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A census of quadratic post-critically finite rational functions defined over $\mathbb{Q}$

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Abstract

We find all quadratic post-critically finite (PCF) rational functions defined over $\mathbb{Q}$, up to conjugation by elements of $\mathrm{PGL}_2(\overline{\mathbb{Q}})$. We describe an algorithm to search for possibly PCF functions. Using the algorithm, we eliminate all but 12 rational functions, all of which are verified to be PCF. We also give a complete description of all possible rational preperiodic structures for quadratic PCF functions defined over $\mathbb{Q}$.

1. *Introduction*

Let $\phi(z) \in \mathbb{Q}(z)$ have degree $d \geqslant 2$. We may regard $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ as a morphism of the projective line. We consider iterates of ϕ:

$$\phi^n(z) = \underbrace{\phi \circ \phi \circ \ldots \circ \phi}_{n\,\text{times}}(z) \quad \text{and} \quad \phi^0(z) = z.$$

The orbit of a point $\alpha \in \mathbb{P}^1$ is the set $\mathcal{O}_\phi(\alpha) = \{\phi^n(\alpha) \mid n \geqslant 0\}$.

Rather than studying individual rational maps, we consider equivalence classes of maps under conjugation by $f \in \mathrm{PGL}_2(\overline{\mathbb{Q}})$; we define $\phi^f = f \circ \phi \circ f^{-1}$. Note that ϕ and ϕ^f have the same dynamical behavior. In particular, f maps the ϕ-orbit of α to the ϕ^f-orbit of $f(\alpha)$.

Critical points of ϕ are the points $\alpha \in \mathbb{P}^1$ such that $\phi'(\alpha) = 0$ as long as α and $\phi(\alpha)$ are finite. To compute the derivative at the excluded values of α, we use a conjugate map. See [**13**, § 1.2] for details.

Definition 1. A rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geqslant 2$ is *post-critically finite* (PCF) if the orbit of each critical point is finite.

A fundamental observation in the study of one-dimensional complex dynamics is that the orbits of the finite set of critical points of ϕ largely determine the dynamics of ϕ on all of $\mathbb{P}^1$. So the study of PCF maps has a long history in complex dynamics, including Thurston's topological characterization of these maps in the early 1980s and continuing to the present day.

In [**2**], for example, the authors find exactly one representative from each conjugacy class of nonpolynomial hyperbolic PCF rational maps of degree 2 and 3 in which the post-critical set, the forward orbit of the critical points, excluding the points themselves, contains no more than four points. In that paper, the authors have no concern for the field of definition of the map, while this is a paramount concern of the present article. In [**14**], Silverman advances the idea of PCF maps as a dynamical analog of abelian varieties with complex multiplication, suggesting that these maps may be of special interest in arithmetic dynamics as well.

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Our main result is inspired by these ideas. Compare this to the statement that, up to isomorphism over $\overline{\mathbb{Q}}$, there are exactly 13 elliptic curves $E/\mathbb{Q}$ with complex multiplication.

THEOREM 1. *Every PCF quadratic map defined over* $\mathbb{Q}$ *is conjugate* (*over* $\overline{\mathbb{Q}}$) *to precisely one of the following* 12 *maps:*

(1) z^2	(2) $\dfrac{1}{z^2}$	(3) z^2-2	(4) z^2-1
(5) $\dfrac{1}{2(z-1)^2}$	(6) $\dfrac{1}{(z-1)^2}$	(7) $\dfrac{-1}{4z^2-4z}$	(8) $\dfrac{-4}{9z^2-12z}$
(9) $\dfrac{2}{(z-1)^2}$	(10) $\dfrac{2z+1}{4z-2z^2}$	(11) $\dfrac{-2z}{2z^2-4z+1}$	(12) $\dfrac{3z^2-4z+1}{1-4z}$

Of these, the first four were well known to researchers in both complex and arithmetic dynamics. Maps (5)–(8) appeared in [**2**]. Maps (9)–(12) did not appear in [**2**] because either they fail to fit the criterion of hyperbolicity or the post-critical set is too large. One major contribution of the present work is the fact that this list is exhaustive. The fact that the list is finite is a consequence of [**1**], where a height bound for PCF maps provides one of the principal preliminary results needed for our analysis. However, translating this height bound into something amenable to reasonable computation is highly nontrivial.

Classifying rational functions by the structure of their rational preperiodic points is a fundamental problem in arithmetic dynamics. In [**12**], Poonen undertakes this task for quadratic polynomials defined over $\mathbb{Q}$, subject to the condition that no rational point is on a cycle of length greater than 3. In [**6**], the second author gives a classification for quadratic rational maps with nontrivial PGL_2 stabilizer, subject to a similar condition. Given the comprehensive list in Theorem 1, we are able to describe all possible rational preperiodic structures for quadratic PCF maps defined over $\mathbb{Q}$ with no additional hypotheses. Difficulty arises only for the first two maps, which have nontrivial twists. We are able to conclude the following.

THEOREM 2. *A quadratic PCF map defined over* $\mathbb{Q}$ *has at most six rational preperiodic points.*

Given the parallels between the set of rational preperiodic points for a rational map and the torsion subgroup of an abelian variety $A(\mathbb{Q})$ (see [**14**, p. 111], for example), this result and the preperiodic structures given in §§ 4 and 5 are analogs of the comprehensive list of torsion subgroups for CM elliptic curves $E/\mathbb{Q}$ in [**11**].

2. *Background*

Since ϕ is quadratic, it has exactly three fixed points in $\mathbb{P}^1$, counting multiplicity. The finite fixed points of ϕ are roots of the polynomial found by setting $\phi(z) = z$. The *multiplier* at a finite fixed point α is $\phi'(\alpha)$. A straightforward computation using the chain rule shows that fixed point multipliers are preserved under conjugation. Hence when ϕ fixes the point at infinity, we may conjugate by any $f \in \mathrm{PGL}_2$ such that $f(\infty)$ and $f(\phi(\infty))$ are both finite. We then take the multiplier for the fixed point at infinity to be $(\phi^f)'(f(\infty))$.

The following result gives a normal form for quadratic rational maps with trivial PGL_2 stabilizer that respects the field of definition. Combined with the height bound in Proposition 1, this allows us to create exactly one map in each equivalence class and test if it is PCF. Rational functions with nontrivial stabilizer are addressed in § 5.

