P. Lappan
Nagoya Math. J.
Vol. 34 (1969), 25–34

CONTINUA WHICH ARE CURVILINEAR CLUSTER SETS

PETER LAPPAN

1. Introduction

Let D be the unit disk $|z| < 1$ and let f be a complex-valued function continuous in D. The *cluster set* $C(f, e^{i\theta})$ *of* f *at* $e^{i\theta}$ is defined by

$$C(f, e^{i\theta}) = \bigcap_{r>0} \overline{f(N(r, e^{i\theta}) \cap D)},$$

where $N(r, e^{i\theta})$ is the disk with center at $e^{i\theta}$ and radius r. By a *path to* $e^{i\theta}$ we mean a Jordan arc γ in $D \cup \{e^{i\theta}\}$ with end point at $e^{i\theta}$. A *curvilinear cluster set of* f *at* $e^{i\theta}$ is a set of the form

$$C_\gamma(f, e^{i\theta}) = \bigcap_{r>0} \overline{f(N(r, e^{i\theta}) \cap D \cap \gamma)},$$

where γ is a path to $e^{i\theta}$. The *intersection curvilinear cluster set of* f *at* $e^{i\theta}$ is the set

$$\Pi(f, e^{i\theta}) = \bigcap_{\gamma} C_\gamma(f, e^{i\theta})$$

where the intersection is taken over all paths to $e^{i\theta}$. Finally, by a *continuum* we mean a closed connected subset of the Riemann sphere W. For simplicity of wording we will allow a set consisting of a single point to be called a continuum here, even though this use of the word is sometimes not allowed by other authors.

It is well known that for an arbitrary continuum K and an arbitrary path γ to $e^{i\theta}$ there exists a meromorphic function f in D such that $C_\gamma(f, e^{i\theta}) = K$ [1, Theorem 4, p. 193]. In this paper we wish to consider the following problem: given a continuous function f in D and a point $e^{i\theta}$, what are necessary and sufficient conditions for a continuum K to be a curvilinear cluster set of f at $e^{i\theta}$? The results in this paper are theorems and examples related to this question.

For any path γ to $e^{i\theta}$ it is necessary that

Received February 12, 1968

$$\Pi(f, e^{i\theta}) \subset C_\gamma(f, e^{i\theta}) \subset C(f, e^{i\theta}),$$

where the containments may be proper or improper. It is known [3, Theorem 4. 6, p. 73] that $C(f, e^{i\theta})$ is a curvilinear cluster set of f at $e^{i\theta}$. The set $\Pi(f, e^{i\theta})$ may be the empty set, but $C_\gamma(f, e^{i\theta})$ is not empty, so that $\Pi(f, e^{i\theta})$ is not necessarily a curvilinear cluster set.

In section 2, we show by example that a continuum K satisfying $\Pi(f, e^{i\theta}) \subsetneqq K \subsetneqq C(f, e^{i\theta})$ need not be a curvilinear cluster set of f at $e^{i\theta}$, even in the case where f is a meromorphic function. In fact, it is shown that even if K is "between" two curvilinear cluster sets of f at $e^{i\theta}$ it need not be a curvilinear cluster set itself. We prove that if $\Pi(f, e^{i\theta}) \neq C(f, e^{i\theta})$ then the cardinality of the set of all curvilinear cluster sets of f at $e^{i\theta}$ is equal to the cardinality of the set of real numbers.

In section 3 we introduce the idea of a permissible continuum for f at $e^{i\theta}$ by using concepts related to the theory of prime ends. We prove that it is necessary, but not sufficient, that a continuum be permissible for f at $e^{i\theta}$ in order that it be a curvilinear cluster set of f at $e^{i\theta}$. Some relationships with other types of cluster sets are given.

In section 4 we give some sufficient conditions for there to exist a curvilinear cluster set of f at $e^{i\theta}$ between two specified sets.

2. The Cardinality of Curvilinear Cluster Sets

If $\Pi(f, e^{i\theta}) \neq C(f, e^{i\theta})$ then there exists at least one continuum K such that $\Pi(f, e^{i\theta}) \subset K \subsetneqq C(f, e^{i\theta})$ and such that K is a curvilinear cluster set of f at $e^{i\theta}$. However, not every such continuum K is a curvilinear cluster set of f at $e^{i\theta}$, as the following example shows.

