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A NOTE ON NIELSEN EQUIVALENCE IN FINITELY GENERATED ABELIAN GROUPS

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Abstract

Nielsen transformations determine the automorphisms of a free group of rank n, and also of a free abelian group of rank n, and furthermore the generating n-tuples of such groups form a single Nielsen equivalence class. For an arbitrary rank n group, the generating n-tuples may fall into several Nielsen classes. Diaconis and Graham ['The graph of generating sets of an abelian group', *Colloq. Math.* **80** (1999), 31–38] determined the Nielsen classes for finite abelian groups. We extend their result to the case of infinite abelian groups.

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1. Introduction

Let G be any group of rank n. The following transformations defined on the set $\Gamma_t(G)$ of all generating t-tuples of G $(t \geq n)$ are called *elementary Nielsen transformations*:

(1) $\pi : \Gamma_t(G) \to \Gamma_t(G)$, defined by

$$\pi(w_1, w_2, \ldots, w_i, \ldots, w_t) = (w_2, w_1, \ldots, w_i, \ldots, w_t);$$

(2) $\sigma : \Gamma_t(G) \to \Gamma_t(G)$, defined by

$$\sigma(w_1, w_2, \ldots, w_t) = (w_2, w_3, \ldots, w_t, w_1);$$

(3) $\mu : \Gamma_t(G) \to \Gamma_t(G)$, defined by

$$\mu(w_1, \ldots, w_i, \ldots, w_t) = (w_1 w_2, w_2, \ldots, w_i, \ldots, w_t);$$

(4) $\tau : \Gamma_t(G) \to \Gamma_t(G)$, defined by

$$\tau(w_1, \ldots, w_i, \ldots, w_t) = (w_1^{-1}, \ldots, w_i, \ldots, w_t).$$

The elementary Nielsen transformations generate a group, $\mathbf{N}_t(G)$, acting on $\Gamma_t(G)$. Two t-tuples from $\Gamma_t(G)$ are said to be *Nielsen equivalent* if one can be transformed into the other by means of a finite sequence of elementary Nielsen transformations.

Nielsen [3] showed that every two generating t-tuples of F_n, the free group of rank n, are Nielsen equivalent, whence it follows in particular that $\mathbf{N}_n(F_n) \cong \mathrm{Aut}(F_n)$.

We are interested here in the case where G is an arbitrary additively written abelian group A of rank $n \geq 1$, that is, in what the Nielsen equivalence classes of $\Gamma_t(A)$ might be for all $t \geq n$. We shall use the standard (and unique for the torsion subgroup) direct decomposition of such an A:

$$A = Z_1 \times \cdots \times Z_k \times Z_{k+1} \times \cdots \times Z_n = \prod_{j=1}^{n} Z_j,$$

where for $1 \leq j \leq k$, $Z_j \cong \mathbb{Z}$, and for $k+1 \leq j \leq n$, $Z_j \cong \mathbb{Z}/m_j\mathbb{Z}$ with the m_j integers exceeding 1 and satisfying $m_{j+1}|m_j$ (so that m_n divides all the m_j).

The following theorem, which in essence extends that of Diaconis and Graham [2] from finite to finitely generated abelian groups, gives the complete answer.

THEOREM 1.1.

(i) *If $t > n$ then all generating t-tuples of A are Nielsen equivalent.*

(ii) *The case $t = n$.*

- (a) *If $k = n$ (so that A is free abelian) then all generating n-tuples of A are Nielsen equivalent. (This case is well known.)*
- (b) *Suppose that $k < n$ (so that A has torsion). Let $z_1, z_2, \ldots, z_n$ be fixed generators of the cyclic summands $Z_1, Z_2, \ldots, Z_n$ of A. Then every generating n-tuple of A is Nielsen equivalent to one and only one n-tuple of the form $(z_1, z_2, \ldots, z_{n-1}, rz_n)$, where $1 \leq r < m_n/2$ and $(r, m_n) = 1$. Hence in the case $m_n > 2$, $\Gamma_n(A)$ falls into $\varphi(m_n)/2$ Nielsen classes, while if $m_n = 2$ there is again just one Nielsen class. (Here φ is the Euler totient function, $\varphi(m) = |\{i : 0 < i \leq m, \gcd(i, m) = 1\}|$, $m \in \mathbb{N}^*$.)*

Our proof follows that of [2] closely. What is new is the inclusion of the case where A is infinite ($k \geq 1$) and the use of matrices from $\mathrm{GL}_t(\mathbb{Z})$. Note also that in [2] only transformations generated by the first three types of elementary Nielsen transformations are used, so that in case (ii)(b) (with, in addition, $k = 0$) they obtain $\varphi(m_n)$ classes.