THEOREM 3 [**8**, Lemma 3.1]. *Let* K *be a field with characteristic different from* 2 *and* 3. *Let* $\psi(z) \in K(z)$ *have degree* 2, *and let* $\lambda_1, \lambda_2, \lambda_3 \in \overline{K}$ *be the multipliers of the fixed points of* ψ

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(counted with multiplicity). Then $\psi(z)$ is conjugate over K to the map

$$\phi(z) = \frac{2z^2 + (2-\sigma_1)z + (2-\sigma_1)}{-z^2 + (2+\sigma_1)z + 2 - \sigma_1 - \sigma_2} \in K(z), \tag{2.1}$$

where σ_1 and σ_2 are the first two symmetric functions of the multipliers. Furthermore, no two distinct maps of this form are conjugate to each other over $\overline{K}$.

The following result gives a crucial bound on $H(\lambda)$, the standard multiplicative height of a fixed point multiplier for a quadratic PCF map. (See [**13**, § 3.1] for background on heights.) In Proposition 1, we use this result to derive height bounds for σ_1 and σ_2.

LEMMA 1 [**1**, Corollary 1.3]. *Let $\phi(z) \in \overline{\mathbb{Q}}(z)$ have degree 2, suppose that ϕ is PCF, and let λ be the multiplier of any fixed point of ϕ. Then $H(\lambda) \leqslant 4$.*

PROPOSITION 1. *Let $\phi(z) \in \overline{\mathbb{Q}}$ be a PCF map of degree 2, and suppose that σ_1 and σ_2 are the first and second symmetric functions on the multipliers of the fixed points. Then $H(\sigma_1) \leqslant 192$ and $H(\sigma_2) \leqslant 12\,288$.*

Proof. We simplify notation by setting $d = [K : \mathbb{Q}]$ for K any field of definition of the fixed point multipliers. By the triangle inequality:

$$|\sigma_1|_v = |\lambda_1 + \lambda_2 + \lambda_3|_v \leqslant \begin{cases} \max\{|\lambda_1|_v, |\lambda_2|_v, |\lambda_3|_v\} & \text{for each finite place} \\ 3\max\{|\lambda_1|_v, |\lambda_2|_v, |\lambda_3|_v\} & \text{for each infinite place.} \end{cases}$$

For an extension of degree d, there are at most d infinite places, so

$$\begin{aligned} H(\sigma_1) &= \prod_{v \in M_K} (\max\{|\sigma_1|_v, 1\}^{n_v})^{1/d} \leqslant 3 \prod_{v \in M_K} \left(\max_{1 \leqslant i \leqslant 3} \{|\lambda_i|_v, 1\}^{n_v} \right)^{1/d} \\ &\leqslant 3 \prod_{v \in M_K} (\max\{|\lambda_1|_v, 1\}^{n_v} \cdot \max\{|\lambda_2|_v, 1\}^{n_v} \cdot \max\{|\lambda_3|_v, 1\}^{n_v})^{1/d} \\ &= 3H(\lambda_1)H(\lambda_2)H(\lambda_3) \leqslant 3 \cdot 4^3 = 192. \end{aligned}$$

The proof for the bound on σ_2 follows similarly:

$$\begin{aligned} H(\sigma_2) &= H(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \leqslant 3 \prod_{v \in M_K} \left(\max_{\substack{i \neq j \\ 1 \leqslant i,j \leqslant 3}} \{|\lambda_i\lambda_j|_v, 1\}^{n_v} \right)^{1/d} \\ &\leqslant 3 \prod_{v \in M_K} (\max\{|\lambda_1\lambda_2|_v, 1\}^{n_v} \cdot \max\{|\lambda_2\lambda_3|_v, 1\}^{n_v} \cdot \max\{|\lambda_1\lambda_3|_v, 1\}^{n_v})^{1/d} \\ &= 3H(\lambda_1\lambda_2)H(\lambda_2\lambda_3)H(\lambda_1\lambda_3) \leqslant 3 \cdot 4^6 = 12\,288. \end{aligned}$$ □

This height bound, together with the normal form given in Theorem 3, reduces the proof of Theorem 1 to testing whether each of a finite set of rational maps is PCF. To accomplish this, we rely on results describing the way the periods of periodic points can change under reduction modulo certain primes.

We fix the following notation: K is a local field with nonarchimedean absolute value $|\cdot|_v$, R is the ring of integers of K, $\mathfrak{p}$ is the maximal ideal of R, $k = R/\mathfrak{p}$ is the residue field, and $\tilde{\ }$ represents reduction modulo $\mathfrak{p}$. A morphism ϕ has good reduction at $\mathfrak{p}$ if $\deg(\phi) = \deg(\tilde{\phi})$.

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THEOREM 4 [**13**, Theorem 2.21]. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function of degree $d \geqslant 2$ defined over K. Assume that ϕ has good reduction, let $P \in \mathbb{P}^1(K)$ be a periodic point of ϕ, and define the following quantities:*

- n *The exact period of P for the map ϕ.*
- m *The exact period of $\widetilde{P}$ for the map $\widetilde{\phi}$.*
- r *The order of $\lambda_{\widetilde{\phi}}(\widetilde{P}) = (\widetilde{\phi}^m)'(\widetilde{P})$ in $k^\times$. (Set $r = \infty$ if $\lambda_{\widetilde{\phi}}(\widetilde{P})$ is not a root of unity.)*
- p *The characteristic of the residue field k.*

Then n has one of the following forms:

$$n = m \quad \text{or} \quad n = mr \quad \text{or} \quad n = mrp^e.$$

The following refinement of Theorem 4 for quadratic maps drives our algorithm.

PROPOSITION 2. *Let $p > 3$ be prime and $\phi(z) \in \mathbb{Q}_p(z)$ be a quadratic function with good reduction. Continue with the other notation of Theorem 4. Then we have the following:*

(i) *the function $\widetilde{\phi}$ has two distinct critical points $\gamma_1, \gamma_2 \in \overline{\mathbb{F}}_p$;*
(ii) *the critical points of ϕ are $\mathbb{Q}_p$-rational if and only if $\gamma_1, \gamma_2 \in \mathbb{F}_p$;*
(iii) *if the critical points of ϕ are $\mathbb{Q}_p$ rational*

$$n = m \quad \text{or} \quad n = mr.$$

Proof. Since ϕ has good reduction at p, $\widetilde{\phi}$ has degree 2. So by the Riemann–Hurwitz Theorem, it must have two critical points counting multiplicity. Corollary 1.3 in [**3**] says that $\widetilde{\phi}$ can have a unique critical point in a field of characteristic p if and only if $\deg \widetilde{\phi} \equiv 0$ or 1 (mod p). Statement (i) follows because $\widetilde{\phi}$ is quadratic and p is odd.

Statement (ii) then follows from (i) (since there is no ramification) and Hensel's lemma.

