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Rotation intervals for a family of degree one circle maps

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Abstract. Let f be a C^0 circle map of degree one with precisely one local minimum and one local maximum, and let $[\rho_-(f), \rho_+(f)]$ be the interval of rotation numbers of f. We study the structure of the function $\rho(\lambda) = \rho_+(R_\lambda \circ f)$, where R_λ is the rotation through the angle λ.

0. *Introduction*

The rotation interval for a degree one circle map was first defined by Newhouse, Palis and Takens in [**9**] and was subsequently shown by Ito ([**6**]) to be closed. Newhouse, Palis and Takens also showed that if f varies continuously in the C^0 topology then $\rho_-(f)$ and $\rho_+(f)$ vary continuously also. In another paper ([**7**]), Ito showed that if $\rho_+(f) \in \mathbb{R}\backslash\mathbb{Q}$, then $\rho_+(R_\lambda \circ f) > \rho_+(f)$ for all $\lambda > 0$.

In this paper we study the behaviour of the function $\rho_+(R_\lambda \circ f)$ near rational values. There are four main theorems, Theorems 2.5, 3.5, 3.7, and 3.8. The first deals with the persistence of $\rho_+(f_\lambda)$ at rational values in any C^0-continuous family of continuous circle maps of degree one. On the assumption that f_λ is never a homeomorphism it is shown that if $\rho_+(f_\lambda) \in \mathbb{Q}$ then there is an $\varepsilon > 0$ such that either $\rho_+(f_\mu) \geq \rho_+(f_\lambda)$ for all $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$ or else $\rho_+(f_\mu) \leq \rho_+(f_\lambda)$ for all $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$. This theorem is a slight generalization of results obtained by Bamon, Malta and Pacífico in [**2**], and it serves to set the stage for the other two main theorems.

Theorems 3.5, 3.7, and 3.8 deal with a more specific family $R_\lambda \circ f$. In addition they require that we impose several differentiability conditions on f. Specifically we assume that f is C^3 with precisely two critical points, a quadratic local minimum and a quadratic local maximum, and that f has a negative Schwarzian derivative. If $b > \frac{1}{2}\pi$ all these conditions are satisfied, for example, by the function family $x + b \sin(2\pi x) \pmod 1$. One further condition, also required for these theorems, ensures that $\rho_-(R_\lambda \circ f) < \rho_+(R_\lambda \circ f)$ for all λ. The function $x + b\sin(2\pi x) \pmod 1$ satisfies this last condition at least when $b > \frac{1}{4}$.

To state the theorems we need a little number theory. For any rational number p/q with $(p, q) = 1$ there is an associated rational number defined as follows: s is the smallest positive integer such that

$$sp + 1 = rq$$

for some $r \in \mathbb{Z}$. Then

$$\frac{p}{q}+\frac{1}{qs}=\frac{r}{s},$$

and so $r/s > p/q$. In fact it is easy to show that r/s is the smallest rational greater than p/q whose denominator satisfies $s < q$. This immediately puts the relationship between p/q and r/s into the context of Farey sequences (see [5]).

We now use the numbers p/q and r/s to construct a decreasing sequence of rationals, each the mediant of p/q and its predecessor:

$$\phi_k\left(\frac{p}{q}\right)=\frac{r+kp}{s+kq}.$$

Thus $\phi_k(p/q)$ is always p/q's nearest neighbour to the right in the Farey sequence of appropriately high order, and every such nearest neighbour is of that form.

Theorem 3.8 may be summarized as follows. Suppose f satisfies the differentiability conditions indicated above, and that $\rho_+(f)=p/q$ with $\rho_+(R_\lambda \circ f) > p/q$ if $\lambda > 0$. Let $\mu_n > 0$ be the smallest value of λ at which $\rho_+(R_{\mu_n} \circ f)=\phi_n(p/q)$. Then

$$\lim_{n\to\infty}\frac{\mu_n}{\mu_{n+1}}$$

exists. In fact we show that this limit is equal to the derivative of the qth iterate of f at a certain periodic orbit.

For Theorem 3.5 we suppose $\lambda_n > 0$ is the value at which the local maximum is periodic, and that its rotation number is $\phi_n(p/q)=\rho_+(R_{\lambda_n} \circ f)$. The theorem concludes that $\lim_{n\to\infty} \lambda_n/\lambda_{n+1}$ exists and is equal to the limit of Theorem 3.8.

For Theorem 3.7 we let ν_n be the greatest parameter value for which

$$\rho_+(R_{\nu_n} \circ f)=\phi_n(p/q).$$

This theorem concludes once again that ν_n/ν_{n+1} has the same limit as $n\to\infty$. In addition, however, the theorem shows that

$$\lim_{n\to\infty}\frac{\nu_{n+1}-\lambda_{n+1}}{\lambda_n-\lambda_{n+1}}=1.$$

This indicates that for small positive λ, $\rho_+(R_\lambda \circ f)$ is nearly always rational.

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1. *Preliminaries*

When studying a continuous degree one circle map f it is convenient to lift it to $\mathbb{R}$ giving a function $F:\mathbb{R}\to\mathbb{R}$ satisfying $F(x+1)=F(x)+1$. In other words, we let $e:\mathbb{R}\to S^1$ be the map $x\mapsto e^{2\pi i x}$, and choose F so that $e\circ F=f\circ e$. F is unique up to an integer, and all the properties of f may be studied by examining F. We will let $\mathrm{End}_1^r(\mathbb{R})$ denote the space of all C^r functions $F:\mathbb{R}\to\mathbb{R}$ with the property $F(x+1)=F(x)+1$.

For $F \in \mathrm{End}_1^0(\mathbb{R})$ the rotation interval is defined as follows: For $x \in \mathbb{R}$ we let

$$\rho(x) = \lim_{n\to\infty} \sup \frac{1}{n} F^n(x).$$

$\rho(x)$ is the *rotation number* of x. The set

$$\rho(F) = \{\rho(x) \mid x \in \mathbb{R}\}$$

is a closed interval ([**9, 6**]) called the *rotation interval* of F. We will write $\rho_-(F)$ and $\rho_+(F)$ for the left and right endpoints of this interval. We also call $\rho(F)$ the rotation interval of f, writing $\rho(f)$, but then we should keep in mind that $\rho(f)$ is unique only up to addition of an integer. However, once a lift F is chosen for one member f in a continuous family of circle maps, the family lifts uniquely to a continuous family in $\mathrm{End}_1^0(\mathbb{R})$. This makes it possible to discuss the variation of the rotation interval of a continuous family.