EXAMPLE 1. *There exists a function f continuous in D together with two continua K_1 and K_2, where both K_1 and K_2 are curvilinear cluster sets of f at $e^{i\theta}$, $K_1 \subset K_2$, and if K_3 is a continuum satisfying $K_1 \subsetneqq K_3 \subsetneqq K_2$, then K_3 is not a curvilinear cluster set of f at $e^{i\theta}$.*

Proof. Let $B(z)$ be a conformal mapping sending the disk D onto the upper half plane U such that $B(e^{i\theta}) = 0$. Let $z = x + iy$ and let F be the function defined on U by

https://doi.org/10.1017/S0027763000024430 Published online by Cambridge University Press

$$F(z)=\begin{cases}\left(\frac{1}{y}\right)\sin\frac{1}{y}, & \text{for } 0<\arg z\leqq\frac{\pi}{4}\\ \left(\frac{y-x}{y}\right)i+\left(\frac{x}{y^2}\right)\sin\frac{1}{y}, & \text{for } \frac{\pi}{4}<\arg z\leqq\frac{\pi}{2}\\ \left(\frac{x+y}{y}\right)i-\left(\frac{x}{y}\right)\sin\frac{1}{y}, & \text{for } \frac{\pi}{2}<\arg z\leqq\frac{3\pi}{4}\\ \sin\frac{1}{y}, & \text{for } \frac{3\pi}{4}<\arg z<\pi.\end{cases}$$

It is easy to see that both the segment $[-1, 1]$ and the whole real line R are curvilinear cluster sets of f at 0. But if $[a, b]$ is any closed interval of the real line such that $[-1, 1]\subsetneqq[a, b]\neq R$, then there is no path γ to 0 contained in $U\cup\{0\}$ such that $C_\gamma(F, 0)=[a, b]$. Thus we have the desired result if we set $f(z)=F(B(z))$, $K_1=[-1, 1]$, and $K_2=R$.

We remark that Example 1 shows that even continua which are "between" two curvilinear cluster sets of a function f at $e^{i\theta}$ need not themselves be curvilinear cluster sets of f at $e^{i\theta}$. Such a result also holds for meromorphic functions, as the following example shows.

EXAMPLE 2. *There exists a meromorphic function f in D and two continua K_1 and K_2 such that $K_1\subset K_2\subset C(f, e^{i\theta})$, where K_1 is a curvilinear cluster set of f at $e^{i\theta}$ and K_2 is not a curvilinear cluster set of f at $e^{i\theta}$.*

Proof. Let L be the left half plane, let

$$V_1=\left\{z=x+iy:\ 0\leqq x<1,\ 2n\pi<y<\left(2n+\frac{1}{2}\right)\pi,\ n=1, 2, 3, \cdots\right\}$$

and let

$$V_2=\left\{z=x+iy:\ 0\leqq x<1,\ \left(2n+\frac{1}{2}\right)\pi<y<(2n+1)\pi,\ n=-1, -2, -3, \cdots\right\}.$$

Let $V=L\cup V_1\cup V_2$, and let $B(z)$ be a conformal mapping of D onto V such that $B(e^{i\theta})=\infty$. Let $F(z)=e^z$ for $z\in V$. Let

$$D(r,\theta_1,\theta_2)=\{z:\ |z|=r,\quad \theta_1\leqq\arg\ z\leqq\theta_2\}$$

let

$$S(a,b,\theta)=\{z:\ a\leqq|z|\leqq b,\ \ \arg z=\theta\}$$

let

$$S = D(1, \pi/2, 2\pi) \cup D(e, 0, \pi/2) \cup S(1, e, 0) \cup S(1, e, \pi/2)$$

and let

$$T = D(1, 0, 2\pi) \cup D(e, 0, \pi) \cup S(1, e, 0) \cup S(1, e, \pi/2).$$

Clearly, $S \subsetneqq T \subsetneqq C(F, \infty)$ and there exists a path Γ to ∞ in V along the upper portion of the boundary of V such that $C_\Gamma(F, \infty) = S$. However, if Γ' were a path to ∞ in V such that $T \subset C_{\Gamma'}(F, \infty)$, then Γ' must meet both the upper and the lower half planes infinitely often so that, since Γ' has no basic point of accumulation on the negative real axis, we must have that $0 \in C_{\Gamma'}(F, \infty)$ and hence $T \neq C_{\Gamma'}(F, \infty)$. It follows that T is not a curvilinear cluster set of F at ∞. Setting $f(z) = F(B(z))$, $K_1 = S$ and $K_2 = T$, we obtain the result claimed.

