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Erratum: The Duality Problem For The Class of AM-Compact Operators On Banach Lattices

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Abstract. It is proved that if a positive operator $S : E \to F$ is AM-compact whenever its adjoint $S' : F' \to E'$ is AM-compact, then either the norm of F is order continuous or E' is discrete.

This note corrects an error in the proof of Theorem 2.3 of B. Aqzzouz, R. Nouira, and L. Zraoula, *The duality problem for the class of AM-compact operators on Banach lattices.* Canad. Math. Bull. **51**(2008).

An operator T from a Banach lattice E into a Banach space X is called AM-compact if $T[-x, x]$ is norm relatively compact for every $x \in E^+$. Hence, the operator $T\colon E \to X$ is AM-compact if and only if for every order bounded sequence (x_n) of E, the sequence $(T(x_n))$ has a norm convergent subsequence in X. The class of AM-compact operators has a shortcoming. In fact, there exist AM-compact operators whose adjoints are not AM-compact, and conversely, there exist operators that are not AM-compact but their adjoints are AM-compact. This problem was studied in [3] as a continuation of the study begun by Zaanen in [8]. However, an error occurred in the demonstration of [3, Theorem 2.3]. In fact, in the proof of this theorem, we used [1, Corollary 21.13] to confirm the existence of $\phi \in (E')^+$ and a sequence (ϕ_n) in $[0, \phi]$, which converges to 0 for the weak topology $\sigma(E', E)$ but does not converge to 0 for the absolute weak topology $|\sigma|(E', E)$. But this is not correct and gives a contradiction. Namely, in this situation such a sequence also converges to 0 for the absolute weak topology $|\sigma|(E', E)$. Indeed, $|\sigma|(E', E)$ is generated by the family of lattice seminorms $\{P_x\colon x \in E\}$, where $P_x(f) = |f|(|x|)$ for each $f \in E'$. Since $\phi_n \to 0$ weakly, $P_x(\phi_n) = \phi_n(|x|) \to 0$ for each $x \in E$. Hence (ϕ_n) converges to 0 for $|\sigma|(E', E)$. More generally, if (ϕ_n) is a positive sequence of E', then $\phi_n \to 0$ for $\sigma(E', E)$ if and only if $\phi_n \to 0$ for $|\sigma|(E', E)$.

The objective of this note is to give a new and correct demonstration for this result. For unexplained terminology on Banach lattices and positive operator theory, we refer the reader to [2].

To give our new and correct proof of [3, Theorem 2.3], we need to recall [4, Lemma 3.4].

Lemma 1 ([4]) *Let $(E, \|\cdot\|)$ be a Banach lattice. If (x_n) is a positive disjoint sequence of E such that $\|x_n\| = 1$ for all n, then there exists a positive disjoint sequence (g_n) of E' such that $\|g_n\| \leq 1$, $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.*

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Let E be a Banach lattice, and let $u \in E^+$. Then the order ideal E_u generated by u and endowed with the norm $\|y\|_\infty = \inf\{\lambda > 0 \colon |y| \leq \lambda.u\}$ is an AM-space having u as unit and $[-u, u]$ as a closed unit ball, and the embedding $i_u \colon (E_u, \|\cdot\|_\infty) \to E$ is continuous. Moreover, for every $f \in E'$ we have $f \circ i_u \in (E_u)'$ and

$$\|f \circ i_u\|_{(E_u)'} = \sup\{|(f \circ i_u)(y)| \colon y \in [-u, u]\} = \sup\{|f(y)| \colon |y| \leq u\} = |f|(u).$$

An operator $T \colon E \to X$ from a Banach lattice E into a Banach space X is AM-compact if and only if for every $x \in E^+$ the composed operator $T \circ i_x \colon E_x \to E \to X$ is compact, where $i_x \colon E_x \to E$ is the natural embedding.

Now, we are in position to give a correct proof of [3, Theorem 2.3].

