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Dimension estimates for badly approximable affine forms

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Abstract. For given $\epsilon > 0$ and $b \in \mathbb{R}^m$, we say that a real $m \times n$ matrix A is ϵ-badly approximable for the target b if

$$\liminf_{q \in \mathbb{Z}^n, \|q\| \to \infty} \|q\|^n \langle Aq - b \rangle^m \geq \epsilon,$$

where $\langle \cdot \rangle$ denotes the distance from the nearest integral vector. In this article, we obtain upper bounds for the Hausdorff dimensions of the set of ϵ-badly approximable matrices for fixed target b and the set of ϵ-badly approximable targets for fixed matrix A. Moreover, we give a Diophantine condition of A equivalent to the full Hausdorff dimension of the set of ϵ-badly approximable targets for fixed A. The upper bounds are established by effectivizing entropy rigidity in homogeneous dynamics, which is of independent interest. For the A-fixed case, our method also works for the weighted setting where the supremum norms are replaced by certain weighted quasinorms.

Key words: Diophantine approximation, effective, Hausdorff dimension, badly approximable
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1. *Introduction*

In classical Diophantine approximation, one wants to approximate an irrational number α by rationals p/q for $p, q \in \mathbb{Z}$. Dirichlet theorem says that for every $N \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ with $0 < q < N$, such that

$$|q\alpha - p| < 1/N < 1/q.$$

In this way, one can see classical Diophantine approximation as studying distribution of $q\alpha$ modulo $\mathbb{Z}$ near zero. Diophantine approximation for irrational numbers has been generalized to investigating vectors, linear forms, and more generally matrices, and have become classical subjects in metric number theory.

In this article, we consider the inhomogeneous Diophantine approximation: the distribution of $q\alpha$ modulo $\mathbb{Z}$ near a 'target' $b \in \mathbb{R}$. Although Dirichlet theorem does not hold anymore, there exist infinitely many $q \in \mathbb{Z}$ such that

$$|q\alpha - b - p| < 1/|q| \quad \text{for some } p \in \mathbb{Z}$$

for almost every $(\alpha, b) \in \mathbb{R}^2$ and, moreover,

$$\liminf_{p,q\in\mathbb{Z},|q|\to\infty} |q||q\alpha - b - p| = 0$$

for almost every $(\alpha, b) \in \mathbb{R}^2$ by inhomogeneous Khintchine theorem [Cas57, Theorem II in Ch. VII].

Similarly to numbers, for an $m \times n$ real matrix $A \in M_{m,n}(\mathbb{R})$, we study $Aq \in \mathbb{R}^m$ modulo $\mathbb{Z}^m$ near the target $b \in \mathbb{R}^m$ for vectors $q \in \mathbb{Z}^n$. In this general situation as well, using inhomogeneous Khintchine–Groshev theorem ([Sch64, Theorem 1] or [Spr79, Ch. 1, Theorem 15]), we have

$$\liminf_{q\in\mathbb{Z}^n,\|q\|\to\infty} \|q\|^n \langle Aq - b\rangle^m = 0$$

for almost every $(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$. Here, $\langle v\rangle \overset{\text{def}}{=} \inf_{p\in\mathbb{Z}^m} \|v - p\|$ denotes the distance from $v \in \mathbb{R}^m$ to the nearest integral vector with respect to the supremum norm $\|\cdot\|$.

The exceptional set of the above equality is our object of interest.

1.1. *Main results.* We will consider the exceptional set with weights in the following sense. Let us first fix, throughout the paper, an m-tuple and an n-tuple of positive reals $\mathbf{r} = (r_1, \ldots, r_m)$, $\mathbf{s} = (s_1, \ldots, s_n)$ such that $r_1 \geq \cdots \geq r_m$, $s_1 \geq \cdots \geq s_n$, and $\sum_{1\leq i\leq m} r_i = 1 = \sum_{1\leq j\leq n} s_j$. The special case where $r_i = 1/m$ and $s_j = 1/n$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$ is called the unweighted case.