When $e(x)$ is periodic under f with period q, then x *is periodic under* F in the following sense:

$$F^q(x) = x + p$$

for some $p \in \mathbb{Z}$. The rotation number of x is then p/q. We may assume that $(p, q) = 1$ here. When we say that x is periodic under F with rotation number p/q we shall always mean it in this sense. We shall also abuse language slightly when speaking of the *F-orbit of a periodic point* x_0. We shall mean by it the preimage under e of the f-orbit of $e(x_0)$, and we shall always index the points of $e^{-1}\{f^k(e(x_0)) \mid k \in \mathbb{Z}_+\}$ in increasing order:

$$\cdots < x_{-1} < x_0 < x_1 < x_2 < \cdots.$$

Then $x_{i+q} = x_i + 1$ for all $i \in \mathbb{Z}$, and $\{x_i\}_{i\in\mathbb{Z}}$ is the union of p disjoint invariant sets, $\{F^k(x_0)\}$, $\{F^k(x_0+1)\}, \ldots, \{F^k(x_0+p-1)\}$, each of them a lift of the f-orbit of $e(x_0)$. In other words, when we speak of a periodic F-orbit it will in actual fact be a set of disjoint periodic orbits.

An F-orbit of period q and rotation number p/q is called a *twist periodic orbit* (TPO, see [**1**]) if $F(x_i) = x_{i+p}$ for all i. Geometrically this means that the effect of f on the points of the orbit is that of an orientation preserving homeomorphism.

If $F^q(x) < x + p$ for all $x \in \mathbb{R}$, then clearly $\rho(x) < p/q$ for all x and so $\rho_+(F) < p/q$. Similarly $\rho_-(F) > p/q$ if $F^q(x) > x + p$ for all x. Thus, if $p/q \in \rho(F)$ then F has a periodic orbit of period q and rotation number p/q. In fact, F will have a TPO of period q and rotation number p/q, and if $p/q \in \mathrm{int}\,(\rho(F))$ it will have infinitely many others of that rotation number ([**8**], see also [**3**]).

2. *Persistence of* $\rho_+(F)$ *at rational values*

If $F \in \mathrm{End}_1^r(\mathbb{R})$, then $\rho_+(F)$ is said to be C^r*-persistent* if $\rho_+(G) = \rho_+(F)$ for all $G \in \mathrm{End}_1^r(\mathbb{R})$ in a neighbourhood of F in the C^r-topology. In their paper [**2**], Bamon, Malta and Pacífico obtained necessary and sufficient conditions for the C^1-persistence of $\rho_+(F)$. One of their results is that $\rho_+(F)$ is not persistent if $\rho_+(F)$ is irrational. This also follows from an earlier paper of Ito ([**7**]) who shows that if $\rho_+(F) \in \mathbb{R}\backslash\mathbb{Q}$ then $\rho_+(R_\lambda \circ F) > \rho_+(F)$ for any rotation λ with $\lambda > 0$. Much of [**2**] is devoted to finding the precise conditions on F with $\rho_+(F)\in\mathbb{Q}$ under which $\rho_+(F)$ is C^1-persistent, and showing that $\rho_+(F)$ is C^1-persistent for generic F. They obtain the following result (Theorem B, (iv)). Letting

$$A_{p,q}=\{x\in\mathbb{R}\,|\,F^q(x)>x+p\},$$

and

$$B_{p,q}=\{x\in\mathbb{R}\,|\,F^q(x)<x+p\},$$

they prove that if $\rho_+(F)=p/q$ then $\rho_+(F)$ is C^1-persistent if and only if $A_{p,q}$ and $B_{p,q}$ are non-empty and there exists $b\in\partial B_{p,q}$ such that $F^q(x)<b+p$ for all $x<b$.

Bamon et al. did not indicate that this persistence of $\rho_+(F)$ at rational values must occur in every one parameter family of degree one circle maps. We will show that it does if the functions in the family are not homeomorphisms. Prior to that we will also present simple necessary and sufficient conditions for C^0-persistence of $\rho_+(F)$ at rational values.

Following Misiurewicz ([**8**]) we associate with $F\in\mathrm{End}_1^0(\mathbb{R})$ the following non-decreasing continuous functions:

$$F_+(x)=\sup_{u\le x}F(u)$$

$$F_-(x)=\inf_{u\ge x}F(u).$$

F_+ may be characterized as the smallest non-decreasing function greater than or equal to F. Similarly F_- may be characterized as the largest non-decreasing function less than or equal to F. It is easy to see that F and F_+ are identical if F is already non-decreasing. Otherwise there will be intervals on which $F_+(x)$ is constant and strictly greater than $F(x)$. Similar comments apply to F_-. Since F_+ and F_- are non-decreasing, they have unique rotation numbers.

Lemma 2.1. (Misiurewicz [**8**], Remark (B)). *$\rho_+(F)=\rho(F_+)$ and $\rho_-(F)=\rho(F_-)$.*

Proof. $F_+(x)\ge F(x)$ for all x. Since F_+ is non-decreasing this implies that $F_+^k(x)\ge F^k(x)$ for all $x\in\mathbb{R}$ and for all $k\in\mathbb{Z}_+$. Thus $\rho(F_+)\ge\rho_+(F)$.

On the other hand, if $p/q>\rho_+(F)$ then $F^q(x)<x+p$ for all x. Choose x_0 arbitrarily and define

$$x=x_{-q+1},x_{-q+2},\dots,x_{-1},x_0$$

inductively by $x_{i-1}=\inf(F^{-1}(x_i))$. Then $F_+(x_{i-1})=F(x_{i-1})=x_i$ for all i. Thus $F_+^q(x)<x+p$ for this particular x. Since $\rho(F_+)$ is a single number it follows that $\rho(F_+)\le p/q$. By choosing a sequence of rationals to converge to $\rho_+(F)$ from above it follows that $\rho(F_+)\le\rho_+(F)$.

The proof that $\rho_-(F)=\rho(F_-)$ is similar.

Lemma 2.2. *$(F_+)^k=(F^k)_+$ and $(F_-)^k=(F^k)_-$.*

Proof. First note that $F_+=F$ if F is already non-decreasing. In particular $((F_+)^k)_+=(F_+)^k$. But then $F(x)\le F_+(x)$ for all x implies $F^k(x)\le(F_+)^k(x)$ for all x. Hence $(F^k)_+\le((F_+)^k)_+=(F_+)^k$.

We prove the reverse inequality by induction. Suppose $(F_+)^k=(F^k)_+$. Then

$$(F_+)^{k+1}(x)=F_+((F_+)^k(x))=\sup\{F(u)\,|\,u\le(F_+)^k(x)\}.$$

Since F is continuous, $\sup\{F(u)\,|\,u\le(F_+)^k(x)\}=F(u)$ for some $u\le(F_+)^k(x)$. But then $u=(F_+)^k(v)=(F^k)_+(v)$ for some $v\le x$. That is, $u=F^k(w)$ for some $w\le x$. Thus $(F_+)^{k+1}(x)=F^{k+1}(w)$ for some $w\le x$. In particular, $(F_+)^{k+1}(x)\le(F^{k+1})_+(x)$.