Theorem 2 *Let E and F be two Banach lattices. If each positive operator $S \colon E \to F$ is AM-compact whenever its adjoint $S' \colon F' \to E'$ is AM-compact, then one of the following statements is valid:*

(i) *the norm of F is order continuous;*

(ii) *E' is discrete.*

Proof Assume by way of contradiction that both the conditions (i) and (ii) fail. Since the norm of F is not order continuous, it follows from Meyer-Nieberg [6, Theorem 2.4.2] that there exists of $y \in F^+$ and a disjoint sequence (y_n) in F such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all n. Hence, by Lemma 1 there exists a positive disjoint sequence (g_n) of F' such that

$$(1) \qquad \|g_n\| \leq 1, \quad g_n(y_n) = 1 \quad \text{for all } n \quad \text{and} \quad g_n(y_m) = 0 \text{ for } n \neq m.$$

On the other hand, if E' is not discrete, Chen–Wickstead [5, Theorem 3.1] implies the existence of a sequence $(f_n) \subset E'$ such that $f_n \to 0$ for $\sigma(E', E)$ and $|f_n| = f > 0$ for all n and some $f \in E'$.

Now, we consider the operators $S, T \colon E \to F$ defined by

$$S(x) = \left(\sum_{n=1}^{\infty} f_n(x) y_n\right) + f(x)y \quad \text{and} \quad T(x) = 2f(x)y \quad \text{for all } x \in E.$$

Note that the sum in the definition of S is norm convergent for each $x \in E$, because $f_n(x) \to 0$ and the sequence (y_n) is disjoint and order bounded. To finish the proof, we have to prove that the positive operator $S \colon E \to F$ is not AM-compact and its adjoint $S' \colon F' \to E'$ is AM-compact.

First, we prove that S is not AM-compact. Choose $u \in E^+$ such that $f(u) > 0$, and note that $(f_n \circ i_u)_n$ has no norm convergent subsequence in $(E_u)'$. In fact, for each $y \in E_u$, we have $f_n \circ i_u(y) = f_n(y) \to 0$ as $n \to \infty$. Then $f_n \circ i_u \to 0$ for $\sigma((E_u)', E_u)$. As $\|f_n \circ i_u\|_{(E_u)'} = |f_n|(u) = f(u) > 0$ for all n, we conclude that $(f_n \circ i_u)$ has no norm convergent subsequence in $(E_u)'$. If S is AM-compact, then the operator $S \circ i_u \colon E_u \to E \to F$ is compact and so is its adjoint $(S \circ i_u)'$. We obtain

$$(S \circ i_u)'(g) = \left(\sum_{n=1}^{\infty} g(y_n) \cdot (f_n \circ i_u)\right) + g(y) \cdot (f \circ i_u) \quad \text{for all } g \in F'.$$

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And, by (1), we have

$$(S \circ i_u)'(g_k) = (f_k \circ i_u) + g_k(y) \cdot (f \circ i_u) \quad \text{for all } k.$$

Hence, $((S \circ i_u)'(g_k))_k$ has a norm convergent subsequence in $(E_u)'$. Since

$$(g_k(y))_k \subset [-\|y\|, \|y\|] \subset \mathbb{R}$$

has a convergent subsequence (because it is a bounded sequence in $\mathbb{R}$), we conclude that $(f_k \circ i_u)_k$ has a convergent subsequence in $(E_u)'$. This is a contradiction, so S is not AM-compact.

Second, we prove that the adjoint S' is AM-compact. For this, we consider the operators $S_1 \colon E \to c_0$, $S_2 \colon c_0 \to F$ and $S_3 \colon E \to F$ defined by

$$S_1(x) = (f_n(x))_n, \quad S_2((a_n)) = \sum_{n=1}^{\infty} a_n y_n, \quad \text{and} \quad S_3(x) = f(x)y$$

for all $x \in E$ and all $(a_n) \in c_0$. Clearly, $S = (S_2 \circ S_1) + S_3$, and hence $S' = ((S_1)' \circ (S_2)') + (S_3)'$. It is clear that S_3 is compact (it has rank one). Then $(S_3)'$ is compact, and hence $(S_3)'$ is AM-compact. Since $S_2 \colon c_0 \to F$ is positive, its adjoint $(S_2)' \colon F' \to l^1$ is also positive. Now, as l^1 is discrete and its norm is order continuous, the regular operator $(S_2)'$ is AM-compact. In fact, $(S_2)'$ maps order intervals of F' to order bounded subsets of l^1 that are norm relatively compact ([7, Theorem 6.1]). Hence, $((S_1)' \circ (S_2)')$ is AM-compact. Finally, we conclude that $S' = ((S_1)' \circ (S_2)') + (S_3)'$ is AM-compact. ■

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