2. Preliminaries

Any t-tuple $\mathbf{g} := (g_1, \ldots g_t)$ of elements of A can be written as a $t \times n$ matrix,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{t1} & g_{t2} & \cdots & g_{tn} \end{pmatrix},$$

where $g_{ij} \in Z_j$, $1 \leq i \leq t$, $1 \leq j \leq n$, and $g_j = g_{j1} \cdots g_{jn}$. Using this representation of t-tuples of A together with the $\mathbb{Z}$-module structure of the subgroups Z_j of A, we have an action of $\mathrm{GL}_t(\mathbb{Z})$ on the set $\Gamma_t(A)$ of generating t-tuples of elements of A,

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namely that given by multiplication of the above matrix $\mathbf{g}$ on the left by the matrices of $\mathrm{GL}_t(\mathbb{Z})$.

Consider the following matrices in $\mathrm{GL}_t(\mathbb{Z})$:

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

These matrices in fact generate $\mathrm{GL}_t(\mathbb{Z})$ [1] and we have an epimorphism $\Phi : \mathbf{N}(F_t) \to \mathrm{GL}_t(\mathbb{Z})$, induced by the natural epimorphism $F_t \to \mathbb{Z}^t$. Thus $\Phi(\pi) = M_1$, $\Phi(\sigma) = M_2$, $\Phi(\mu) = M_3$, and $\Phi(\tau) = M_4$, taking $G = F_t$ in the definitions of π, σ, μ, τ above.

The following lemma is immediate from the fact that on the one hand M_1, M_2, M_3, M_4 act on the t-tuples of $\Gamma(A)$ like π, σ, μ, τ respectively, and on the other they generate $\mathrm{GL}_t(\mathbb{Z})$.

LEMMA 2.1. *Let A be, as above, a finitely generated abelian group of rank n, and let $\mathbf{g}$ and $\mathbf{h}$ be generating t-tuples of A, written, as above, as $t \times n$ matrices. Then $\mathbf{g}$ is Nielsen equivalent to $\mathbf{h}$ if and only if there exists a matrix $S \in \mathrm{GL}_t(\mathbb{Z})$ such that $S\mathbf{g} = \mathbf{h}$.*

The next lemma is key.

LEMMA 2.2. *Let C be an (additively written) nontrivial cyclic group and $(a_1, \ldots, a_t)$ a generating t-tuple of C, $t \geq 2$. Then for any generator z of C, there exists $S \in \mathrm{GL}_t(\mathbb{Z})$ such that $S(a_1, \ldots, a_t)^T = (z, 0, \ldots, 0)^T$. (Here T denotes the transpose.) Equivalently, there exists a sequence of elementary Nielsen transformations taking $(a_1, \ldots, a_t)$ to $(z, 0, \ldots, 0)$ for any generator z of C.*

PROOF. We use induction on t. We identify C with the additive group of the ring $\mathbb{Z}/m\mathbb{Z}$, where $m = |C|$ (including the case $m = \infty$, $C \cong \mathbb{Z}$).