Define the $\mathbf{r}$-quasinorm of $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{s}$-quasinorm of $\mathbf{y} \in \mathbb{R}^n$ by

$$\|\mathbf{x}\|_{\mathbf{r}} \overset{\text{def}}{=} \max_{1\leq i\leq m} |x_i|^{1/r_i} \quad \text{and} \quad \|\mathbf{y}\|_{\mathbf{s}} \overset{\text{def}}{=} \max_{1\leq j\leq n} |y_j|^{1/s_j}.$$

Denote $\langle \mathbf{x}\rangle_{\mathbf{r}} \overset{\text{def}}{=} \inf_{p\in\mathbb{Z}^m} \|\mathbf{x} - p\|_{\mathbf{r}}$. We call A, ϵ-*bad* for $b \in \mathbb{R}^m$ if

$$\liminf_{q\in\mathbb{Z}^n,\|q\|_{\mathbf{r}}\to\infty} \|q\|_{\mathbf{s}} \langle Aq - b\rangle_{\mathbf{r}} \geq \epsilon.$$

Denote

$$\mathbf{Bad}(\epsilon) \overset{\text{def}}{=} \{(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m : A \text{ is } \epsilon\text{-bad for } b\},$$

$$\mathbf{Bad}_A(\epsilon) \overset{\text{def}}{=} \{b \in \mathbb{R}^m : A \text{ is } \epsilon\text{-bad for } b\}, \quad \mathbf{Bad}_A \overset{\text{def}}{=} \bigcup_{\epsilon>0} \mathbf{Bad}_A(\epsilon),$$

$$\mathbf{Bad}^b(\epsilon) \overset{\text{def}}{=} \{A \in M_{m,n}(\mathbb{R}) : A \text{ is } \epsilon\text{-bad for } b\}, \quad \mathbf{Bad}^b \overset{\text{def}}{=} \bigcup_{\epsilon>0} \mathbf{Bad}^b(\epsilon).$$

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The set $\mathbf{Bad}^0$ can be seen as the set of badly approximable systems of m linear forms in n variables. This set is of Lebesgue measure zero [Gro38], but has full Hausdorff dimension mn [Sch69]. See [KTV06, KW10, PV02] for the weighted setting.

For any b, $\mathbf{Bad}^b$ also has zero Lebesgue measure [Sch66] and full Hausdorff dimension for every b [ET11]. Indeed, it is shown that $\mathbf{Bad}^b$ is a winning set [ET11] and even a hyperplane winning set [HKS20], a property which implies full Hausdorff dimension. However, the set $\mathbf{Bad}_A$ also has full Hausdorff dimension for every A [BHKV10]. See [BM17, Har12, HM17] for the weighted setting.

The sets $\mathbf{Bad}^b$ and $\mathbf{Bad}_A$ are unions of subsets $\mathbf{Bad}^b(\epsilon)$ and $\mathbf{Bad}_A(\epsilon)$ over $\epsilon > 0$, respectively, and thus a more refined question is whether the Hausdorff dimension of $\mathbf{Bad}^b(\epsilon)$, $\mathbf{Bad}_A(\epsilon)$ could still be of full dimension. For the homogeneous case ($b = 0$), the Hausdorff dimension $\mathbf{Bad}^0(\epsilon)$ is less than the full dimension mn (see [BK13, Sim18] for the unweighted case and [KM19] for the weighted case). Thus, a natural question is whether $\mathbf{Bad}^b(\epsilon)$ can have full Hausdorff dimension for some b. Our first main result says that in the unweighted case, $\mathbf{Bad}^b(\epsilon)$ cannot have full Hausdorff dimension for any b. We provide an effective bound on the dimension in terms of ϵ as well.

THEOREM 1.1. *For the unweighted case, that is, $r_i = 1/m$ and $s_j = 1/n$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, there exist $c_0 > 0$ and $M_0 > 0$ depending only on d such that for any $\epsilon > 0$ and $b \in \mathbb{R}^m$,*

$$\dim_H \mathbf{Bad}^b(\epsilon) \le mn - c_0\epsilon^{M_0}.$$

As for the set $\mathbf{Bad}_A(\epsilon)$, the third author, together with U. Shapira and N. de Saxcé, showed that Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ is less than the full dimension m for almost every A [LSS19]. In fact, it was shown that one can associate to A a certain point x_A in the space of unimodular lattices $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ such that if x_A has no escape of mass on average for a certain diagonal flow (see §1.2 for more details), which is satisfied by almost every point, then the Hausdorff dimension of $\mathbf{Bad}_A(\epsilon)$ is less than m.