The proof that $(F_-)^k=(F^k)_-$ is similar. □

LEMMA 2.3. *If $F_+^q(x)-p\ge x$ for all x, then $\rho_+(F+\lambda)>p/q$ for all $\lambda>0$. If $F_+^q(x)-p\le x$ for all x, then $\rho_+(F-\lambda)<p/q$ for all $\lambda>0$.*

Proof. Suppose $\lambda>0$ and assume $F_+^q(x)-p\ge x$ for all x. Then

$$(F_++\lambda)^{q-1}(x)\ge(F)^{q-1}(x)$$

because F_+ is non-decreasing. Therefore, for all x,

$$\begin{aligned}(F_++\lambda)^q(x)-p&\ge(F_++\lambda)(F_+)^{q-1}(x)-p\\&>F_+^q(x)-p\ge x.\end{aligned}$$

Since $(F_++\lambda)^q$ satisfies $(F_++\lambda)^q(x+1)=(F_++\lambda)^q(x)+1$, therefore there is an $\varepsilon>0$ such that $(F_++\lambda)^q(x)-p>x+\varepsilon$ for all x. It now follows immediately that $\rho(F_++\lambda)>p/q$. The second assertion is proved similarly. □

PROPOSITION 2.4. *If $F\in\mathrm{End}_1^0(\mathbb{R})$, and $\rho_+(F)=p/q$, then $\rho_+(F)$ is C^0-persistent if and only if $\rho(F_+)$ is C^0-persistent in the sense that for every $G\in\mathrm{End}_1^0(\mathbb{R})$ in a C^0-neighbourhood of F_+ we have $p/q\in\rho(G)$.*

Proof. The transformations $\mathrm{End}_1^0(\mathbb{R})\to\mathrm{End}_1^0(\mathbb{R})$ given by $F\mapsto F_+$ and $F\mapsto F_+^q-p$ are clearly continuous in the C^0 topology.

If $\rho(F_+)$ is C^0-persistent in the sense defined above, then for G in a C^0-neighbourhood of F, G_+ will be near F and so $\rho(G_+)=p/q$. This proves that $\rho(F)$ is then C^0-persistent.

To prove the converse, note that if there exist x_1, x_2 such that $F_+^q(x_1)-p<x_1$ and $F_+^q(x_2)-p>x_2$, then these conditions hold on a C^0-neighbourhood of F_+, and so $\rho(F_+)$ is C^0-persistent. Thus, if $\rho(F_+)=p/q$ but is not C^0-persistent, then either $F_+^q(x)-p\ge x$ for all x or $F_+^q(x)-p\le x$ for all x. Lemma 2.3 implies that $\rho(F_+)$ fails to be C^0-persistent in either case. □

The conditions obtained in [**2**] for C^1-persistence of $\rho_+(F)$ are necessary but not quite sufficient for C^0-persistence. We have instead the following parallel result.

PROPOSITION 2.5. *If $F\in\mathrm{End}_1^0(\mathbb{R})$ and $\rho_+(F)=p/q$, then $\rho_+(F)$ is C^0-persistent if and only if $A_{p,q}$ and $B_{p,q}$ are non-empty and there exists a point b on the boundary of a connected component of $B_{p,q}$ such that $F^q(x)<b+p$ for all $x<b$.*

Proof. First note that $A_{p,q}(F)=\varnothing$ if and only if $A_{p,q}(F_+)=\varnothing$ and that $B_{p,q}(F_+)=\varnothing$ if $B_{p,q}(F)=\varnothing$. For $B_{p,q}$ this follows immediately from the fact that $F_+^q(x)\ge F^q(x)$ for all x. The proof that $A_{p,q}(F_+)=\varnothing$ implies $A_{p,q}(F)=\varnothing$ is similar. On the other hand, if $A_{p,q}(F)=\varnothing$ then the function F^q-p is less than or equal to the increasing function $i(x)=x$. Therefore $(F^q-p)_+=F_+^q-p<i$. Thus $A_{p,q}(F_+)=\varnothing$.

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Suppose $\rho_+(F)$ is C^0-persistent. Then by Proposition 2.4, $\rho(F_+)$ is C^0-persistent and so by Lemma 2.3, $A_{p,q}(F_+)\neq\emptyset$ and $B_{p,q}(F_+)\neq\emptyset$. Hence $A_{p,q}(F)\neq\emptyset$ and $B_{p,q}(F)\neq\emptyset$. Furthermore, if $F_+^q(x_1)-p<x_1$ and $F_+^q(x_2)-p>x_2$ let J be the connected component of $B_{p,q}(F_+)$ containing x_1 and let $b=\sup J$. Then $F_+^q(b)-p=b$, and $F_+^q(x)-p<b$ for $x<b$. Therefore $F^q(x)-p<b$ for $x<b$. But $F_+^q(b)-p=b$ implies $F^q(u)-p=b$ for some $u\leq b$. Since $F^q(u)-p\leq F_+^q(u)-p$ this implies $u=b$. Thus $F^q(b)-p=b$. Since also $F^q(x)-p\leq F_+^q(x)-p<x$ for $x\in J$ we conclude that b is on the boundary of a connected component of $B_{p,q}(F)$.

Now for the converse suppose $A_{p,q}(F)$ and $B_{p,q}(F)$ are non-empty and that there is a point b on the boundary of a connected component J of $B_{p,q}(F)$ such that $F^q(x)<b+p$ for all $x<b$. The latter condition implies $F_+^q(b)=b+p$. Also, $A_{p,q}(F_+)\neq\emptyset$. It remains only to show that $B_{p,q}(F_+)\neq\emptyset$, for then $\rho_+(F)$ will be C^0-persistent by Proposition 2.4. But if $B_{p,q}(F_+)$ is empty, then for $x>b$, $F_+^q(x)-p\geq x$ and so for each such x there is a number $u\leq x$ at which $F^q(u)-p\geq x>b$. Since $F^q(u)\leq b+p$ for $u\leq b$, therefore $u\in(b,x)$. In particular J cannot lie above b. Thus $J=(a,b)$ for some $a<b$. Let $C=\sup\{F^q(x)-p\,|\,x<a\}$. Then $c<b$ by our assumptions. Let

$$a_1=\sup\{x\in J\,|\,F^q(x)-p=c\}.$$

Then $a_1<b$. Now suppose $x\in(a_1,b)$. Then $F_+^q(x)-p=F^q(u)-p$ for some $u\leq x$. If $u<a_1$ this means $F_+^q(x)-p\leq c<x$, and if $u\geq a_1$ then