For (iii), we use [**13**, Theorem 2.28] which bounds the exponent e in Theorem 4 when K has characteristic 0. If $v : K^* \to \mathbb{Z}$ is the normalized valuation on K, we have

$$p^{e-1} \leqslant \frac{2v(p)}{p-1}.$$

Since $K = \mathbb{Q}_p$, $v(p) = 1$. Then $p > 3$ gives $e = 0$, leaving only the first two possible periods. □

To apply Proposition 2 in our algorithm, we consider $\phi(z) = f(z)/g(z) \in \mathbb{Q}(z)$ to be defined over $\mathbb{Q}_p$ for primes $p > 3$ of good reduction. We ensure that the critical points of $\widetilde{\phi}$ are in $\mathbb{F}_p$, and hence that the critical points of ϕ are in $\mathbb{Q}_p$, by eliminating primes p where the Wronskian $f'g - g'f$ is an irreducible quadratic over $\mathbb{F}_p$.

If ϕ is PCF, then some iterate $\phi^j(\gamma_i)$ has exact period n. Since the $\mathbb{F}_p$-orbit $\mathcal{O}_{\tilde{\phi}}(\tilde{\gamma}_i)$ is necessarily finite, some iterate $\tilde{\phi}^k(\tilde{\gamma}_i)$ has exact period m. Proposition 2 gives a set of possible n values based on the more easily computed m. We intersect these possible values for various primes, and discard the map as not PCF if that intersection becomes empty.

3. *Algorithm*

We first describe the overall flow of the algorithm for determining if a map with trivial stabilizer is potentially PCF. We then provide more detailed pseudocode and explanation for each piece of the algorithm. Our actual code is provided in the arXiv distribution of this paper.

(1) Build a database containing, for each p in a list of primes, all quadratic rational maps mod p in the form given in (2.1), along with their critical points and corresponding possible global periods as given by Theorem 4 (see Algorithm 1).

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(2) For σ_1 and σ_2 values within the height bounds in Proposition 1, create $\phi(z) \in \mathbb{Q}(z)$ as in (2.1), reduce the map modulo primes p, and iteratively intersect the possible global periods (found in the database) for the critical points at each prime. If the intersection becomes empty, the map is not PCF (see Algorithm 2 and Subroutines 2A and 2B).

Using this algorithm, we identified ten potentially PCF quadratic rational functions. For each of these functions, we calculated the forward orbit of each critical point and verified that each critical orbit was, indeed, eventually periodic. In other words, all ten 'potentially PCF' functions were verified to be PCF. These maps and their critical orbits are given in § 4.

Algorithm 1 builds the databases; our implementation used the first 99 primes greater than 3. We construct a rational function $\psi = F/G \in \mathbb{F}_p(z)$ based on the normal form in Theorem 3. The critical points of ψ are the roots of the Wronskian $F'G - G'F$; when the Wronskian is linear, we also include the point at infinity. If the critical points are not defined over $\mathbb{F}_p$, that is, if the Wronskian is an irreducible quadratic over $\mathbb{F}_p$, the map is not included in the database.

Algorithm 1 — Build Database

Input: `pList`, a list of primes > 3

Output: a database of quadratic rational maps over $\mathbb{F}_p$ together with critical point data for each prime p in `pList`

for p in `pList`:
- for each pair $(b, c) \in \mathbb{F}_p^2$:
  - create the morphism $\psi := (2z^2 + bz + b)/(-z^2 + (4-b)z + cz^2)$
  - set $\mathtt{crit}_1, \mathtt{crit}_2 :=$ the critical points of ψ
  - if $\deg(\psi) = 2$ and $\mathtt{crit}_1, \mathtt{crit}_2$ are defined over $\mathbb{F}_p$:
    - add an entry for ψ to the database
    - for $i = 1, 2$:
      - find m_i, the length of the cycle into which $\mathtt{crit_i}$'s orbit eventually falls
      - find λ_i, the multiplier of that cycle
      - if $\lambda_i = 0$:
        - append the pair $(\mathtt{crit_i}, \{m_i\})$ to the database entry for ψ
      - else:
        - find $r_i :=$ the multiplicative order of λ_i in $\mathbb{F}_p^\times$
        - append the pair $(\mathtt{crit_i}, \{m_i, m_i r_i\})$ to the database entry for ψ

Algorithm 2 filters out functions which are certainly not PCF, but does not guarantee that the functions which remain are PCF. The algorithm uses the resultant of ϕ, meaning the resultant of the relatively prime polynomials f and g such that $\phi = f/g$. The resultant of a map given in the form (2.1) is nonzero if and only if the map has trivial stabilizer [**8**, Remark 3.2], and primes dividing the resultant are precisely the primes of bad reduction for ϕ [**13**, §§ 2.4 and 2.5]. Our implementation used the height bounds from Proposition 1, namely $H_1 = 192$ and $H_2 = 12\,288$.

Subroutine 2A is called when ϕ has rational critical points γ_1 and γ_2, meaning we can easily reduce them modulo primes p. (If p divides the denominator, the point reduces to the point at infinity on $\mathbb{P}^1_{\mathbb{F}_p}$.) We must keep track of the possible periods for each critical point independently, because it is possible that they terminate in cycles of different lengths.

Subroutine 2B is called when ϕ has irrational critical points. The equation defining the critical points is a quadratic polynomial in $\mathbb{Q}[z]$, so in this case γ_1 and γ_2 must be Galois

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Algorithm 2 — Find PCF maps up to height bound

Input: `pList`, a list of primes included in the database; height bounds H_1 and H_2

Output: a set of possibly PCF maps with $\sigma_1 \leqslant H_1$ and $\sigma_2 \leqslant H_2$

for $\sigma_1 \in \mathbb{Q}$ of height $\leqslant H_1$ and $\sigma_2 \in \mathbb{Q}$ of height $\leqslant H_2$:
  create the rational map $\phi(z) := \frac{2z^2+(2-\sigma_1)z+(2-\sigma_1)}{-z^2+(2+\sigma_1)z+(2-\sigma_1-\sigma_2)}$
  normalize the coordinates of ϕ (clearing denominators so all coefficients are in $\mathbb{Z}$)
  calculate `res` := resultant of ϕ

  if `res` $\neq 0$:
    calculate γ_1, γ_2 := critical points of ϕ

    if $\gamma_1, \gamma_2 \in \mathbb{Q}$:
      if `Check_Rational_Periods`(ϕ, γ_1, γ_2, `pList`, `res`):
        add ϕ to set of possibly PCF maps
    else:
      if `Check_Irrational_Periods`(ϕ, `pList`, `res`):
        add ϕ to set of possibly PCF maps