Let $t = 2$, and let (a_1, a_2) be any pair generating C. Since z is a generator of C, we have $a_1 = n_1 z$, $a_2 = n_2 z$, for some $n_1, n_2 \in \mathbb{Z}$. Let $d = \gcd(n_1, n_2)$, let $k, l \in \mathbb{Z}$ be such that $kn_1 + ln_2 = d$, and define $S_1 \in \mathrm{GL}_2(\mathbb{Z})$ by

$$S_1 = \begin{pmatrix} k & l \\ -n_2/d & n_1/d \end{pmatrix}.$$

One verifies directly that $S_1(a_1, a_2)^T = (dz, 0)^T$. Since $(dz, 0)$ is a generating pair,

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we have $(d, m) = 1$ ($d = \pm 1$ if $m = \infty$), and so for some $c \in \mathbb{Z}$ we have $cd \equiv 1 \bmod m$. A possible two further S_2 and S_3 are defined as follows: first

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} dz \\ 0 \end{pmatrix} = \begin{pmatrix} dz \\ z \end{pmatrix},$$

and then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dz \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

Taking the product of the matrices S_1, S_2, S_3 (right to left) we obtain a matrix $S \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$S \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

The inductive step from t to $t + 1$ proceeds as follows: if $(a_1, \ldots, a_t, a_{t+1})$ is a generating $(t + 1)$-tuple of C, then by the induction hypothesis there exists $M \in \mathrm{GL}_t(\mathbb{Z})$ such that

$$M(a_1, \ldots, a_t)^T = (u, 0, \ldots, 0)^T,$$

where u is any generator of the subgroup $\langle a_1, \ldots, a_t \rangle$. Define the $(t + 1) \times (t + 1)$ matrix W by

$$W := \begin{pmatrix} M & \mathbf{0}^T \\ \mathbf{0} & 1 \end{pmatrix}.$$

Then $W \in \mathrm{GL}_{t+1}(\mathbb{Z})$, and

$$W \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \\ a_{t+1} \end{pmatrix} = \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \\ a_{t+1} \end{pmatrix},$$

where now $(u, 0, \ldots, 0, a_{t+1})$ is a generating $(t + 1)$-tuple for C. Defining $Q_{(t+1)\times(t+1)}$ by

$$Q := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

we have

$$Q \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \\ a_{t+1} \end{pmatrix} = \begin{pmatrix} u \\ a_{t+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

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Let $S \in \mathrm{GL}_2(\mathbb{Z})$ be obtained as in the case $t = 2$ above, that is, such that

$$S(u, a_{t+1})^T = (z, 0)^T,$$

where z is our chosen arbitrary generator of C. If we define the matrix $R \in \mathrm{GL}_{t+1}(\mathbb{Z})$ by

$$R := \begin{pmatrix} S & \bigcirc \\ \bigcirc & I_{t-1} \end{pmatrix},$$

then $RQW(a_1, \ldots, a_t, a_{t+1})^T = (z, 0, \ldots, 0)^T$. □

3. Proof of the theorem

PROOF. The proof follows that for finite abelian groups given by Diaconis and Graham [2], except that we present it in a somewhat modified form, using matrices in $\mathrm{GL}_t(\mathbb{Z})$ to execute the Nielsen transformations. We may assume without loss of generality that $t \geq 2$ since the case $t = 1$ (implying $n = 1$) is obvious.

As above, we write an arbitrary generating t-tuple of A in the form of a $t \times n$ matrix:

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{t1} & g_{t2} & \cdots & g_{tn} \end{pmatrix},$$

with $g_{ij} \in Z_j$, $1 \leq i \leq t$, $1 \leq j \leq n$.

We show by induction on s, $1 \leq s < t$, that for any choice of generators $z_1, z_2, \ldots, z_s$ of $Z_1, Z_2, \ldots, Z_s$ respectively, there is a matrix $R_s \in \mathrm{GL}_t(\mathbb{Z})$ such that

$$R_s\mathbf{g} = \begin{pmatrix} z_1 & 0 & \cdots & 0 & f_{1,s+1} & \cdots & f_{1n} \\ 0 & z_2 & \cdots & 0 & f_{2,s+1} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & f_{s,s+1} & \cdots & f_{sn} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & f_{t,s+1} & \cdots & f_{tn} \end{pmatrix}. \tag{3.1}$$

The initial step: $s = 1$.