$$F_+^q(x)-p=F^q(u)-p<u\leq x.$$

Thus $(a_1,b)\subset B_{p,q}(F_+)$. □

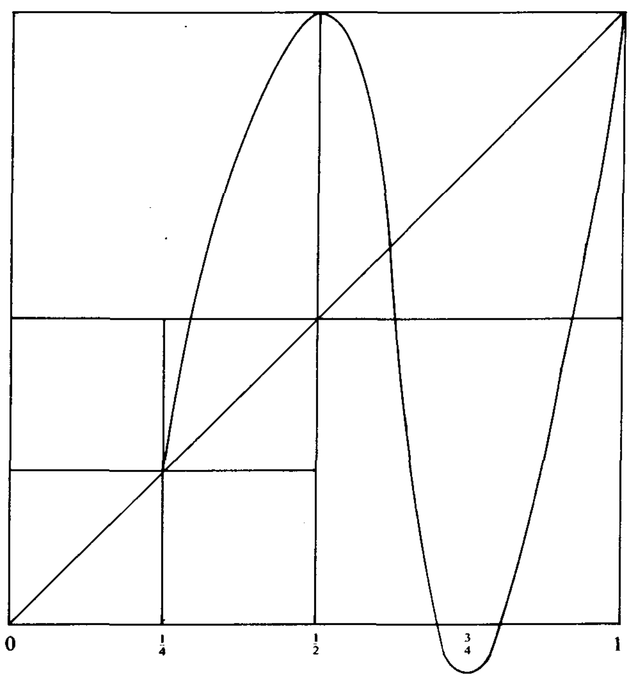

FIGURE 1

Although the differences between the conditions presented in Proposition 2.5 and those presented in [2] are very slight, the latter are not in fact sufficient for C^0-persistence, as the following example will show.

EXAMPLE 2.6. *We will define a function F on $[0, 1]$ with $F(0)+1=F(1)$, and then extend it to $\mathbb{R}$ using the identity $F(x+1)=F(x)+1$. First we define a function $\alpha(x)$ on $[\frac{1}{4}, 1]$ with the graph shown in figure 1.*

We only require the following of α: α is continuous, $\alpha(x)\le 1$ on $[\frac{1}{4}, 1]$, $\alpha(x)>x$ on $[\frac{1}{4}, \frac{1}{2}]$, $\alpha(\frac{1}{4})=\frac{1}{4}$, $\alpha(\frac{1}{2})=1$, $\alpha(x)<x$ on $(\frac{3}{4}, 1)$ and $\alpha(1)=1$. We now define F on $[0, 1]$ as follows:

$$F(x)=\begin{cases} x \text{ if } x\in[\frac{1}{4}, 1] \text{ and if } x=0 \\ \dfrac{1}{4^n}\alpha(4^n x) \text{ if } x\in\left[\dfrac{1}{4^{n+1}}, \dfrac{1}{4^n}\right] \text{ and } n\ge 1. \end{cases}$$

It is easy to see that if this F is extended to $\mathbb{R}$ then $\rho_+(F)=0$ is not C^0-persistent, even though $A_{0,1}$ and $B_{0,1}$ are non-empty, and $1\in\partial B_{0,1}$ with $F(x)<1$ for $x<1$.

The main theorem of this section proves that the persistence of $\rho_+(F)$ at rational values occurs in every continuous one-parameter family, provided the members of the family are not homeomorphisms.

THEOREM 2.7. *Let F_λ, $\lambda\in[\alpha, \beta]$ be a family of functions in $\mathrm{End}_1^0(\mathbb{R})$, none of which are homeomorphisms. We assume that $\lambda\mapsto F_\lambda$ is continuous in the C^0 topology on $\mathrm{End}_1^0(\mathbb{R})$. Then for every $\lambda\in(\alpha, \beta)$ with $\rho_+(F_\lambda)\in\mathbb{Q}$ one of the following is true, and possibly both:*

(i) *There is an $\varepsilon>0$ such that $\rho_+(F_\mu)\ge\rho_+(F_\lambda)$ for all μ with $|\mu-\lambda|<\varepsilon$.*

(ii) *There is an $\varepsilon>0$ such that $\rho_+(F_\mu)\le\rho_+(F_\lambda)$ for all μ with $|\mu-\lambda|<\varepsilon$.*

In particular, if $\rho_+(F_\mu)<p/q$ and $\rho_+(F_\nu)>p/q$, then there is a non-trivial parameter interval J between μ and ν such that $\rho_+(F_\lambda)=p/q$ for all $\lambda\in J$.

Proof. If (i) and (ii) are true the rest of the theorem clearly follows. Suppose $\rho_+(F_\lambda)=p/q$.

In the C^0 topology $(F_{\lambda+})$ depends continuously on λ, as does $(F^q_{\lambda+})$. Suppose (ii) is false. Thus there is a sequence $\lambda_i\to\lambda$ such that $\rho_+(F_{\lambda_i})>\rho_+(F_\lambda)$. That is, for each i $(F_{\lambda_i^+})^q(x)-p>x$ for all $x\in\mathbb{R}$. Therefore, by continuity, $(F_{\lambda+})^q(x)-p\ge x$ for all $x\in\mathbb{R}$. Now by hypothesis there is at least one interval (x_1, x_2) on which $(F_{\lambda+})^q-p$ is locally constant. Thus

$$(F^q_\lambda)_+(x_1)-p=(F^q_\lambda)_+(x_2)-p\ge x_2>x_1.$$

But then, again by continuity, there is an $\varepsilon>0$ such that $(F^q_\mu)_+(x_1)-p>x_1$ for all μ with $|\mu-\lambda|<\varepsilon$. Thus $\rho_+(F_\mu)\ge p/q$ for all μ with $|\mu-\lambda|<\varepsilon$. □

We end this section with two further results about the relationship between F and F_+.