Subroutine 2A — **Check_Rational_Periods** filters out maps ϕ which are not PCF

Input: A quadratic rational map ϕ with integer coefficients, resultant `res`, and rational critical points γ_1, γ_2. A list of primes `pList` for which the database has been built

Output: `False` if ϕ is definitely not PCF and `True` otherwise

initialize empty lists $\texttt{PossPer}_1$ and $\texttt{PossPer}_2$

for p in `pList`:
  if $p \nmid$ `res` (p is a prime of good reduction):
    set $\psi := \phi \pmod p$, $\texttt{crit}_1 := \gamma_1 \pmod p$, $\texttt{crit}_2 := \gamma_2 \pmod p$
    look up ψ in the database

    for $i = 1, 2$:
      retrieve the set of possible global periods for $\texttt{crit}_\texttt{i}$
      if $\texttt{PossPer}_i$ is empty (this is the first good prime):
        set $\texttt{PossPer}_i = \{$possible global periods for $\texttt{crit}_\texttt{i}\}$
      else:
        set $\texttt{PossPer}_i = \texttt{PossPer}_i \cap \{$possible global periods for $\texttt{crit}_\texttt{i}\}$

      if $\texttt{PossPer}_i$ is empty:
        return `False`

return `True`

conjugates. Since $\phi(z) \in \mathbb{Q}(z)$, the same is true of $\phi^i(\gamma_1)$ and $\phi^i(\gamma_2)$ for every $i \geqslant 0$. The subroutine takes advantage of this symmetry. The orbits of γ_1 and γ_2 are either both finite or both infinite. If the orbits are finite, they will terminate in cycles of the same length, and that length must be in the intersection of the possible periods for each critical point at every good prime.

Algorithm 1 was implemented in Sage [**15**], using built-in functionality for morphisms on projective spaces over finite fields. The database used throughout was GNU dbm [**10**]. Algorithm 2 and Subroutines 2A and 2B were prototyped in Sage and eventually implemented

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Subroutine 2B — Check_Irrational_Periods filters out maps ϕ which are not PCF

Input: A quadratic rational map ϕ which has irrational critical points and integer coefficients; `res`, the resultant of ϕ; and a list of primes `pList` for which the database has been built

Output: `False` if ϕ is definitely not PCF and `True` otherwise

initialize empty list `PossPer`

for p in `pList`:
  if $p \nmid$ `res` (p is a prime of good reduction):
    set $\psi := \phi \pmod p$
    look up ψ in the database
    let $\mathtt{Poss}_1$ and $\mathtt{Poss}_2$ be the sets of possible global periods for the critical points of ψ

    if `PossPer` is empty (this is the first good prime):
      set $\mathtt{PossPer} = \mathtt{Poss}_1 \cap \mathtt{Poss}_2$
    else:
      set $\mathtt{PossPer} = \mathtt{PossPer} \cap \mathtt{Poss}_1 \cap \mathtt{Poss}_2$

    if `PossPer` is empty:
      return `False`

return `True`

in C to improve speed of computation. They use the GNU Multiple Precision Arithmetic Library [4]. The program was run on two 6-core Intel® Xeon® CPUs at 2.80 GHz, with 12 GB of RAM and running Linux (CentOS 5.10).

4. *PCF maps with trivial* PGL_2 *stabilizer*

Table 1 lists the output of our algorithm: all quadratic PCF maps defined over $\mathbb{Q}$ with trivial PGL_2 stabilizer. In the critical portraits, an arrow from P to Q indicates that $\phi(P) = Q$; an integer over the arrow indicates the ramification index of the map at that point. In particular, the critical points are the initial points for arrows where the integer is 2. The portraits demonstrate that for each map found by our algorithm both critical points have finite forward orbit, so the map is definitely PCF. The final column gives a conjugate map in simpler form, reflecting the statement of Theorem 1.

Remark. This list of PCF maps raises some questions.

(i) All maps except the sixth one and the last one satisfy $\sigma_1 \in \{\pm 2, -6\}$. (This is also true of the maps with nontrivial stabilizer described in § 5.) The line $\sigma_1 = 2$ in the moduli space of quadratic rational maps corresponds to the quadratic polynomials. What (if anything) is special about these other two lines?

(ii) Similarly, all maps except the sixth one and the last one correspond to integer values of (σ_1, σ_2). What is special about these the two anomalous maps?

(iii) For the two anomalous maps we have $(\sigma_1, \sigma_2) = (-\frac{2}{3}, \frac{4}{3})$ and $(\sigma_1, \sigma_2) = (-\frac{10}{3}, \frac{20}{3})$. In other words, the symmetric functions of the multipliers have denominator at most 3 for all quadratic PCF maps defined over $\mathbb{Q}$. Is there some general phenomenon here that extends to maps defined over number fields?

From [13, Proposition 4.73], functions with trivial PGL_2 stabilizer have no nontrivial twists. That is, any quadratic PCF map defined over $\mathbb{Q}$ with trivial stabilizer must be conjugate to

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one of the ten maps in Table 2, and the conjugacy must also be defined over $\mathbb{Q}$. Hence the rational preperiodic structures for these maps are invariant within the conjugacy class. The possible structures, computed with Sage [15], appear in Table 2.

TABLE 1. *All quadratic PCF maps defined over* $\mathbb{Q}$ *with trivial* PGL_2 *stabilizer.*