In spite of the two previous examples, the following theorem shows that there are, except in trivial cases, many curvilinear cluster sets of f at $e^{i\theta}$.

THEOREM 1. *If f is a continuous function in D and if $\Pi(f, e^{i\theta}) \neq C(f, e^{i\theta})$ then the cardinality of the set of curvilinear cluster sets of f at $e^{i\theta}$ is equal to the cardinality of the set of real numbers.*

Proof. The cardinality of the set of all continua on the Riemann sphere is equal to the cardinality of the set of real numbers, and since each curvilinear cluster set of f at $e^{i\theta}$ must be a continuum, the cardinality of the set of all curvilinear cluster sets of f at $e^{i\theta}$ is not greater than the cardinality of the set of real numbers.

If $\Pi(f, e^{i\theta}) \neq C(f, e^{i\theta})$, then there exists a path γ to $e^{i\theta}$ such that $C_\gamma(f, e^{i\theta}) \neq C(f, e^{i\theta})$. Let w_0 be a point in $C(f, e^{i\theta}) - C_\gamma(f, e^{i\theta})$ and let $K = C_\gamma(f, e^{i\theta})$. Let $r_0 = \mathscr{X}(w_0, K)$, the chordal distance on W between w_0 and K, and let r be a fixed real number such that $0 < r < r_0$. For each natural number n such that $r + 1/n < r_0$, set

$$G_n = \{w \in W\colon \ \mathscr{X}(w, K) < r + 1/n\}$$

and set

$$H_n = \{z \in D\colon f(z) \in G_n\} \cap \{z \in D\colon |z - e^{i\theta}| < 1/n\}.$$

Finally, let J_n be the component of H_n containing γ (where γ is shortened, if necessary, so that γ is contained in H_n). Since $C(f, e^{i\theta})$ is a connected

https://doi.org/10.1017/S0027763000024430 Published online by Cambridge University Press

set containing both K and w_0, there exists a point $z_n \in J_n$ such that $\mathscr{H}(f(z_n), K) = r$. We may assume that $z_n \neq z_m$ for $n \neq m$. Let $\{\gamma_n\}$ be a collection of disjoint subarcs of γ such that the sets $\{\gamma_n\}$ converge to $e^{i\theta}$, $\gamma_n \subset J_n$, and the chordal diameter of $f(\gamma_n)$ on W is less than $1/n$. Since $z_n \in J_n$ and J_n is a connected set, we can replace γ_n by a simple arc γ_n', where $\gamma_n' \subset J_n$, $z_n \in \gamma_n'$, and the end points of γ_n' are the end points of γ_n. Replacing each γ_n by γ_n' but leaving all other points of γ fixed, and by then judiciously deleting the loops from the resulting path, we obtain a path Γ to $e^{i\theta}$ with the property that $K \subset C_\Gamma(f, e^{i\theta})$ and that:

$$C_\Gamma(f, e^{i\theta}) \subset \{w \in W : \mathscr{H}(w, K) \leqq t\}$$

if and only if $t \geqq r$. Thus we see that our construction of Γ depended on the choice of r, and that a different choice of r would lead to a different curvilinear cluster set $C_\Gamma(f, e^{i\theta})$. Thus the cardinality of the set of curvilinear cluster sets of f at $e^{i\theta}$ is not less than the cardinality of the set of real numbers in the open interval $(0, r_0)$. Thus the theorem is proved.

3. The Sets K_S

The Jordan arc s is a *crosscut of D at $e^{i\theta}$* if s has endpoints $e^{i\theta_1}$ and $e^{i\theta_2}$, where

$$\theta - \pi < \theta_1 < \theta < \theta_2 < \theta + \pi$$

and $s \subset D \cup \{e^{i\theta_1}\} \cup \{e^{i\theta_2}\}$. If $S = \{s_n\}$ is a sequence of crosscuts of D at $e^{i\theta}$, we say that *S converges to $e^{i\theta}$* if the diameter of S_n tends to zero as n goes to ∞. For each sequence S of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$, let

$$K_S(f) = \cap_{k=1}^{\infty} (\overline{\cup_{n=k}^{\infty} f(s_n)}),$$

where f is a function defined in D. As we will be considering specific functions, $K_S(f)$ will be abbreviated simply by K_S.

If L is a continuum and f is a continuous function in D, then L will be called a *permissible continuum for f at $e^{i\theta}$* if

(1) $\Pi(f, e^{i\theta}) \subset L \subset C(f, e^{i\theta})$, and

(2) $L \cap K_S \neq \phi$ for each sequence S of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$.