LEMMA 2.8. *If $\{x_i\}$ is a periodic orbit of F_+ then $\{x_i\}$ is a TPO (of F_+).*

Proof. Suppose $F^q_+(x_i)-p=x_i$ for all $i\in\mathbb{Z}$. Since F_+ is non-decreasing it follows that $F_+(x_{i+1})>F_+(x_i)$ for all i. Say $F_+(x_i)=x_{i+\sigma(i)}$, $\sigma(i+1)\ge\sigma(i)$. Since $x_{i+q}=x_i+1$, therefore $\sigma(i+q)=\sigma(i)$. From this it follows immediately that $\sigma(i)$ is independent

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of i. Say $\sigma(i)=m$. But then $F^q(x_i)=x_{i+qm}$. On the other hand $F^q(x_i)=x_i+p=x_{i+qp}$. This shows that $m=p$, and thus that $\{x_i\}$ is a TPO. □

COROLLARY 2.9. *If $\{x_i\}$ is a periodic orbit of F at which F and F_+ agree, then $\{x_i\}$ is a TPO.*

3. *Rate of increase of $\rho_+(F)$ from rational values*

In this section we will assume that $\rho_+(F)=p/q$ and $\rho_+(F+\lambda)>p/q$ for $\lambda>0$. To obtain our main results about the function $\rho_+(F+\lambda)$ we have to make certain differentiability assumptions about F. We let $\mathscr{C}$ be the subset of $\mathrm{End}_1^3(\mathbb{R})$ consisting of all functions F which on each unit interval have precisely one local maximum and one local minimum and for which $|F'(x)|>0$ everywhere else. In this section we assume that F belongs to $\mathscr{C}$. We let C and D be a pair of adjacent critical points with C a local maximum and D a local minimum and $C<D$. Note that F and F_+ are identical except on the integer translates of an interval (C, C'), with $D<C'<C+1$, on which F_+ is constant. Similarly F and F_- differ only on the integer translates of an interval (D', D) with $D-1<D'<C<D$.

In addition to $F\in\mathscr{C}$ we shall also assume that $SF<0$ where $SF(x)$ is the Schwarzian derivative

$$SF(x)=\frac{F'''(x)}{F'(x)}-\frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2.$$

We will make use of the following subclasses $\mathscr{S}$ and $\mathscr{T}$ of $\mathscr{C}$. $\mathscr{S}$ consists of all functions $F\in\mathscr{C}$ satisfying the following conditions:

S1. $SF(x)<0$ if x is not a critical point;
S2. $\rho_-(F)<\rho_+(F)$.

$\mathscr{T}$ consists of all functions $F\in\mathscr{C}$ satisfying these conditions:

T1. $SF(x)<0$ if x is not a critical point;
T2. $F'(x)>1$ if $x\notin(D', C')+\mathbb{Z}$.

Conditions S2 and T2 serve to prevent bifurcations of F_-^q from interfering with bifurcations of F_+^q. T2 is satisfied, for example, by $F(x)=x+b\sin(2\pi x)$ provided $b>\frac{1}{4}$. Clearly, if F satisfies T2 then so does $F+\lambda$. The disadvantage of condition S2 is that it seems hard to verify.

For background information on the properties of functions with negative Schwarzian derivative the reader should consult [**4**] or [**10**]. Let us simply note that these functions constitute an open subset in the class of C^3 functions, and that if F has negative Schwarzian derivative then so does F^q-p for any $q\in\mathbb{Z}_+$ and $p\in\mathbb{Z}$. It follows that the derivative of F^q-p does not have a negative local maximum or a positive local minimum. Consider the following two lemmas as applications of this property. We let $G_\lambda=(F_++\lambda)^q-p$.

LEMMA 3.1. *If $F\in\mathscr{S}$ or $F\in\mathscr{T}$ and if $\rho_+(F)=p/q$ and $\rho_+(F+\lambda)>p/q$ for all $\lambda>0$, then $G_0(C')=C'$ and $G_0'(C')>1$.*

Proof. Suppose $G_0(C')\neq C'$. Then $G_0(C')>C'$, because otherwise $\rho_+(F+\lambda)\le p/q$ for small λ. Let $B=\inf\{x>C'\colon G_0(x)=x\}$. We claim that G_0 is strictly increasing between C' and B. Otherwise there would exist $k, \ell \in \mathbb{Z}$ such that $F_+^k(C') < C' + \ell \leq F_+^k(B)$ and $0 < k \leq q-1$. Then $C' \leq F_+^k(B) - \ell$. Since $F_+^k(B) - \ell$ is a fixed point of G_0, we have $F_+^k(B) - \ell \geq B$, which together with $F_+^k(C') \leq C' + \ell$ gives $\ell/k = p/q$. This is a contradiction, since $(p, q) = 1$.

Note that $G_0'(B) = 1$ and $G_0''(B) \geq 0$, for otherwise $\rho_+(F+\lambda) \leq p/q$ for small λ. Let

$$E = \sup \{x < C' : (F^q)'(x) = 0\}.$$

Then because $SF^q < 0$ we have that $(F^q - p)(x) > x$ on $[E, B)$. Either $F^k(E) - \ell = C$ or $F^k(E) - \ell = D$ for some $k, \ell \in \mathbb{Z}$, $k = 0, 1, 2, \ldots, q-1$. Since G_0 is strictly increasing between C' and B we know that $F^k(E) - \ell = D$. Since B is the smallest element larger than D in the TPO $\{B_i\}$ generated by $B = B_0$, therefore if $k > 0$ there must be a point $E_1 \in [E, B)$ such that $F^k(E_1) - \ell = B_{pk-1} - \ell$. But then $F^q[E_1, B] - p \supset [B_{-1}, B]$, contradicting the fact that $(F^q - p)(x) > x$ on $[E, B)$. Thus $k = 0$ and $E = D$. That is, $F^q(x) - p > x$ on the whole interval $[D', B)$.

We now show that this is also the case if $G_0(C') = C'$ and $G_0'(C') \leq 1$. Then $G_0'(C') = 1$, for otherwise $\rho_+(F+\lambda) \leq p/q$ for small λ. Again, because $SF < 0$, it follows that $F^q(x) - p > x$ on $[D', C')$. In this case we put $B = C'$.

If $F \in \mathcal{S}$, then $F^q - p$ has a fixed point on (C, C'). If $F \in \mathcal{T}$, the fact that $G_0'(B) = 1$ implies that some point of the form $F_+^k(B) - \ell$ lies inside (D', C'). But $F_+^k(B) - \ell$ is fixed under G_0 while $G_0(x) > x$ on $[D', B)$. □

It follows from Lemma 3.1 and Corollary 2.9 that the orbit of C' is a TPO under F. We let $C_0' = C'$ and index the orbit $\{C_i'\}$ as usual. Note that $C_{-1}' < C < D < C_0'$, so that F is strictly increasing on each interval $[C_{i-1}', C_i']$, $i \not\equiv 0 \pmod q$.

Lemma 3.2. *Under the assumptions of Lemma* 3.1, $G_0(x) > x$ *for all* $x \in (C_{-1}', C')$.

Proof. Let $A = \sup \{x < C' : G_0(x) = x\}$. Then $A < C$ because G_0 is constant on $[C, C']$. Therefore, if $C_{-1}' < A$ then F^q is strictly increasing on $[C_{-1}', A]$. On the other hand, F^q then has slope less than 1 at a point between C_{-1} and A even though $(F^q)'(A) \geq 1$ and $(F^q)'(C_{-1}') > 1$. This is impossible since $SF < 0$. Thus $A = C_{-1}$. □

The first theorem in this section deals with parameter values λ at which C is periodic under $F + \lambda$ with rotation number $\phi_n(p/q)$. We begin with the following observation.