$\phi(z)$	Critical portrait	Conjugate map
$\dfrac{2z^2}{-z^2+4z+8}$	$0 \overset{2}{\circlearrowleft}$ $-4 \xrightarrow{2} -4/3 \xrightarrow{1} 4 \overset{1}{\circlearrowleft}$	z^2-2
$\dfrac{2z^2}{-z^2+4z+4}$	$0 \overset{2}{\circlearrowleft}$ $-2 \underset{2}{\overset{1}{\leftrightarrows}} -1$	z^2-1
$\dfrac{2z^2+8z+8}{-z^2-4z+4}$	$\infty \xrightarrow{2} -2 \xrightarrow{2} 0 \xrightarrow{1} 2 \underset{1}{\overset{1}{\leftrightarrows}} -4$	$\dfrac{1}{2(z-1)^2}$
$\dfrac{2z^2+8z+8}{-z^2-4z}$	$\infty \xrightarrow{2} -2 \xrightarrow{2} 0 \xrightarrow{1} \infty$	$\dfrac{1}{(z-1)^2}$
$\dfrac{2z^2+4z+4}{-z^2}$	$0 \xrightarrow{2} \infty \xrightarrow{1} -2 \underset{2}{\overset{1}{\leftrightarrows}} -1$	$\dfrac{-1}{4z^2-4z}$
$\dfrac{6z^2+8z+8}{-3z^2+4z+4}$	$0 \xrightarrow{2} 2 \xrightarrow{1} \infty \xrightarrow{1} -2 \underset{2}{\overset{1}{\leftrightarrows}} -1$	$\dfrac{-4}{9z^2-12z}$
$\dfrac{2z^2+8z+8}{-z^2-4z-2}$	$\infty \xrightarrow{2} -2 \xrightarrow{2} 0 \xrightarrow{1} -4 \overset{1}{\circlearrowleft}$	$\dfrac{2}{(z-1)^2}$
$\dfrac{2z^2+4z+4}{-z^2+4}$	$-3-\sqrt{5} \xrightarrow{2} -\frac{1}{2}(1+\sqrt{5}) \xrightarrow{1} 2$ $-3+\sqrt{5} \xrightarrow{2} \frac{1}{2}(-1+\sqrt{5}) \xrightarrow{1} 2 \xrightarrow{1} \infty \underset{1}{\overset{1}{\leftrightarrows}} -2$	$\dfrac{2z+1}{4z-2z^2}$
$\dfrac{2z^2+4z+4}{-z^2+2}$	$-2-\sqrt{2} \xrightarrow{2} -\sqrt{2} \xrightarrow{1} \infty$ $-2+\sqrt{2} \xrightarrow{2} \sqrt{2} \xrightarrow{1} \infty \xrightarrow{1} -2 \overset{1}{\circlearrowleft}$	$\dfrac{-2z}{2z^2-4z+1}$
$\dfrac{6z^2+16z+16}{-3z^2-4z-4}$	$0 \xrightarrow{2} -4 \xrightarrow{1} -4/3 \overset{1}{\circlearrowleft}$ $-2 \underset{2}{\overset{1}{\leftrightarrows}} -1$	$\dfrac{3z^2-4z+1}{1-4z}$

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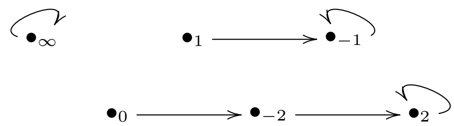
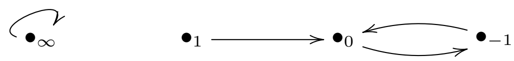
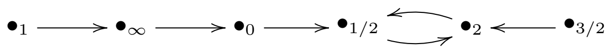
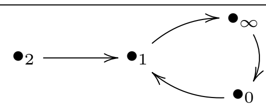
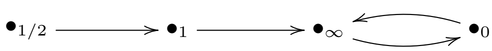
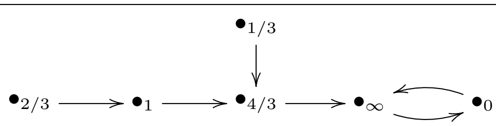
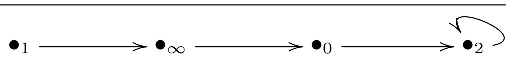
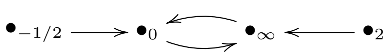
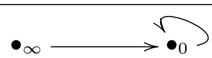
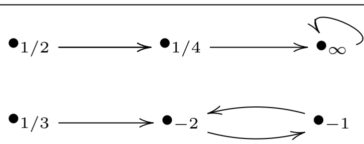
5. *PCF maps with nontrivial* PGL_2 *stabilizer*

Quadratic rational maps with nontrivial PGL_2 stabilizer have been extensively studied. In [9], Milnor described the symmetry locus for quadratic rational maps; the second author investigated the arithmetic of these maps in [6, 7]. Jones and the second author found a height bound on PCF maps with nontrivial stabilizer and used that bound to show that over $\mathbb{Q}$, the only maps meeting these criteria must be conjugate to either $\psi_1(z) = z^2$ or $\psi_2(z) = 1/z^2$ [5, Proposition 5.1].

Unlike the six maps described in § 4, these two maps have nontrivial twists. That is, there are infinitely many $\mathrm{PGL}_2(\mathbb{Q})$-conjugacy classes within each of these two $\mathrm{PGL}_2(\overline{\mathbb{Q}})$-conjugacy classes of maps. The different $\mathbb{Q}$-conjugacy classes may have very different structures for their rational preperiodic points. In this section, we find all of the possible rational preperiodic structures for these two $\overline{\mathbb{Q}}$ conjugacy classes. This determination of preperiodic structures completes the proof of Theorem 2 from the introduction.

Throughout this section, ζ_n represents a primitive nth root of unity.

TABLE 2. *Preperiodic structures for quadratic maps with trivial stabilizer.*

$\phi(z)$	Rational preperiodic points graph
z^2-2	
z^2-1	
$\dfrac{1}{2(z-1)^2}$	
$\dfrac{1}{(z-1)^2}$	
$\dfrac{-1}{4z^2-4z}$	
$\dfrac{-4}{9z^2-12z}$	
$\dfrac{2}{(z-1)^2}$	
$\dfrac{2z+1}{4z-2z^2}$	
$\dfrac{-2z}{2z^2-4z+1}$	
$\dfrac{3z^2-4z+1}{1-4z}$	

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DEFINITION 2. If a point $\alpha \in \mathbb{P}^1$ enters a cycle of least period m after n iterations (that is, if $\phi^n(\alpha)$ has period m with n and m minimal), then α is called a periodic point of type m_n.

5.1. *Maps conjugate to $\psi_1(z) = z^2$*

Twists of ψ_1 are described completely in [6]. They are given by

$$\phi_b(z) = \frac{z}{2} + \frac{b}{z},$$

where $b \neq 0$ is defined up to squares in $\mathbb{Q}$. Applying propositions from [6], we easily conclude:

(1) The map ϕ_b always has a rational fixed point at infinity and a rational point of type 1_1 at 0 [6, Propositions 1 and 5].

(2) The map ϕ_b has finite rational fixed points if and only if $b = c^2/2$ for $c \in \mathbb{Q}^\times$ [6, Proposition 1], and all such maps are conjugate over $\mathbb{Q}$. Taking $b = \frac{1}{2}$ yields

$$\phi_{1/2}(z) = \frac{z^2+1}{2z}.$$

In this case, there are no additional points of type 1_1 [6, Proposition 5].