We note that if L is a permissible continuum for f at $e^{i\theta}$ and if $L \subset K \subset$

https://doi.org/10.1017/S0027763000024430 Published online by Cambridge University Press

$C(f, e^{i\theta})$, where K is a continuum, then K is also a permissible continuum for f at $e^{i\theta}$.

THEOREM 2. *If a continuum L is a curvilinear cluster set of the continuous function f at $e^{i\theta}$, then L is a permissible continuum for f at $e^{i\theta}$.*

Proof. Clearly condition (1) above must be satisfied by L. If $S = \{s_n\}$ is a sequence of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$, and if $L = C_\gamma(f, e^{i\theta})$, where γ is a path to $e^{i\theta}$, then $\gamma \cap s_n \neq \phi$ for n sufficiently large, so that $K_S \cap L \neq \phi$. Thus condition (2) is satisfied and the theorem is proved.

Unfortunately, the condition that L is a permissible continuum for f at $e^{i\theta}$ is not sufficient for L to be a curvilinear cluster set of f at $e^{i\theta}$, as can easily be verified from Examples 1 and 2 above. As Example 2 demonstrates, the added condition that f be a meromorphic function does not help.

It should be noted that two permissible continua for a function f at $e^{i\theta}$ may be disjoint, even if f is meromorphic. Any example of an ambiguous point (in the sense of Bagemihl) will suffice (see [2] or [3, p. 86]).

Although it is not essential to our main purpose, we note some interesting relationships between various types of cluster sets and certain of the sets K_S. Let $Z(f, e^{i\theta})$ denote the set of all points w of the Riemann sphere W for which there exists a sequence S of crosscuts of D converging to $e^{i\theta}$ such that K_S is the singleton set $\{w\}$.

THEOREM 3. *If f is a continuous function in D, then $Z(f, e^{i\theta}) \subset \Pi(f, e^{i\theta})$.*

Proof. Let w be such that $\{w\} = K_S$, where $S = \{s_n\}$ is a sequence of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$. If γ is any path to $e^{i\theta}$, then $\gamma \cap s_n \neq \phi$ for n sufficiently large, and hence $w \in C_\gamma(f, e^{i\theta})$. Thus $w \in \Pi(f, e^{i\theta})$ and $Z(f, e^{i\theta}) \subset \Pi(f, e^{i\theta})$.

Let $A(f, e^{i\theta})$ be the set of asymptotic values of f at $e^{i\theta}$, i.e., $A(f, e^{i\theta})$ is the set of all $w \in W$ for which there exists a path γ to $e^{i\theta}$ for which $C_\gamma(f, e^{i\theta})$ is the singleton set $\{w\}$.

THEOREM 4. *If f is a continuous function in D, then*

$$A(f, e^{i\theta}) \subset \cap K_S,$$

where the intersection is taken over all sequences S of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$.

Proof. Let $S=\{s_n\}$ be any sequence of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$ and let $w\in A(f,e^{i\theta})$. Then there exists a path γ to $e^{i\theta}$ such that $\{w\}=C_\gamma(f,e^{i\theta})$. But for n sufficiently large, $\gamma\cap s_n\neq\phi$ and hence $w\in K_S$. Thus, since S is arbitrary, $w\in\cap K_S$ and hence $A(f,e^{i\theta})\subset\cap K_S$.

That equality need not occur in either Theorems 3 or 4 is seen by the following example.

EXAMPLE 3. *There exists a function f continuous in D and a point $e^{i\theta}$ such that $Z(f,e^{i\theta})=\phi$, $A(f,e^{i\theta})=\phi$, $\Pi(f,e^{i\theta})$ is the closed interval* $[-1, 1]$, *and* $K_S=[-1, 1]$ *for each sequence S of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$.*

Proof. Let $B(z)$ be the conformal mapping of the disk D onto the upper half plane U such that $B(e^{i\theta})=0$. Let $F(z)=\sin 1/y$, where $z=x+iy\in U$. Then it is easy to verify that the function in D defined by $f(z)=F(B(z))$ has the desired properties.

4. Sufficient Conditions

In this section we investigate some sufficient conditions for there to exist a continuum which is a curvilinear cluster set for a continuous function and which is between two other specified continua.