Since $\mathbf{g}$ is a generating t-tuple, it follows that $(g_{1j}, g_{2j}, \ldots, g_{tj})$ is a generating t-tuple for Z_j, $1 \leq j \leq n$. Consider the first column of (g_{ij}), whose entries are in Z_1. By Lemma 2.2, there exists a matrix $R_1 \in \mathrm{GL}_t(\mathbb{Z})$ such that

$$R_1 \begin{pmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{t1} \end{pmatrix} = \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

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where z_1 is the arbitrarily chosen generator of the cyclic group Z_1 ($z_1 = \pm 1$ if $Z_1 \cong \mathbb{Z}$). Thus

$$R_1\mathbf{g} = \begin{pmatrix} z_1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{t2} & \cdots & a_{tn} \end{pmatrix},$$

for some $a_{ij} \in Z_j$. As always, since $R_1 \in \mathrm{GL}_t(\mathbb{Z})$, $R_1\mathbf{g}$ is still a generating t-tuple of A.

The inductive step from s to $s+1 \leq n$.

We assume inductively that there is a matrix R_s such that $R_s\mathbf{g}$ has the form (3.1) above. Consider the $(t-s)$-tuple $\mathbf{f}_{s+1} := (f_{s+1,s+1}, \ldots, f_{t,s+1})$ consisting of the entries in (3.1) below $f_{s,s+1}$. This generates a subgroup $\langle d \rangle$ of the cyclic group Z_{s+1}. Thus by Lemma 2.2 there exists a $(t-s) \times (t-s)$ matrix $P \in \mathrm{GL}_{t-s}(\mathbb{Z})$ such that

$$P \begin{pmatrix} f_{s+1,s+1} \\ f_{s+2,s+1} \\ \vdots \\ f_{t,s+1} \end{pmatrix} = \begin{pmatrix} d \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence defining the $t \times t$ matrix

$$S := \begin{pmatrix} I_s & \bigcirc \\ \bigcirc & P \end{pmatrix}, \qquad S \in \mathrm{GL}_t(\mathbb{Z}),$$

where I_s is the $s \times s$ identity matrix, we have

$$S \begin{pmatrix} z_1 & 0 & \cdots & 0 & f_{1,s+1} & \cdots & f_{1n} \\ 0 & z_2 & \cdots & 0 & f_{2,s+1} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & f_{s,s+1} & \cdots & f_{sn} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & f_{t,s+1} & \cdots & f_{tn} \end{pmatrix}$$

$$= \begin{pmatrix} z_1 & 0 & \cdots & 0 & f_{1,s+1} & f_{1,s+2} & \cdots & f_{1n} \\ 0 & z_2 & \cdots & 0 & f_{2,s+1} & f_{2,s+2} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & f_{s,s+1} & f_{s,s+2} & \cdots & f_{sn} \\ 0 & 0 & \cdots & 0 & d & h_{s+1,s+2} & \cdots & h_{s+1,n} \\ 0 & 0 & \cdots & 0 & 0 & h_{s+2,s+2} & \cdots & h_{s+2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & h_{t,s+2} & \cdots & h_{tn} \end{pmatrix}.$$

Now since the elements of A represented by the rows of the submatrix

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$$\begin{pmatrix} z_1 & 0 & \dots & 0 & f_{1,s+1} \\ 0 & z_2 & \dots & 0 & f_{2,s+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_s & f_{s,s+1} \\ 0 & 0 & \cdots & 0 & d \end{pmatrix}$$

generate $Z_1 \times Z_2 \times \cdots \times Z_{s+1}$, there must exist integers $v_1, v_2, \dots, v_{s+1}$ such that

$$v_1(z_1, 0, \dots, 0, f_{1,s+1}) + v_2(0, z_2, 0, \dots, 0, f_{2,s+1}) + \cdots + v_{s+1}(0, \dots, 0, d) \\ = (0, \dots, 0, z_{s+1}),$$

where z_{s+1} is the arbitrarily chosen generator of Z_{s+1}. It follows that $v_j \equiv 0 \bmod m_j$ for $j = 1, \dots, s$ (where, as usual, we interpret this to mean $v_j = 0$ for those j, if any, for which $Z_j \cong \mathbb{Z}$). In view of the ordering of the Z_j so that $m_{j+1} | m_j$, we also have that all of $v_1, \dots, v_s \equiv 0 \bmod m_{s+1}$ (or are all zero if $Z_{s+1} \cong \mathbb{Z}$). Hence $v_{s+1}d = z_{s+1}$, so that in fact d generates Z_{s+1}. Thus there exist integers a_i, $1 \le i \le s$, such that $f_{i,s+1} = a_i d$. We now proceed in three steps.