In this article, we provide an effective bound on the dimension in terms of ϵ and a certain Diophantine property of A as follows. We say that an $m \times n$ matrix A is *singular on average* if for any $\epsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N}|\{l \in \{1, \ldots, N\} : \text{ there exists } q \in \mathbb{Z}^n \text{ such that } \langle Aq\rangle_{\mathbf{r}} < \epsilon 2^{-l} \text{ and } 0 < \|q\|_{\mathbf{s}} < 2^l\}| = 1.$$

THEOREM 1.2. *For any $A \in M_{m,n}(\mathbb{R})$ which is not singular on average, there exists a constant $c(A) > 0$ depending on A such that for any $\epsilon > 0$, $\dim_H \mathbf{Bad}_A(\epsilon) \le m - c(A)\epsilon/\log(1/\epsilon)$.*

Here, the constant $c(A)$, which depends on η_A in Proposition 4.1 and H in equation (4.7), encodes the quantitative singularity on average.

However, the third author, together with Y. Bugeaud, D. H. Kim, and M. Rams, showed that in the one-dimensional case ($m = n = 1$), $\mathbf{Bad}_\alpha(\epsilon)$ has full Hausdorff dimension for

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some $\epsilon > 0$ if and only if $\alpha \in \mathbb{R}$ is singular on average [**BKLR21**]. We generalize this characterization to the general dimensional setting.

THEOREM 1.3. *Let $A \in M_{m,n}(\mathbb{R})$ be a matrix. Then the following are equivalent.*

(1) *For some $\epsilon > 0$, the set $\mathbf{Bad}_A(\epsilon)$ has full Hausdorff dimension.*

(2) *A is singular on average.*

Note that the implication (1) $\implies$ (2) of Theorem 1.3 follows from Theorem 1.2. The other direction will be shown in §6.

1.2. *Idea of the proofs.* We mainly use entropy rigidity in homogeneous dynamics, a principle that the measure of maximal entropy is invariant for a suitable group [**EL10**]. The main tool in [**LSS19**] is a relative version of entropy rigidity. In this article, we effectivize this phenomenon (Theorem 2.12) in terms of the static entropy and conditional measures. To use the effective version of the entropy rigidity, for each invariant measure, we construct a 'well-behaved' partition and a σ-algebra, well-behaved in the sense that the 'dynamical δ-boundary' has a small measure which is controlled uniformly (see Definition 2.6 and Lemma 2.7). We then compare the associated dynamical entropy and the static entropy. Section 2 consists of these results in the general setting of real Lie groups as in [**EL10**], which are of independent interest.

To describe the scheme of the proofs for main theorems, we consider a more specific homogeneous space as follows. For $d = m + n$, let us denote by $\mathrm{ASL}_d(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R}) \ltimes \mathbb{R}^d$ the set of area-preserving affine transformations and denote by $\mathrm{ASL}_d(\mathbb{Z}) = \mathrm{SL}_d(\mathbb{Z}) \ltimes \mathbb{Z}^d = \mathrm{Stab}_{\mathrm{ASL}_d(\mathbb{R})}(\mathbb{Z}^d)$ the stabilizer of the standard lattice $\mathbb{Z}^d$. We view $\mathrm{ASL}_d(\mathbb{R})$ as a subgroup of $\mathrm{SL}_{d+1}(\mathbb{R})$ by $\mathrm{ASL}_d(\mathbb{R}) = \{\left(\begin{smallmatrix} g & v \\ 0 & 1 \end{smallmatrix}\right) : g \in \mathrm{SL}_d(\mathbb{R}), v \in \mathbb{R}^d\}$,and take a lift of the element $g \in \mathrm{SL}_d(\mathbb{R})$ to $\mathrm{ASL}_d(\mathbb{R}) \subset \mathrm{SL}_{d+1}(\mathbb{R})$ by $g \longmapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$,denoted again by g. For given weights $\mathbf{r} \in \mathbb{R}^m_{>0}$ and $\mathbf{s} \in \mathbb{R}^n_{>0}$, we consider the 1-parameter diagonal subgroup

$$\{a_t = \mathrm{diag}(e^{r_1 t}, \ldots, e^{r_m t}, e^{-s_1 t}, \ldots, e^{-s_n t})\}_{t \in \mathbb{R}}$$

in $\mathrm{SL}_d(\mathbb{R})$ and let $a \overset{\mathrm{def}}{=} a_1$ be the time-one map of the diagonal flow a_t. We consider

$$U = \left\{ \begin{pmatrix} I_m & A & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} : A \in M_{m,n}(\mathbb{R}) \right\}; \quad W = \left\{ \begin{pmatrix} I_m & 0 & b \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{R}^m \right\},$$

both of which are unstable horospherical subgroups in $\mathrm{ASL}_d(\mathbb{R})$ for a.