THEOREM 5. *Let K be a permissible continuum for the continuous function f at $e^{i\theta}$ such that there exists a sequence S of crosscuts converging to $e^{i\theta}$ for which $K_S\subset K$. Then there exists a path γ to $e^{i\theta}$ such that*

$$K_S\subset C_\gamma(f,e^{i\theta})\subset K.$$

Proof. Let $\{G_n\}$ be a descending chain of open connected sets such that $K=\cap G_n$. Let $S=\{s_n\}$, $S_k^*=\{s_n\colon n\geqq k\}$ and let

$$H_n=\{z\in D\colon\ f(z)\in G_n \text{ and } |z-e^{i\theta}|<1/n\}.$$

We note that since $K_S\subset G_n$ for each n, then for a fixed k we have that $s_n\subset H_k$ for n sufficiently large. We claim that for each k there exists an integer n_k such that $S_{n_k}^*$ is contained in a single component of H_k.

Suppose the claim is false. Let C_n be the component of H_k containing s_n, and suppose there exists a number n' greater than n such that $s_{n'}$ is not contained in C_n. Then $D-H_k$ has a component which separates s_n from $s_{n'}$. But then there exists a crosscut which we will call s_n^* of D at $e^{i\theta}$ such that

$$f(s_n^*) \subset \{w\colon \mathscr{X}(w, W - G_k) < 1/n\}$$

where s_n^* lies between s_n and $s_{n'}$. Since s_n^* will exist for infinitely many different choices of n (where k is kept fixed) we have that $S^* = \{s_n^*\}$ is a sequence of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$ and $K_{S^*} \subset W - G_k$. Since $K \subset G_k$, we would have that $K_{S^*} \cap K = \phi$, in violation of our hypothesis that K is a permissible continuum for f at $e^{i\theta}$. Thus the claim is valid.

Let J_k be the component of H_k containing $S_{n_k}^*$. Since $H_{k+1} \subset H_k$, we have that $J_{k+1} \subset J_k$ and that $\cap \bar{J}_k = \{e^{i\theta}\}$. Let $\{z_k\}$ be a sequence of points such that $z_k \in J_k$ and $\{f(z_k)\}$ is dense in K_S. Since z_k and z_{k+1} are both in J_k, they can be joined by an arc γ_k contained in J_k. Let γ be the path resulting from the union of all the γ_k, where loops are eliminated, if necessary, such that γ is a path to $e^{i\theta}$ for which

$$K_S \subset C_\gamma(f, e^{i\theta}) \subset G_n$$

for each n. Thus we have

$$K_S \subset C_\gamma(f, e^{i\theta}) \subset K.$$

We note that we cannot guarantee the existence of a path γ to $e^{i\theta}$ such that $C_\gamma(f, e^{i\theta})$ will be precisely either K or K_S under the hypothesis of Theorem 5, as the following example shows.

EXAMPLE 4. *There exists a continuous function f in D, a permissible continuum K for f at $e^{i\theta}$, and a sequence S of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$ such that $K_S \subset K$ and neither K nor K_S is a curvilinear cluster set of f at $e^{i\theta}$.*

Proof. Let $B(z)$ be a conformal mapping of D onto the upper half plane U such that $B(e^{i\theta}) = \infty$. For each natural number n let

$$U_n = \{z \in U\colon n - 1/4 \leqq |z| \leqq n + 1/4,\ |z - n| \geqq 1/4\},$$
$$V_n = \{z \in U\colon |z - n| = 1/5\},$$
$$Y_n = \{z \in U\colon |z - n| \leqq 1/6\},$$

and

$$T_n = \{z \in U\colon |z| = n + 1/2\}.$$

By the Tietze Extension Theorem there exists a function F continuous in U such that $F(z) = \operatorname{Arg} z$ for $z \in U_n$, $F(z) = i$ for $z \in V_n$, $F(z) = 2\pi$ for $z \in T_n$, and $F(z) = -6 + 1/(|z - n|)$ for $z \in Y_n$, $|F(z)| \leqq 2\pi$ for $z \notin (\cup_{n=1}^{\infty} Y_n)$, and

$F(z)$ is real valued for each $z \in U$ such that $|z-n| \geqq 1/4$ for each natural number n.