- Step 1 (valid also if $s + 1 = t$). Let W_s be the matrix in $\mathrm{GL}_t(\mathbb{Z})$ obtained from the identity matrix by replacing the $(i, s+1)$ entries with a_i for $1 \le i \le s$. Then

$$W_s \begin{pmatrix} z_1 & 0 & \cdots & 0 & * & * & \cdots & * \\ 0 & z_2 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & d & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}$$

$$= \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & z_2 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & d & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}.$$

  The $*$ entries are placeholders and their values are not important for the argument; we are using them to simplify notation.

  The next two steps require $s + 1 < t$.
- Step 2. Let X_s be the matrix in $\mathrm{GL}_t(\mathbb{Z})$ obtained from the identity matrix by replacing the $(s+2, s+1)$ entry with v_{s+1}. Then

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$$X_s \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & z_2 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & d & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}$$

$$= \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & z_2 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & d & * & \cdots & * \\ 0 & 0 & \cdots & 0 & z_{s+1} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}.$$

- Step 3. Let Y_s be the matrix in $\mathrm{GL}_t(\mathbb{Z})$ obtained from the identity matrix by modifying four of the entries as follows: $(s+1, s+1) : 0$, $(s+1, s+2) : 1$, $(s+2, s+1) : 1$, $(s+2, s+2) : -b$, where b is an integer such that $bz_{s+1} = d$. An immediate calculation gives

$$Y_s \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & z_2 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & d & * & \cdots & * \\ 0 & 0 & \cdots & 0 & z_{s+1} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}$$

$$= \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & z_2 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_s & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & z_{s+1} & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \end{pmatrix}. \tag{3.2}$$

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We have thus reached the form required by the induction step, completing the induction.

Part (i) of the theorem (the case $t > n$) now follows, since for $s = n$ $(<t)$ the matrix on the right of Equation (3.2) becomes

$$\mathbf{h} = \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 \\ 0 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & z_n \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For part (ii) of the theorem (the case $t = n$), the above argument (up to and including Step 1) shows that for $s = n - 1$ our initial generating t-tuple can be transformed by means of matrices from $\mathrm{GL}_n(\mathbb{Z})$ to

$$\mathbf{h} = \begin{pmatrix} z_1 & 0 & \cdots & 0 & 0 \\ 0 & z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & z_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & d \end{pmatrix}, \tag{3.3}$$

and the generator d of Z_n can be written as rz_n for a unique r with $1 \le r < m_n$ satisfying $(r, m_n) = 1$. If $r \ge m_n/2$, then premultiplication by the matrix

$$\begin{pmatrix} I_{n-1} & \mathbf{0}^T \\ \mathbf{0} & -1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

will cause rz_n to be replaced by $-rz_n = r'z_n$ for r' satisfying $0 < r' < m_n/2$.

Finally, we show that if $\mathbf{h}_1, \mathbf{h}_2$ are as in (3.3) with entries $d_1 = r_1 z_n$, $d_2 = r_2 z_n$ in place of d, where $0 < r_1 < r_2 < m_n/2$, then $\mathbf{h}_1, \mathbf{h}_2$ cannot be transformed into one another by any matrix from $\mathrm{GL}_n(\mathbb{Z})$. If $A \in \mathrm{GL}_n(\mathbb{Z})$, with $A\mathbf{h}_1 = \mathbf{h}_2$, one can easily see that modulo m_n, A is a diagonal matrix, with entries $a_{ii} = 1$ for $1 \le i \le n-1$, and with $a_{nn} r_1 = r_2$. Since $\det(A) \in \{-1, 1\}$, $A \in \mathrm{GL}_n(\mathbb{Z})$, then also modulo m_n, $\det(A) = a_{nn} \in \{-1, 1\}$. It follows that $r_1 = r_2$ or $r_1 = -r_2$. □

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