The homogeneous spaces $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ and $\mathrm{ASL}_d(\mathbb{R})/\mathrm{ASL}_d(\mathbb{Z})$ can be seen as the space of unimodular lattices and the space of unimodular grids, that is, unimodular lattices translated by a vector in $\mathbb{R}^d$, respectively. We say that a point $x \in \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ has *δ-escape of mass on average* (with respect to the diagonal flow a_t) if for any compact set Q in $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$,

$$\liminf_{N\to\infty} \frac{1}{N} |\{\ell \in \{1, \ldots, N\} : a_\ell x \notin Q\}| \geq \delta.$$

A point $x \in X$ has *no escape of mass on average* if it does not have δ-escape of mass on average for any $\delta > 0$.

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For $A \in M_{m,n}(\mathbb{R})$ and $(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$, we associate points

$$x_A \overset{\text{def}}{=} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \mathrm{SL}_d(\mathbb{Z}) \quad \text{and} \quad y_{A,b} \overset{\text{def}}{=} \begin{pmatrix} I_m & A & -b \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{ASL}_d(\mathbb{Z}),$$

respectively. In [**LSS19**], it was shown that $\dim_H \mathbf{Bad}_A(\epsilon) < m$ for all $\epsilon > 0$ if x_A is *heavy*, which is a condition equivalent to no escape of mass on average. Note that x_A is heavy for almost every $A \in M_{m,n}(\mathbb{R})$. However, we remark that A is singular on average if and only if the corresponding point x_A has 1-escape of mass on average (with respect to the diagonal flow a_t) by Dani's correspondence (see also [**KKLM17**]).

Now we give the outline of the proofs for Theorems 1.1 and 1.2. From the Dani correspondence, we characterize the Diophantine property $(A, b) \in \mathbf{Bad}(\epsilon)$ by the dynamical property that the orbit $(a_t y_{A,b})_{t\geq 0}$ is eventually in some target $\mathcal{L}_\epsilon$ (see §3.2). Using this characterization, we construct a-invariant measures with large dynamical entropies relative to W and U (Propositions 4.1 and 5.4), which are related to the Hausdorff dimensions of $\mathbf{Bad}_A(\epsilon)$ and $\mathbf{Bad}^b(\epsilon)$, respectively. Here, we use 'well-behaved' σ-algebra constructed in Proposition 2.8. Then we associate the dynamical entropies with the static entropies (Lemma 2.10). Finally, we obtain effective upper bounds for the Hausdorff dimensions of $\mathbf{Bad}_A(\epsilon)$ and $\mathbf{Bad}^b(\epsilon)$ using an effective version of the variational principle (Theorem 2.12).

To treat $\mathbf{Bad}_A(\epsilon)$ and $\mathbf{Bad}^b(\epsilon)$ at the same time, we need to consider the entropy relative to an arbitrary expanding closed subgroup L normalized by a, which is more general than [**LSS19**]; in [**LSS19**], the special case $L = W$ whose orbits stay in the compact fiber of $\mathrm{ASL}_d(\mathbb{R})/\mathrm{ASL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ is considered.

For $\mathbf{Bad}_A(\epsilon)$, we treat the case when x_A has some escape of mass on average as well, whereas x_A has no escape of mass on average in [**LSS19**]. We need to consider $\mathcal{L}_\epsilon \subset \mathrm{ASL}_d(\mathbb{R})/\mathrm{ASL}_d(\mathbb{Z})$, which is non-compact, whereas in [**LSS19**], for heavy x_A, it was enough to consider the set of fibers over a compact part of $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. In the case of $\mathbf{Bad}^b(\epsilon)$, as fixing b does not determine the amount of excursion in the cusp, we need an additional step (Proposition 5.3) to control the measure near the cusp allowing a small amount of escape of mass.

Another new feature of this article is the use of the effective equidistribution of expanding translates under the diagonal action on $\mathrm{ASL}_d(\mathbb{R})/\mathrm{ASL}_d(\mathbb{Z})$ and $\mathrm{SL}_d(\mathbb{R})/\Gamma_q$, where Γ_q is a congruence subgroup of $\mathrm{SL}_d(\mathbb{Z})$, in the case of $\mathbf{Bad}^b(\epsilon)$. The former result is proved by the second author in [**Kim**], and the latter result is a slight modification of [**KM23**].