Let K be the real line and let $S = \{s_n\}$ where

$$s_n = \{z \in U: \ |z| = n + 1/4\}.$$

Then since $F(z) = \text{Arg } z$ for $z \in s_n$ we have that $K_S = [0, \pi]$. But for any path γ to ∞, we have that $\gamma \cap T_n \neq \phi$ for n sufficiently large, so that $2\pi \in C_\gamma(F, \infty)$ and hence $K_S \neq C_\gamma(F, \infty)$. However, if $K \subset C_\gamma(F, \infty)$, then γ must meet infinitely many of the sets Y_n. But this means that γ must meet infinitely many of the sets V_n, so that $i \in C_\gamma(F, \infty)$. Thus $C_\gamma(F, \infty) \neq K$, and neither K nor K_S is a curvilinear cluster set of F at ∞. Setting $f(z) = F(B(z))$ we have the corresponding result for f at $e^{i\theta}$.

As an immediate result of Theorem 5, we obtain the following corollary.

COROLLARY. *If f is a continuous function in D and there exists a sequence S of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$ such that K_S is a permissible continuum for f at $e^{i\theta}$, then K_S is a curvilinear cluster set for f at $e^{i\theta}$.*

THEOREM 6. *Let f be a continuous function in D and let K be a continuum which is a curvilinear cluster set of f at $e^{i\theta}$. If S is a sequence of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$ such that $K_S - K \neq \phi$, then there exists a path γ to $e^{i\theta}$ such that*

$$K \subsetneqq C_\gamma(f, e^{i\theta}) \subset K \cup K_S.$$

Proof. Let $S = \{s_n\}$ and let $\{G_n\}$ be a descending chain of open connected sets such that $K_S = \cap G_n$. Let $H(n, k)$ be the component of $f^{-1}(G_n)$ such that s_k is contained in $H(n, k)$. For a fixed n and for k sufficiently large we have that $f(s_k) \subset G_n$ so that for each natural number n, $H(n, k)$ will exist for k sufficiently large.

Let Γ be a path to $e^{i\theta}$ such that $C_\Gamma(f, e^{i\theta}) = K$, let $w \in K_S - K$, and let $\{z_k\}$ be a sequence of points such that $z_k \in s_k$ and $f(z_k) \to w$. For each natural number n choose an integer k_n such that $H(n, k_n)$ exists, $k_n \geqq n$, and choose points p_n and q_n on $\Gamma \cap H(n, k_n)$ such that p_n comes before q_n on Γ, where Γ is oriented toward $e^{i\theta}$, and such that the chordal diameter in W of the image of Γ between p_n and q_n is less than $1/n$. It is possible to alter Γ by replacing the portions of Γ between the points p_n and q_n by arcs γ_n, where $\gamma_n \subset H(n, k_n)$ and $z_{k_n} \in \gamma_n$. Infinitely many replacements can be made so that Γ is altered into a curve γ such that $C_\Gamma(f, e^{i\theta}) \subset C_\gamma(f, e^{i\theta})$ and $w \in C_\gamma(f, e^{i\theta})$. Since all the new points of γ are contained in $H(n, k_n)$ for some integer n, and since $K_S = \cap G_n$, we have that

$$K \subsetneqq C_\gamma(f, e^{i\theta}) \subset K \cup K_S.$$

Thus γ is the desired path to $e^{i\theta}$.

We note that the method of the proof can be extended to construct a path γ such that $C_\gamma(f, e^{i\theta}) = K \cup K_S$, but we do not present the details here.

THEOREM 7. *Let f be a continuous function in D for which $\Pi(f, e^{i\theta}) \neq C(f, e^{i\theta})$. If there exists an ascending chain of permissible continua $\{K_n\}$ for f at $e^{i\theta}$ and a sequence $\{S_n\}$ of sequences of crosscuts of D at $e^{i\theta}$ converging to $e^{i\theta}$ such that K_1 is a curvilinear cluster set of f at $e^{i\theta}$ and for each natural number n, $K_{n+1} = K_n \cup K_{S_n}$, then there exists a sequence $\{K'_n\}$ of continua together with a sequence $\{\gamma_n\}$ of paths to $e^{i\theta}$ such that $K_n \subset K'_n \subset K_{n+1}$ and $C_{\gamma_n}(f, e^{i\theta}) = K'_n$.*

Theorem 7 follows immediately from Theorem 6 and is, in some sense, a clarification of Theorem 1.

REFERENCES

[1] F. Bagemihl and W. Seidel, *Some boundary properties of analytic functions*, Math. Z. **61** (1954), 186–199.

[2] ——————————, *Functions of bounded characteristic with prescribed ambiguous points*, Michigan Math. J. **3** (1955–56), 77–81.

[3] E.F. Collingwood and A.J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge University Press, 1966.

Michigan State University