(3) The map ϕ_b has rational points of primitive period 2 if and only if $b = -3c^2/2$ for $c \in \mathbb{Q}^\times$ [6, Proposition 2], and all such maps are conjugate over $\mathbb{Q}$. So we take $b = -\frac{3}{2}$ to get

$$\phi_{-3/2}(z) = \frac{z^2-3}{2z}.$$

In this case, we have two rational points of type 2_1 [6, Proposition 5] but no rational points of type 2_n for $n > 1$ [6, Proposition 8] and no finite rational fixed points [6, Proposition 9].

(4) The map ϕ_b cannot have rational points of primitive period 3 or 4 [6, Theorems 3 and 4]. This will also follow Theorem 5 below.

(5) The map ϕ_b has rational points of type 1_2 if and only if $b = -c^2/2$ for $c \in \mathbb{Q}^\times$, and all such maps are conjugate over $\mathbb{Q}$. So we take $b = -\frac{1}{2}$ to get the map

$$\phi_{-1/2}(z) = \frac{z^2-1}{2z}.$$

In this case, there are no finite rational fixed points [6, Proposition 6] and no rational points of period 2 [6, Proposition 9].

(6) The map ϕ_b cannot have rational points of type 1_n for $n \geqslant 3$ [6, Propositions 7 and 8].

The description above yields four possible rational preperiodic structures, shown in Table 3. In order to claim we have a complete description of the possible rational preperiodic structures, we need the following result.

THEOREM 5. *Let*

$$\phi_b(z) = \frac{z}{2} + \frac{b}{z}.$$

Then ϕ has no rational point of least period $n > 2$.

Proof. Consider a point $\alpha \in \mathbb{Q}$ so that α is periodic for $\phi_b(z)$. Let

$$f(z) = \frac{z}{\sqrt{2b}}, \quad \text{so } \phi_b^f(z) = \phi_{1/2}(z) = \frac{z^2+1}{2z}.$$

Then we have that $f(\alpha) = \alpha/\sqrt{2b}$ is periodic for $\phi_{1/2}(z)$.

Now let $g = (z-1)/(z+1)$. It is a simple matter to check that $\psi_1(z) := \phi_{1/2}^g = z^2$, so that $g(f(\alpha)) \in \mathbb{Q}[\sqrt{2b}]$ is periodic for $\psi_1(z)$.

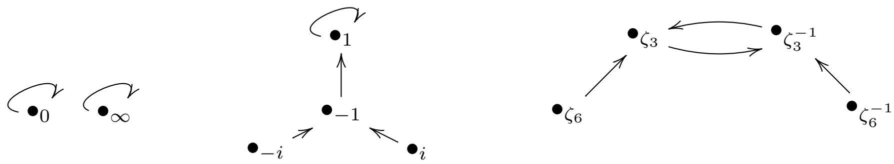


FIGURE 1. *All possible quadratic periodic points for* $\psi(z) = z^2$.

We will now categorize periodic points for $\psi_1(z) = z^2$ that lie in quadratic fields, showing that none of them have period of length more than 2. The result will follow.

The map ψ_1 has a totally ramified fixed point at ∞. Any finite periodic point of $\psi_1(z) = z^2$ is a root of $z^{2^n} - z$, so it is either 0 or a root of $z^{2^n-1} - 1$, that is, a root of unity. Since we seek periodic points that lie in quadratic fields, we can restrict our search to roots of unity that lie in quadratic fields, namely $\{\pm 1, \pm i, \zeta_3, \zeta_3^{-1}, \zeta_6, \zeta_6^{-1}\}$.

A computation verifies that the preperiodic structures for ψ_1 containing these points are the ones shown in Figure 1. So the only quadratic periodic points have period 1 or 2 as desired. $\square$

5.2. *Maps conjugate to* $\psi_2(z) = 1/z^2$

From [8], all such maps are conjugate over $\mathbb{Q}$ to a map of the form

$$\theta_{d,k}(z) = \frac{kz^2 - 2dz + dk}{z^2 - 2kz + d}, \quad \text{with } k \in \mathbb{Q}, d \in \mathbb{Q}^\times, \text{ and } k^2 \neq d. \tag{5.1}$$

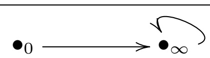
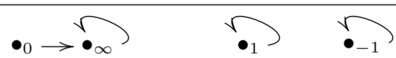
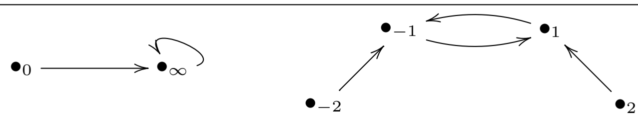
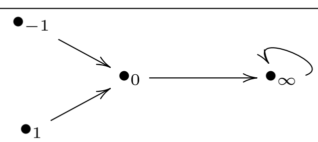
Conjugating this map by

$$f(z) = \frac{z - \sqrt{d}}{z + \sqrt{d}} \quad \text{yields } \theta^f_{d,k}(z) = \frac{t}{z^2} \text{ where } t = \frac{k - \sqrt{d}}{k + \sqrt{d}}.$$

Conjugating this by $g(z) = t^{-1/3} z$ gives

$$(\theta^f_{d,k})^g(z) = \frac{1}{z^2}.$$

TABLE 3. *Preperiodic structures for twists of* $\psi_1(z) = z^2$.

$\phi_b(z) = \dfrac{z}{2} + \dfrac{b}{z}$	Rational preperiodic points graph
$\phi_1(z) = \dfrac{z}{2} + \dfrac{1}{z}$	
$\phi_{1/2}(z) = \dfrac{z}{2} + \dfrac{1}{2z}$	
$\phi_{-3/2}(z) = \dfrac{z}{2} - \dfrac{3}{2z}$	
$\phi_{-1/2}(z) = \dfrac{z}{2} - \dfrac{1}{2z}$	

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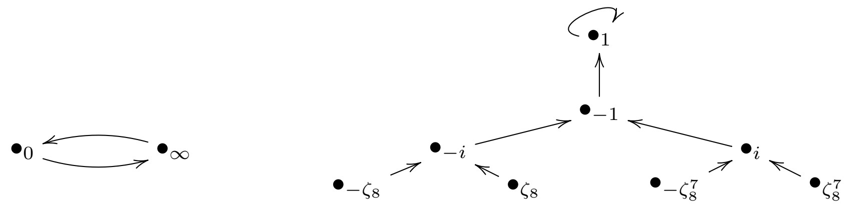

FIGURE 2. $\psi_2(z) = 1/z^2$: *2-cycle and one fixed point.*

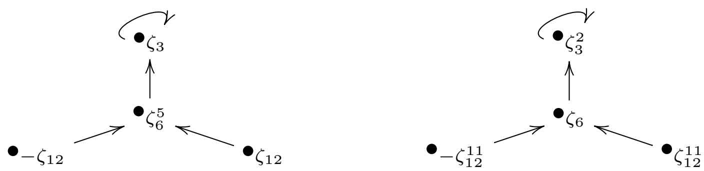

FIGURE 3. $\psi_2(z) = 1/z^2$: *two additional fixed points.*

If $\alpha \in \mathbb{Q}$ is preperiodic for $\theta_{d,k}$, then $\beta = g^{-1}f^{-1}(\alpha) \in \mathbb{Q}(t^{1/3})$ is a preperiodic point for $\psi_2(z)$. Since $[\mathbb{Q}(\beta) : \mathbb{Q}] \leqslant 6$, we may find all rational preperiodic structures for this family of maps by describing preperiodic points for ψ_2 of degree at most 6. Conjugating these points to lie in the rationals, we will find a map in the family with specified rational preperiodic points or show that none exists.

