each $i \in [r]$ (see Cases 1 and 2 above) is a semitotal forcing set of G_i. We now let $S' = \cup_{i=1}^r S_i$. Thus,

$$|S'| = \sum_{i=1}^{r} |S_i| \leq \sum_{i=1}^{r} \frac{\Delta - 1}{\Delta} n_i = \frac{\Delta - 1}{\Delta} \sum_{i=1}^{r} n_i.$$

If $R = \emptyset$, then $V(G) = \cup_{i=1}^r V(G_i)$. We claim that $S = S'$ is a semitotal forcing set of G. As shown earlier, each set S_i is a semitotal forcing set of G_i for all $i \in [r]$. By the construction, S is semitotal. We colour all vertices in S black and the other vertices white. When we apply the colour-change rule, all vertices of G_i will become black in the order i and

$$|S| = |S'| \leq \frac{\Delta - 1}{\Delta} \sum_{i=1}^{r} n_i = \frac{\Delta - 1}{\Delta} n.$$

Now we consider $R \neq \emptyset$. Suppose $G[R]$ has order n_R. Recall that every component of $G[R]$ is either an edge or an isolated vertex and there is no edge between R and $A \cup A' \cup B \cup B'$. Since G is connected, every component of $G[R]$ is adjacent to some vertex of C. If there exists a vertex $v \in R$ which is not adjacent to some vertex of C, then v belongs to a P_2-component of $G[R]$ and its neighbour is adjacent to some vertex of C. Take all the vertices that are the same as v and put them into T. Let $|T| = t$ and $W = R \setminus T$. Note that $W \neq \emptyset$. Let $D \subseteq C$ be a minimum dominating set of W and $|D| = d$, $D = \{x_1, \ldots, x_d\}$. By the minimality of the set D, each vertex x_j in D dominates

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a vertex y_j in W that is not dominated by the other vertices in D, where $j \in [d]$. Let $D' = \{y_1, \ldots, y_d\}$ and $L = W \setminus D'$. Let $|L| = l$ so that $n_R = d + l + t$.

If $l = 0$, then $S = S'$ is a semitotal forcing set of G. Additionally,

$$|S| = |S'| \leq \frac{\Delta - 1}{\Delta} \sum_{i=1}^{r} n_i < \frac{\Delta - 1}{\Delta} n.$$

Now assume that $l \neq 0$. If $d(v, L \cup S') \leq 2$ for any $v \in L$, then set $S'' = L$. Then $S = S' \cup S''$ is a semitotal forcing set of G; we will justify this claim at the end of the proof. Since each vertex in D $(\subseteq C)$ is adjacent to a vertex of B and to exactly one vertex in D'_i, we have $l \leq d(\Delta - 2)$. Recall that $n_R = d + l + t \geq d + l$, so $|S''| = l \leq d(\Delta - 2) \leq (n_R - |S''|)(\Delta - 2)$. This implies that $|S''| \leq ((\Delta - 2)/(\Delta - 1))n_R < ((\Delta - 1)/\Delta)n_R$. Thus,

$$|S| = |S'| + |S''| < \frac{\Delta - 1}{\Delta} \sum_{i=1}^{r} n_i + \frac{\Delta - 1}{\Delta} n_R = \frac{\Delta - 1}{\Delta}\Big(\sum_{i=1}^{r} n_i + n_R\Big) = \frac{\Delta - 1}{\Delta} n.$$