Note that [**Kim**, **KM23**] hold in the weighted setting and the only reason we consider the unweighted setting for the Hausdorff dimension of $\mathbf{Bad}^b(\epsilon)$ is the covering estimate in Theorem 5.1 ([**KKLM17, Theorem 1.5**]).

The article is organized as follows. In §2, we introduce entropy, relative entropy, and a general setup. In this general setup, we construct a partition with a well-behaved 'dynamical δ-boundary' and a σ-algebra in a quantitative sense. From this construction, we compare the dynamical entropy and the static entropy. Finally, we prove an effective

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version of the variational principle for relative entropy in the spirit of [EL10, §7.55]. In §3, we introduce preliminaries for the proofs of dimension upper bounds including properties of dimensions with respect to quasi-metrics. We also reduce badly approximable properties to dynamical properties in the space of grids in $\mathbb{R}^{m+n}$. In §§4 and 5, we construct a-invariant measures on $\mathrm{ASL}_d(\mathbb{R})/\mathrm{ASL}_d(\mathbb{Z})$ with large relative entropy and estimate dimension upper bounds of Theorems 1.2 and 1.1 using the effective variational principle. We conclude the paper with §6, characterizing the singular on average property in terms of best approximations and show the (2) $\implies$ (1) part in Theorem 1.3 using a modified version of the Bugeaud–Laurent sequence in [BL05].

2. *Effective version of entropy rigidity*

In this section, we will establish an effective version of entropy rigidity in [EL10, §7]. There have been effective uniqueness results along the line of [EL10] in various settings: [Pol11] for toral automorphisms, [Kad15] for hyperbolic maps on Riemannian manifolds, [Rüh16] on p-adic homogeneous spaces, and [Kha17] for a p-adic diagonal action in the S-arithmetic setting. However, in all of the above results as well as in [KLP23], there exists a partition compatible with the given map or flow in the sense that images under the iteration have boundaries of small measure with respect to any invariant measure of interest.

In our setting of a diagonal action on a quotient of real Lie groups, one of the main technical difficulties is that there is no such partition for all the invariant measures we consider. We thus construct a partition $\mathcal{P}$ for each invariant measure μ and control the μ-measure of its 'dynamical δ-boundary' E_δ constructed out of images of thickenings of the boundary $\mathcal{P}$. The value $\mu(E_\delta)$ is bounded above uniformly over the partition $\mathcal{P}$ and the measure μ. See Lemma 2.7.

2.1. *Entropy and relative entropy.*

In this subsection, we recall the definitions of the entropy and the relative entropy for σ-algebras which we use in the later sections. We refer the reader to [ELW, Chs. 1 and 2] for basic properties of the entropy.

Definition 2.1. Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system on a Borel probability space, and let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be sub-σ-algebras. Suppose that $\mathcal{C}$ is countably generated. Note that there exists an $\mathcal{A}$-measurable conull set $X' \subset X$ and a system $\{\mu_x^{\mathcal{A}} | x \in X'\}$ of measures on X, referred to as *conditional measures*, given for instance by [ELW, Theorem 2.2]. The *information function* of $\mathcal{C}$ given $\mathcal{A}$ with respect to μ is defined by

$$I_\mu(\mathcal{C}|\mathcal{A})(x) = -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{C}}),$$

where $[x]_{\mathcal{C}}$ is the atom of $\mathcal{C}$ containing x.

(1) The *conditional (static) entropy of* $\mathcal{C}$ given $\mathcal{A}$ is defined by

$$H_\mu(\mathcal{C}|\mathcal{A}) \stackrel{\text{def}}{=} \int_X I_\mu(\mathcal{C}|\mathcal{A})(x)\, d\mu(x),$$

which is the average of the information function $\mathcal{C}$ given $\mathcal{A}$. If the σ-algebra $\mathcal{A}$ is trivial, then we denote by $H_\mu(\mathcal{C}) = H_\mu(\mathcal{C}|\mathcal{A})$, which is called the *(static) entropy*

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of $\mathcal{C}$. Note that the entropy of the countable partition $\xi = \{A_1, A_2, \ldots\}$ of X is given by

$$H_\mu(\xi) = H(\mu(A_1), \ldots) = -\sum_{i \geq 1} \mu(A_i) \log \mu(A_i) \in [0, \infty],$$

where $0 \log 0 = 0$.