Lemma 3.3. *Suppose* $F \in \mathcal{C}$ *and suppose* $\rho_+(F) = p/q$ *with* $\rho_+(F+\lambda) > p/q$ *for any* $\lambda > 0$. *If* λ_n *is the smallest parameter value at which C is periodic of period* $nq + s$ *and rotation number* $\phi_n(p/q)$ *under* $F_+ + \lambda$, *then it is also the smallest value at which this happens under* $F + \lambda$.

Proof. If $(F+\lambda)^{nq+s}(C) - np - r = C$ then $(F_+ + \lambda)^{nq+s}(C) - np - r \geq C$. Thus λ_n is no greater than the first parameter value at which C is periodic under $F + \lambda$ with rotation number $\phi_n(p/q)$.

Suppose k is the smallest integer greater than 0 such that

$$(F_+ + \lambda_n)^k(C) = (F + \lambda_n)^k(C) \in [C, C'] + \mathbb{Z}.$$

Say $E=(F_+ +\lambda_n)^k(C)\in[C, C']+1$. If $E>C+1$, then $k<nq+s$ and

$$(F_+ +\lambda_n)^{nq+s}(C)=(F_+ +\lambda_n)^{nq+s}(E-1)=E-1+np+r>C+np+r,$$

which is false. Thus $E=C+1$ and $k=nq+s$. □

The following technical result provides the estimate required for the proof of Theorem 3.5 below.

PROPOSITION 3.4. *Let $H:[0,a]\times[0,\theta]\to\mathbb{R}$ be a C^2 map and let $H_\lambda=H(\cdot,\lambda)$. Suppose that $H_0'(x)>1$ for all $x\in[0,a]$, $(\partial H/\partial\lambda)(0,0)>0$, and $H(0,0)=0$. Let $c:[0,\theta]\to\mathbb{R}$ be a C^1 map with $c(0)\in(0,a)$. Then for n large enough and τ small enough there is a unique $\lambda_n\in(0,\tau)$ such that $H^n_{\lambda_n}(0)=c(\lambda_n)$. Moreover,*

$$\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}}=H_0'(0).$$

Proof. Let

$$u=\frac{\partial H}{\partial\lambda}(0,0)\Big/H_0'(0)-1$$

and suppose $0<\varepsilon<\frac{1}{2}u$. We will consider the curves $x=H_\lambda^{-n}(c(\lambda))$ and identify the parameters λ_n as their λ-intercepts. Define $\hat{H}:[0,a]\times[0,\theta]\to\mathbb{R}^2$ by $\hat{H}(x,\lambda)=(H_\lambda(x),\lambda)$. Note that

$$d\hat{H}(x,\lambda)=\begin{pmatrix} H_\lambda'(x) & \partial H/\partial\lambda(x,\lambda)\\ 0 & 1\end{pmatrix}.$$

In particular,

$$d\hat{H}(x,\lambda)(a_1,a_2)^{\mathrm{T}}=\left(a_1H_\lambda'+a_2\frac{\partial H}{\partial\lambda},a_2\right)^{\mathrm{T}}.$$

We may assume that $H_\lambda'(x)>b>1$ throughout $[0,a]\times[0,\theta]$. Let $L=d\hat{H}(0,0)$ and let w be the eigenvector of L with eigenvalue 1, given by $w=(-u,1)^{\mathrm{T}}$. Now let

$$\kappa=2\|w\|\frac{b}{b-1}$$

and choose δ so small that for all $x\in[0,\delta]$ we have

$$1-\varepsilon<\frac{H_0'(x)}{H_0'(0)}<1+\varepsilon, \tag{1}$$

and such that for all $(x,\lambda)\in[0,\delta]^2$ we have

$$\|d\hat{H}^{-1}-L^{-1}\|<\frac{\varepsilon}{\kappa}.$$

Now suppose $\hat{H}^{-k}(x,\lambda)\in[0,\delta]^2$ for $k=0,1,2,\ldots,n$. Then for any tangent vector v of the form $(v_1,1)^{\mathrm{T}}$ we have

$$\begin{aligned}\|d\hat{H}^{-1}v-w\|&=\|d\hat{H}^{-1}v-L^{-1}w\|\le\|d\hat{H}^{-1}(v-w)\|+\|d\hat{H}^{-1}w-L^{-1}w\|\\&\le\frac{1}{b}\|v-w\|+\frac{\varepsilon}{\kappa}\|w\|.\end{aligned}$$

Applying this n times, we get

$$\|d\hat{H}^{-n}v-w\|\le\frac{1}{b^n}\|v-w\|+\frac{\varepsilon}{\kappa}\|w\|\frac{b}{b-1}. \tag{2}$$

Now choose m so large that $H_0^{-m}(c(0))<\delta$, and choose $\tau\in(0,\delta)$ so small that $0<H_\lambda^{-m}(c(\lambda))<\delta$ for all $\lambda\in[0,\tau]$. Let $c_k(\lambda)=H_\lambda^{-k}(c(\lambda))$ (where defined), and choose K' so that $K'\geq|c'_m(\lambda)|$ for all $\lambda\in[0,\tau]$. Now choose n so large that

$$\frac{1}{b^n}K'+\frac{\varepsilon}{\kappa}\|w\|\frac{b}{b-1}<\varepsilon.$$

Suppose $v=(v_1,1)$ is a tangent at $(c_m(\lambda),\lambda)$ to the graph of $c_m(\lambda)$ and suppose $\hat{H}^{-k}(c_m(\lambda),\lambda)$ is defined for $k=0,1,2,\dots n$. Then by (2) we have $\|d\hat{H}^{-n}v-w\|<\varepsilon$. Consequently, the graph of $c_{m+n}(\lambda)$ on $[0,\tau]$ has negative slope for n large, and lies between the lines 1_1 and 1_2 through $(c_{m+n}(0),0)$ of slopes $dx/d\lambda=-u\pm\varepsilon$. Note that both these slopes are less than $-\frac{1}{2}u$. Let $k=m+n$. Since $c_k(0)=H_0^{-k}(c(0))\to 0$ as $k\to\infty$, therefore for k sufficiently large the graph of $c_k(\lambda)$ intercepts the λ-axis just once in $[0,\tau]$, at $\lambda=\lambda_k$. Furthermore, the λ-intercepts of 1_1 and 1_2 are given by $\lambda=c_k(0)/u\pm\varepsilon$. Thus

$$\frac{c_k(0)}{u+\varepsilon}\leq\lambda_k\leq\frac{c_k(0)}{u-\varepsilon}.$$

Therefore,

$$\frac{c_k(0)}{c_{k+1}(0)}\cdot\frac{u-\varepsilon}{u+\varepsilon}\leq\frac{\lambda_k}{\lambda_{k+1}}\leq\frac{c_k(0)}{c_{k+1}(0)}\cdot\frac{u+\varepsilon}{u-\varepsilon}.$$