Suppose that there exists $v \in L$ such that $d(v, L \cup S') \geq 3$. Take all the vertices that are the same as v and put them into X. For any $w \in X$, there exists $u \in D$ such that u is adjacent to w and $u \in C_i$ for some i as in Subcase 1.2. Here, u is adjacent to x_1^i, that is, $u = y_1^i$ and x_1^i is its neighbour in G_i. Since $d(w, L \cup S') \geq 3$, we have $N_R(y_1^i) = \{w, w'\}$, where $w' \in D'$. Take all the vertices that are the same as w' and put them into Y. Now replace G_i with $G'_i = G_i \cup \{w, w'\}$ and again divide into two cases. In the case $b_i \geq 2$, we set $x \in B_i \setminus \{x_1^i\}$ and $S'_i = (S_i \setminus \{x\}) \cup \{x_1^i, w\}$. In the case $b_i = 1$, $c_i \geq 2$, clearly, $D_i = \{x_1^i\}$, $D'_i = \{y_1^i\}$ and $L_i \neq \emptyset$. Since $L_i \subseteq S_i$, we set $y \in L_i$ and $S'_i = (S_i \setminus y) \cup \{y_1^i, w\}$. In both cases, it is not hard to check that S'_i is a semitotal forcing set of G'_i. Then for G'_i, $n'_i = n_i + 2 = b_i + d_i + l_i + 3$ and $|S'_i| = |S_i| + 1 = b_i + l_i + 1 \leq b_i + d_i(\Delta - 2) + 1 = b_i - d_i + d_i(\Delta - 1) + 1 \leq \Delta - 1 + d_i(\Delta - 1) + 1 = (d_i + 1)(\Delta - 1) + 1$, which implies that $d_i + 1 \geq (|S'_i| - 1)/(\Delta - 1)$. Thus, $n'_i = b_i + l_i + d_i + 3 = |S'_i| + d_i + 2 \geq |S'_i| + (|S'_i| - 1)/(\Delta - 1) + 1 = (\Delta|S'_i| + \Delta - 2)/(\Delta - 1) > \Delta|S'_i|/(\Delta - 1)$ and further $|S'_i| < ((\Delta - 1)/\Delta)n'_i$.

Now return to W. Let $W' = W \setminus (X \cup Y)$, $D'' = D' \setminus Y$ and $S'' = L \setminus X$. Let $R' = W' \cup T (= (D' \setminus Y) \cup (L \setminus X) \cup T)$ and $G_{R'}$ have order $n_{R'}$. Then $n_{R'} = d'' + |S''| + t \geq d'' + |S''|$, where $d'' = |D''|$. Thus, $S = S' \cup S''$ is a semitotal forcing set of G, where some S_i in S' is replaced by S'_i. We have $|S''| \leq (\Delta - 2)d'' \leq (\Delta - 2)(n_{R'} - |S''|) = (\Delta - 2)n_{R'} - (\Delta - 2)|S''|$. This implies $|S''| \leq ((\Delta - 2)/(\Delta - 1))n_{R'} < ((\Delta - 1)/\Delta)n_{R'}$. Thus,

$$|S| = |S'| + |S''| < \frac{\Delta - 1}{\Delta} \sum_{i=1}^{r} n_i + \frac{\Delta - 1}{\Delta} n_{R'} = \frac{\Delta - 1}{\Delta}\Big(\sum_{i=1}^{r} n_i + n_{R'}\Big) = \frac{\Delta - 1}{\Delta} n.$$

We now show that the set S is a semitotal forcing set in G. By the construction, S is semitotal. In the first stage of the forcing process, we colour all vertices in G_i for $i \in [r]$ black. As shown earlier, when we apply the colour-change rule to S_i in G_i with the order from small to large, all vertices of G_i turn black.

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In the second stage of the forcing process, we colour all vertices of R black. Now we play each of the vertices of D in turn, thereby colouring all vertices in D' black. Finally, all vertices of T can be forced and all vertices of G are coloured black.

Thus, $F_{t2}(G) \leq |S| \leq ((\Delta - 1)/\Delta)n$, as desired. Suppose next that $F_{t2}(G) = ((\Delta - 1)/\Delta)n$. Then S is a minimum semitotal forcing set in G and $|S| = ((\Delta - 1)/\Delta)n$. Recall that by our earlier assumptions, $\Delta \geq 3$. If $R \neq \emptyset$, then, as shown above, $|S| < ((\Delta - 1)/\Delta)n$, which is a contradiction. Hence, $R = \emptyset$, implying that $|S_i| = ((\Delta - 1)/\Delta)n_i$. For all $i \in [k]$, the set S_i must have been constructed as in Subcases 1.1 and 1.2 and equality holds in (4.1) and (4.2), which implies that ($\Delta = 3$, $G_i = P_3$) and ($b_i = \Delta$, $d_i = 1$, $l_i = \Delta - 2$), respectively.