LEMMA 2. *All preperiodic points for $\psi_2(z) = 1/z^2$ of degree at most 6 are given in Figures 2–5.*

Proof. For n even, $\psi_2^n(z) = z^{2^n}$, so the points with period dividing n are 0, ∞, and $(2^n - 1)$th roots of unity. For n odd, $\psi_2^n(z) = z^{-2^n}$, so the points with period dividing n are $(2^n + 1)$th roots of unity. Hence all strictly periodic points other than 0 and ∞ are roots of unity of odd order. The only roots of unity of odd order with degree no more than 6 are powers of $\{1, \zeta_3, \zeta_5, \zeta_7, \zeta_9\}$. We may find their periodic structures by iterating ψ_2 with the appropriate seed values.

Let β be a preperiodic point for ψ_2. Then $[\mathbb{Q}(\beta) : \mathbb{Q}] \leqslant 6$ if and only if all powers of β also satisfy $[\mathbb{Q}(\beta^n) : \mathbb{Q}] \leqslant 6$. In particular, the orbit of β lands in some cycle, and the points of that cycle have degree no more than 6. Hence we can find all preperiodic points for ψ_2 having degree no more than 6 by finding preimages of the periodic points described above, and continuing until the field generated by the preimages has degree greater than 6. It is a simple matter to verify that this process yields the diagrams given. $\square$

PROPOSITION 3. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2. Then ϕ has no points of type 2_n for $n \geqslant 1$. For $m \neq 2$, ϕ has the same number of rational points of type m_1 as it has rational points of primitive period m.*

Proof. The critical points of ψ_2 lie on a 2-cycle, and this property is preserved under conjugation. Therefore each critical point is also a critical value, so if the critical points of ϕ are $\{\gamma_1, \gamma_2\}$ we have $\phi^{-1}(\gamma_i) = \{\gamma_j\}$ for $i \neq j$. Hence ϕ has no points of type 2_1 and it follows that ϕ has no points of type 2_n for $n \geqslant 1$.

Let α be a rational point of primitive period m for ϕ. Then all points on the m-cycle containing α are also rational since $\phi(z) \in \mathbb{Q}(z)$. Therefore the quadratic $\phi(z) = \alpha$ has one

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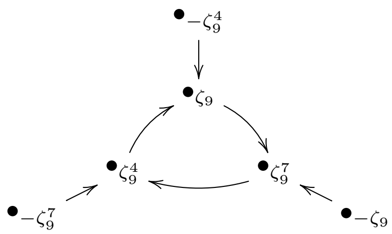

FIGURE 4. $\psi_2(z) = 1/z^2$*: two* 3*-cycles.*

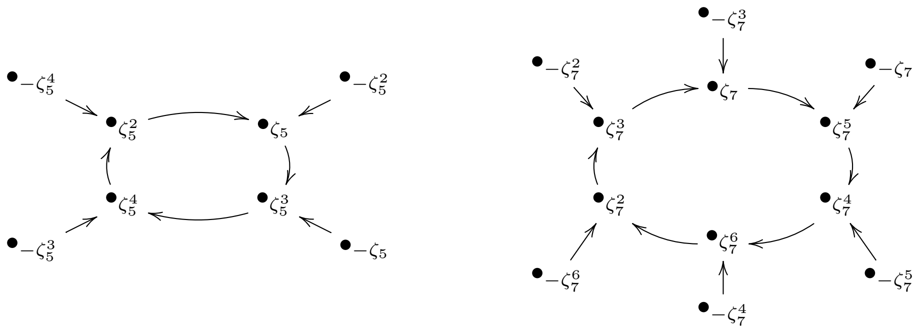

FIGURE 5. $\psi_2(z) = 1/z^2$*: a* 4*-cycle and a* 6*-cycle.*

rational root. Since $m \neq 2$, α is not one of the critical values of ϕ by the argument above. Hence, the quadratic $\phi(z) = \alpha$ has two distinct roots, so both must be rational. That is, there is a rational point β not on the m-cycle satisfying $\phi(\beta) = \alpha$, and β is a point of type m_1. □

PROPOSITION 4. *Let $\phi(z) \in \mathbb{Q}(z)$ be conjugate to ψ_2. If ϕ has a rational 2-cycle then it may have either no rational fixed points or one rational fixed point. In either case, it has no other rational preperiodic points except the required point of type 1_1.*

Proof. From [8, Lemma 5.1], we see that ϕ has a rational 2-cycle if and only if it is conjugate over $\mathbb{Q}$ to $\theta_t(z) = t/z^2$ for some $t \in \mathbb{Q}^\times$. Solving $\theta_t(z) = z$, we see that there is a rational fixed point if and only if $t \in (\mathbb{Q}^\times)^3$, and all such maps are conjugate over $\mathbb{Q}$.

Furthermore, if $f(z) = t^{-1/3}z$, then $\theta_t^f(z) = \psi_2$. Applying f to the preperiodic structures given in Lemma 2, we find no other rational preperiodic points. □

By Proposition 3, we have only two rational preperiodic structures for maps conjugate to $\phi_2(z)$ that contain rational points of primitive period 2. These are the first two maps represented in Table 4.

PROPOSITION 5. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2. Suppose ϕ has no rational points of period $n > 1$. Then ϕ has one of the following rational preperiodic structures:*
- (i) *ϕ has no rational fixed points (hence no rational preperiodic points at all);*
- (ii) *ϕ has exactly one rational fixed point and one point of type 1_1 but no other rational preperiodic points;*
- (iii) *ϕ has exactly one rational fixed point, one rational point of type 1_1, and two rational points of type 1_2, with no other rational preperiodic points; or*
- (iv) *ϕ has exactly three rational fixed points and three rational points of type 1_1, with no other rational preperiodic points.*

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Proof. Choosing $k = 1$ and $d = 2$ in the normal form from equation (5.1) yields the map

$$\frac{z^2 - 4z + 2}{z^2 - 2z + 2}.$$

One can check computationally that this map has no rational points of primitive period 1, 2, 3, 4, or 6. By Lemma 2, these are the only possibilities.