Finally, from (1) we see that

$$(1-\varepsilon)\cdot H_0'(0)<\frac{c_k(0)}{c_{k+1}(0)}<(1+\varepsilon)\cdot H_0'(0),$$

whence,

$$(1-\varepsilon)\cdot\frac{u-\varepsilon}{u+\varepsilon}\cdot H_0'(0)\leq\frac{\lambda_k}{\lambda_{k+1}}\leq(1+\varepsilon)\cdot\frac{u+\varepsilon}{u-\varepsilon}\cdot H_0'(0)$$

for sufficiently large k. Since ε is arbitrary, this proves that

$$\lim_{k\to\infty}\frac{\lambda_k}{\lambda_{k+1}}=H_0'(0).$$ □

THEOREM 3.5. *Suppose that $F\in\mathcal{S}$ or $F\in\mathcal{T}$ and that $\rho_+(F)=p/q$ and $\rho_+(F+\lambda)>p/q$ for all $\lambda>0$. Suppose as well that s is the smallest positive integer such that $sp+1=rq$ for some $r\in\mathbb{Z}$. Then for sufficiently large n there is a unique $\lambda_n>0$ such that under $F+\lambda_n$ the critical point C is periodic with period $q+ns$ and rotation number $\phi_n(p/q)$ and $\rho_+(F+\lambda_n)=\phi_n(p/q)$. Furthermore,*

$$\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}}=\sigma,$$

where $\sigma=(F^q)'(C')$.

Proof. Note that $C_i'\notin[C,C']+\mathbb{Z}$ if $i\not\equiv 0\pmod q$. Furthermore, a simple calculation reveals that

$$\frac{d}{d\lambda}G_\lambda(C')|_{\lambda=0}=\sum_{k=0}^{q-1}(F^k)'(F^{q-k}(C'))>0.$$

It follows that for small positive a and θ, the function

$$H(x,\lambda)=G_\lambda(x+C')-C'$$

is C^2 on $[0,a]\times[0,\theta]$. Furthermore,

$$\frac{\partial H}{\partial\lambda}(0,0)=\frac{d}{d\lambda}G_\lambda(C')|_{\lambda=0}>0.$$

Also, $H(0,0)=G_0(C')-C'=0$, *and*

$$\frac{\partial H}{\partial x}(0,0)=G_0'(C')>1.$$

We may choose a so small that $H_0'(x)>1$ for all $x\in[0,a]$.

To apply Proposition 3.4 to the function H, we must now identify the curve $c(\lambda)$. If a is sufficiently small, then F^s-r is a C^1 diffeomorphism of $[C',C'+a]$ into $[C'_{-1},C)$ with $(F^s-r)(C')=C'_{-1}$, $(F^s-r)(x)=(F^s_+-r)(x)$, and $(F^s-r)'(x)>K_1>0$ for some K_1 and for all $x\in[C',C'+a]$. We let $a_1=(F^s-r)(C'+a)$. Furthermore, by Lemma 3.2, the sequence $\{G_0^k(a_1)\}$ is such that for some k,

$$a_1<G_0(a_1)<G_0^2(a_1)<\cdots<G_0^{k-1}(a_1)<C<G_0^k(a_1).$$

Therefore $G_0^k=F^{qk}-pk$ on $[C'_{-1},a_1]$ and $G_0^k(\gamma_1)=C$ for some $\gamma_1\in(C'_{-1},a_1)$, and $(F^{qk}-pk)'(x)>K_2>0$ for some K_2 and for all $x\in[0,a_1]$. Now define $\gamma\in(C',C'+a)$ by $(F^s-r)(\gamma)=\gamma_1$. Then $\{(F+\lambda)^l_+(\gamma)\}$, $l=0,1,2,\ldots,s+kq-1$, is disjoint from $[C,C']+\mathbb{Z}$ if $\lambda=0$. By continuity this will be true for all $\lambda\in[0,\theta]$ if θ is sufficiently small. In particular, $(F+\lambda)_+^{s+kq-1}(x)=(F+\lambda)^{s+kq-1}(x)$ for all x in a neighbourhood of γ. That is, the function

$$(x,\lambda)\mapsto G_\lambda^k\circ((F+\lambda)^s_+-r)(x)$$

is C^2 on a neighbourhood of $(\gamma,0)$, with

$$\{G_0^k\circ(F^s_+-r)\}'(\gamma)>K_1K_2>0.$$

It follows, therefore, by the Implicit Function Theorem that there is a C^2 curve $c(\lambda)$ with $c(0)\in(0,a)$ such that

$$(G_\lambda^k\circ((F+\lambda)^s_+-r))(C'+c(\lambda))=C,$$

for all $\lambda\in[0,\theta]$, if $\theta>0$ is sufficiently small. Furthermore, by continuity we may assume that

$$(G_\lambda^k\circ((F+\lambda)^s_+-r))'(C'+c(\lambda))>0$$

for all $\lambda\in[0,\theta]$ (two-sided derivative), from which it follows immediately that $C'+c(\lambda)$ is the unique preimage of C under $G_\lambda^k\circ((F+\lambda)^s_+-r)$ for all $\lambda\in[0,\theta]$.

Thus C is periodic under $(F+\lambda)_+$ with rotation number $\phi_n(p/q)$, $n>k$, if and only if

$$G_\lambda^n\circ((F+\lambda)^s_+-r)(C)=C,$$

which is true if and only if

$$G_\lambda^n\circ((F+\lambda)^s_+-r)(C')=C.$$

That is,

$$G_\lambda^{n-k}(C')=C'+c(\lambda),$$

or

$$H_\lambda^{n-k}(0) = c(\lambda).$$

Since λ_n is a solution of this equation, then Proposition 3.4 asserts that λ_n is unique for large n, and that

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = \sigma. \qquad \square$$

We will now study the largest parameter values at which $\rho_+(F+\lambda) = \phi_n(p/q)$. The main result is contained in Theorem 3.7 below. First we must extend the estimates of Proposition 3.4 a little.

PROPOSITION 3.6. *Let H and $c(\lambda)$ be as in Proposition 3.4, and suppose that $1 < b < H_0'(0)$ as in the proof of Proposition 3.4. Suppose moreover that $c^n(\lambda)$ is a sequence of C^1 curves $[0, \theta] \to \mathbb{R}$ which converges to $c(\lambda)$ from below, uniformly on $[0, \theta]$, with $\|c - c^n\| < A/\sqrt{b^n}$ for some constant A. Then for n large enough and τ small enough the equation $H^n_{\nu_{n+1}}(0) = c^n(\nu_{n+1})$ has a solution $\nu_{n+1} \in (0, \tau)$, and*

$$\lambda_n - \nu_{n+1} \le B/b^n\sqrt{b^n}$$

for some $B > 0$.