We claim that G is a regular graph. Otherwise, $\delta < \Delta$ and we can choose a weak partition (A, B, C, A', B', R) of V such that v_1 is a vertex of minimum degree. Thus, $b_1 \neq \Delta$. Further, $\Delta = 3$ and $G_1 = P_3$, where $d(v_1) = 1$. Let $B_1 = \{z\}$. We find that $d(z) = 2$. Now we reselect a weak partition (A, B, C, A', B', R) of V such that $v_1 = z$. Then $d(v_1) = b_i = 2 < \Delta$ and, by the previous analysis, equality holds in (4.1) and (4.2) for $i = 1$, which is a contradiction. Thus, $\delta = \Delta \geq 3$.

Now consider $i = 1$. With S_1 constructed as in Subcase 1.2, we have $b_1 = \Delta$, $d_1 = 1$, $l_1 = \Delta - 2$. Then, $d(v_1) = d(x_1^1) = \Delta$. First, we show that B_1 is an independent set. Otherwise, there exist $u, v \in B_1$ different from x_1 such that u is adjacent to v. Since Δ is the maximum degree, there exists $w \in C_1$ such that w is not adjacent to v. Let $S_1' = V(G_1) \setminus \{u, x_1, w\}$. Then $v \to u$ and further $v_1 \to x_1^1 \to w$. Thus, S_1' is a semitotal forcing set of G_1 smaller than S_1, and so $(S \setminus S_1) \cup S_1'$ is a semitotal forcing set of G smaller than S, which is a contradiction. Since Δ is the maximum degree, it is not hard to see that $N(v) = \{v_1\} \cup C_1$ for each $v \in B_1$. Therefore, $G = K_{\Delta,\Delta}$, as desired.

This completes the proof. □

As an immediate consequence of Theorem 4.3, we have the following result.

THEOREM 4.4. *If G is a connected graph of order $n \geq 3$ with maximum degree $\Delta \geq 2$, then*

$$F_{t2}(G) \leq \frac{(\Delta - 1)n + 1}{\Delta},$$

with equality if and only if $G = K_n$ or $G = P_3$.

PROOF. Let G be a connected graph of order $n \geq 3$ with maximum degree $\Delta \geq 2$. If $G = K_n$, then $F_{t2}(G) = n - 1 = ((\Delta - 1)n + 1)/\Delta$. Now consider $G \neq K_n$. If $n = 3$, then $G = P_3$ and $F_{t2}(G) = 2 = ((\Delta - 1)n + 1)/\Delta$, as desired. If $n \geq 4$, then $F_{t2}(G) \leq (\Delta - 1)n/\Delta < ((\Delta - 1)n + 1)/\Delta$ by Theorem 4.3. Thus, $F_{t2}(G) \leq ((\Delta - 1)n + 1)/\Delta$, with equality if and only if $G = K_n$ or $G = P_3$. □

If G is a connected graph of order n with maximum degree Δ, then $n \geq \Delta + 1$ and

$$\frac{(\Delta - 1)n + 1}{\Delta} \leq \frac{\Delta}{\Delta + 1}n. \tag{4.3}$$

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The equality holds in (4.3) if and only if $n = \Delta + 1$. Thus, $F_{t2}(G) = ((\Delta - 1)n + 1)/\Delta = \Delta n/(\Delta + 1)$ if and only if $G = K_n$ or $G = P_3$. Thus, the upper bound of Theorem 4.2 follows as an immediate consequence of the upper bound of Theorem 4.4.

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