Choosing $k = 0$ and $d = 2$ in the normal form from equation (5.1) yields the map

$$-\frac{4z}{z^2 + 2}.$$

One can check computationally that this map has fixed point 0 and no other rational points of primitive period 1, 2, 3, 4, or 6. By Lemma 2, these are the only possibilities. We also have $\infty \mapsto 0$, a rational point of type 1_1. The preimages of ∞ are not rational, so there are no other rational preperiodic points.

Beginning with the preperiodic structure described in Lemma 2, we see that conjugating ϕ_2 by any $f \in \mathrm{PGL}_2$ which maps three arbitrary rational points to 1, i, and $-i$ creates a map with rational type 1_2 points. Choose

$$f(z) = \frac{iz + 1}{z + i} \quad \text{which yields } \psi_2^f(z) = \frac{-z^2 + 2z + 1}{z^2 + 2z - 1}.$$

TABLE 4. *Preperiodic structures for twists of $\psi_2(z) = 1/z^2$.*

$\phi(z)$	Rational preperiodic points graph
$\frac{1}{z^2}$	$\bullet_{-1} \longrightarrow \bullet_{1}\circlearrowleft \qquad \bullet_0 \rightleftarrows \bullet_\infty$
$\frac{2}{z^2}$	$\bullet_0 \rightleftarrows \bullet_\infty$
$\frac{z^2-4z+2}{z^2-2z+2}$	no rational preperiodic points
$-\frac{4z}{z^2+2}$	$\bullet_\infty \longrightarrow \bullet_0\circlearrowleft$
$\frac{-z^2+2z+1}{z^2+2z-1}$	$\bullet_0 \to \bullet_{-1},\ \bullet_\infty \to \bullet_{-1},\ \bullet_{-1} \longrightarrow \bullet_1\circlearrowleft$
$-\frac{(z-2)z}{2z-1}$	$\bullet_2 \longrightarrow \bullet_0\circlearrowleft \qquad \bullet_{-1} \longrightarrow \bullet_1\circlearrowleft \qquad \bullet_{\frac{1}{2}} \longrightarrow \bullet_\infty\circlearrowleft$
$\frac{2z-1}{z^2-1}$	$\bullet_{\frac{1}{2}} \to \bullet_0,\ \bullet_0 \to \bullet_1,\ \bullet_1 \to \bullet_\infty,\ \bullet_\infty \to \bullet_0,\ \bullet_{-1} \to \bullet_\infty,\ \bullet_2 \to \bullet_1$

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One can check computationally that this map has no rational point of period 2, 3, 4, or 6. The only rational fixed point is 1; -1 is a type 1_1 point; and 0 and ∞ are type 1_2 points. There are no rational type 1_3 points.

Again, beginning with the preperiodic structure described in Lemma 2, we see that conjugating ϕ_2 by any $f \in \mathrm{PGL}_2$ which maps three arbitrary rational points to 1, ζ_3, and ζ_3^2, yields a map with three rational fixed points. Choose

$$f(z) = \frac{(1+\zeta_3)z+1}{z-\zeta_3^2} \quad \text{which yields } \psi_2^f(z) = -\frac{(z-2)z}{2z-1}.$$

This map has fixed points at 0, 1, and ∞ and the corresponding rational type 1_1 points. One can check computationally that this map has no rational point of period 2, 3, 4, or 6, and no rational type 1_2 points.

We have shown that each of the possibilities listed are possible for maps conjugate to ψ_2. It remains to check that no other possibilities exist.

Since ϕ is defined over $\mathbb{Q}$, the cubic polynomial $\phi(z) = z$ has either zero, one, or three rational roots. Hence we cannot have exactly two rational fixed points.

If a map ϕ is conjugate to ψ_2 and has rational points of type 1_3, then it is conjugate over $\mathbb{Q}$ to a map with a fixed point at 1 and the type 1_2 points at 0 and ∞. We found such a map above, and it does not have rational type 1_3 points.

Similarly, if a map ϕ is conjugate to ψ_2 and has three rational fixed points and rational points of type 1_2, then it is conjugate over $\mathbb{Q}$ to a map with one fixed point at 1 and its type 1_2 points at 0 and ∞. We found such a map above, and it does not have additional rational fixed points. We have now exhausted all possibilities. □

By Proposition 5, the third through sixth rational preperiodic structures in Table 4 are the only ones possible for maps conjugate to ψ_2 that have no rational points of least period $n > 1$.

PROPOSITION 6. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2. Suppose ϕ has a rational point of period* 3. *Then ϕ has exactly three such points and three points of type* 3_1. *The map ϕ has no other rational preperiodic points.*

Proof. If ϕ is conjugate to ψ_2 and has a rational point of period 3, then it is conjugate over $\mathbb{Q}$ to a map with the 3-cycle $0 \mapsto 1 \mapsto \infty \mapsto 0 \mapsto \dots$. This conjugacy completely specifies the map. Given the preperiodic structure described in Lemma 2, we may begin with $f \in \mathrm{PGL}_2$ which maps 0, 1, and ∞ to ζ_9, ζ_9^7, and ζ_9^6. This is

$$f = \frac{\zeta_9^6 z - \zeta_9^7}{-z+\zeta_9^4} \quad \text{which yields } \psi_2^f(z) = \frac{2z-1}{z^2-1}.$$

One may verify computationally that this map has the desired 3-cycle and no other rational points of period 1, 2, 3, 4, or 6. It has rational type 3_1 points mapping into the 3-cycle, but the type 3_2 points are not rational. □

By Proposition 6, there is only one rational preperiodic structure for maps conjugate to ψ_2 that have a rational point of primitive period 3. This is the last map in Table 4.

PROPOSITION 7. *Let $\phi(z) \in \mathbb{Q}$ be conjugate over $\overline{\mathbb{Q}}$ to ψ_2. Then ϕ has no rational points of period $n > 3$.*

Proof. If ϕ has rational points of period 4, then it is conjugate over $\mathbb{Q}$ to a map where three of those points are at 0, 1, and ∞. Applying Lemma 2, we choose $f \in \mathrm{PGL}_2$ mapping these three rational points to three powers of ζ_5. Conjugating ψ_2 by this map does not yield a map defined over $\mathbb{Q}$. The argument for points of period 6 is the same, but using powers of ζ_7. □

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