Proof. Letting $c_k^n(\lambda) = H_\lambda^{-k}(c^n(\lambda))$, where defined, we have

$$0 < c_k(\lambda) - c_k^n(\lambda) < \frac{1}{b^k} \cdot \frac{A}{\sqrt{b^n}}.$$

Clearly, $x = c_n^n(\lambda)$ will have a λ-intercept if $x = c_n(\lambda)$ does. Furthermore, it follows directly from the proof of Proposition 3.4 that if $c_n^n(\nu_{n+1}) = 0$, then

$$0 < c_n(\nu_{n+1}) < \frac{1}{b^n} \cdot \frac{A}{\sqrt{b^n}},$$

and $x = c_n(\lambda)$ lies below the line through

$$\left(\frac{1}{b^n} \cdot \frac{A}{\sqrt{b^n}}, \nu_{n+1}\right)$$

with slope $dx/d\lambda = -u + \varepsilon$ for $\lambda > \nu_{n+1}$. Therefore,

$$\lambda_n - \nu_{n+1} \le \frac{1}{(u-\varepsilon)} \cdot \frac{1}{b^n} \cdot \frac{A}{\sqrt{b^n}}. \qquad \square$$

THEOREM 3.7. *Suppose F is as in Theorem 3.5 with the added assumption that $F''(C) < 0$. Let $\lambda = \nu_n$ be the largest parameter value for which $\rho_+(F+\lambda) = \phi_n(p/q)$. Then*

$$\lim_{n\to\infty} \frac{\nu_{n+1} - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = 1,$$

and consequently

$$\lim_{n\to\infty} \frac{\nu_n}{\nu_{n+1}} = (F^q)'(C').$$

Proof. Note that $G_0''(C) = -\beta < 0$ because $F''(C) < 0$. In the proof of Theorem 3.5 we noted that

$$\frac{d}{d\lambda} G_\lambda(C')|_{\lambda=0} = \alpha > 0,$$

and by Lemma 3.1 we have that $G_0'(C') = \sigma > 1$. Suppose $\alpha_1 > \alpha$, $0 < \beta_1 < \beta$, and $0 < \sigma_1 < \sigma < \sigma_2$, with $(\sigma_2/\sigma_1) < \sqrt{\sigma_1}$. Then there are ε, $\theta > 0$ such that for $C - \varepsilon \leq x \leq C$ and $0 \leq \lambda \leq \theta$

$$\beta_1(C - x)^2 < G_\lambda(C) - G_\lambda(x),$$

and

$$G_\lambda(C) - C' < \alpha_1 \lambda.$$

Note that $(F_+ + \nu_n)^s \circ G_{\nu_n}^n(C') = C' + r$. Clearly $\lambda_n > \nu_{n+1} > 0$, so $G_{\nu_{n+1}}(C) > C'$ and $\lim_{n\to\infty} \nu_{n+1} = 0$. For large n, therefore, there is a unique $C_n'' < C$ such that $G_{\nu_{n+1}}(C_n'') = C'$. The above inequalities give

$$(C - C_n'')^2 < \frac{\alpha_1}{\beta_1} \cdot \nu_{n+1}.$$

To complete the proof we define $H(x, \lambda)$ and $c(\lambda)$ as in the proof of Theorem 3.5 and we let

$$c^n(\lambda) = \{(F + \lambda)^s \circ G_\lambda^k\}^{-1}(C_n'' + r) - C'.$$

To apply Proposition 3.6 we must show that $c^n(\lambda)$ is C^1 and that

$$\|c - c^n\| < A/\sqrt{\sigma_1^n}$$

for some A. We saw in the proof of Theorem 3.5 that for some $a > 0$ and for sufficiently small λ

$$((F + \lambda)^s - r) \circ G_\lambda^k$$

is a C^1 diffeomorphism of $[C', C' + a]$ onto an interval containing C in its interior, with

$$(((F + \lambda)^s - r) \circ G_\lambda^k)'(x) > K_1 K_2 > 0$$

throughout that interval. It follows immediately that $c^n(\lambda)$ is C^1 and that

$$\|c - c^n\| < \frac{1}{K_1 K_2} \cdot \sqrt{\frac{\alpha_1}{\beta_1} \nu_{n+1}} < \frac{1}{K_1 K_2} \cdot \sqrt{\frac{\alpha_1}{\beta_1} \lambda_n}.$$

Since $\lambda_n < A_1/\sigma_1^n$ for some $A_1 > 0$ by Theorem 3.5, we have

$$\|c - c^n\| < A/\sqrt{\sigma_1^n}.$$

Thus, by Proposition 3.6 $\lambda_n - \nu_{n+1} \leq B/\sigma_1^n \sqrt{\sigma_1^n}$. But then

$$\lim_{n\to\infty} \left(\frac{\nu_{n+1} - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}}\right) = \lim_{n\to\infty} \left(1 - \frac{\lambda_n - \nu_{n+1}}{\lambda_n - \lambda_{n+1}}\right) = 1,$$

since

$$\frac{\lambda_n - \nu_{n+1}}{\lambda_n - \lambda_{n+1}} < \frac{B/\sigma_1^n \sqrt{\sigma_1^n}}{B_1/\sigma_2^n}$$

and $(\sigma_1/\sigma_2) < \sqrt{\sigma_1}$. The result follows. □

The final theorem now follows as an easy corollary:

THEOREM 3.8. *Suppose F is as in Theorem* 3.7. *Let* $\lambda = \mu_n$ *be the smallest parameter value at which* $\rho_+(F+\lambda) = \phi_n(p/q)$. *Then*

$$\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = (F^q)'(C').$$

Proof. Clearly $\lambda_{n+1} < \nu_{n+1} < \mu_n < \lambda_n$. Therefore by Theorem 3.7

$$\lim_{n\to\infty} \frac{\mu_n - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} = 1.$$

Therefore

$$\lim_{n\to\infty} \frac{\mu_n}{\lambda_n} = \lim_{n\to\infty} \left\{ \left(\frac{\mu_n - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} \right) \left(1 - \frac{\lambda_{n+1}}{\lambda_n} \right) + \frac{\lambda_{n+1}}{\lambda_n} \right\} = 1.$$

Hence

$$\lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} = \lim_{n\to\infty} \left(\frac{\mu_n/\lambda_n}{\mu_{n+1}/\lambda_{n+1}} \cdot \frac{\lambda_n}{\lambda_{n+1}} \right) = \sigma. \qquad \square$$

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