(2) Let $\mathcal{A} \subseteq \mathcal{B}$ be a sub-σ-algebra such that $T^{-1}\mathcal{A} = \mathcal{A}$. For any countable partition ξ of X, let

$$h_\mu(T, \xi) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_0^{n-1}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\xi_0^{n-1}),$$

$$h_\mu(T, \xi|\mathcal{A}) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_0^{n-1}|\mathcal{A}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\xi_0^{n-1}|\mathcal{A}),$$

where $\xi_0^{n-1} = \bigvee_{i=0}^{n-1} T^{-i}\xi$. The *(dynamical) entropy of T* is

$$h_\mu(T) \stackrel{\text{def}}{=} \sup_{\xi : H_\mu(\xi)<\infty} h_\mu(T, \xi).$$

The *conditional (dynamical) entropy of T given* $\mathcal{A}$ is

$$h_\mu(T|\mathcal{A}) \stackrel{\text{def}}{=} \sup_{\xi : H_\mu(\xi)<\infty} h_\mu(T, \xi|\mathcal{A}).$$

2.2. *General setup.* Let G be a closed real linear group (or connected, simply connected real Lie group) and let $\Gamma < G$ be a lattice subgroup. We consider the quotient $Y = G/\Gamma$ with a G-invariant probability measure m_Y and call it the Haar measure on Y. Let d_G be a right invariant metric on G, which induces the metric d_Y on the space $Y = G/\Gamma$, which is locally isometric to G. Let r_y be the maximal injectivity radius at $y \in Y$, which is the supremum of $r > 0$ such that the map $g \mapsto gy$ is an isometry from the open r-ball B_r^G around the identity in G onto the open r-ball $B_r^Y(y)$ around $y \in Y$. For any $r > 0$, we denote

$$Y(r) \stackrel{\text{def}}{=} \{y \in Y : r_y \geq r\}.$$

It follows from the continuity of the injectivity radius that $Y(r)$ is compact. Since Γ is a lattice, we may assume that

$$r_{\max} \stackrel{\text{def}}{=} \inf\{r > 0 : r_y \leq r \text{ for all } y \in Y\} \leq 1 \tag{2.1}$$

by rescaling the right invariant metric d_G on G. It follows that for any $r > 1$, $Y(r) = \varnothing$. For any closed subgroup $L < G$, we consider the right invariant metric d_L by restricting d_G on L, and similarly denote by B_r^L the open r-ball around the identity in L.

In this section, we fix an element $a \in G$ which is Ad-diagonalizable over $\mathbb{R}$. Let

$$G^+ = \{g \in G | a^k g a^{-k} \to \text{id as } k \to -\infty\}$$

be the unstable horospherical subgroup associated to a (or equivalently the stable horospherical subgroup associated to a^{-1}), which is always a closed subgroup of G in our setting.

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2.3. *Construction of a^{-1}-descending, subordinate algebra and its entropy properties.* In this subsection, our goal is to strengthen the results of [**EL10**, §7] for our quantitative purposes.

Definition 2.2. [**EL10**, Definition 7.25] Let $G^+ < G$ be the unstable horospherical subgroup associated to a. Let μ be an a-invariant measure on Y and $L < G^+$ be a closed subgroup normalized by a.

(1) We say that a countably generated σ-algebra $\mathcal{A}$ is *subordinate to* L (mod μ) if for μ-almost every (a.e.) y, there exists $\delta > 0$ such that

$$B_\delta^L \cdot y \subset [y]_{\mathcal{A}} \subset B_{\delta^{-1}}^L \cdot y. \tag{2.2}$$

(2) We say that $\mathcal{A}$ is a^{-1}*-descending* if $(a^{-1})^{-1}\mathcal{A} = a\mathcal{A} \subseteq \mathcal{A}$.

For each $L < G^+$ and a-invariant ergodic probability measure μ on Y, there exists a countably generated σ-algebra $\mathcal{A}$ which is a^{-1}-descending and subordinate to L [**EL10**, Proposition 7.37]. We will prove that such a σ-algebra can be constructed so that we also have an explicit upper bound of the measure of the set violating equation (2.2) for fixed $\delta > 0$. To prove an effective version of the variational principle later, we need this quantitative estimate independent of μ.

We first introduce some notation that will be used in this subsection. For a subset $B \subset Y$ and $\delta > 0$, we denote by $\partial_\delta B$ the δ-neighborhood of the boundary of B, that is,

$$\partial_\delta B \overset{\text{def}}{=} \left\{ y \in Y : \inf_{z \in B} d_Y(y, z) + \inf_{z \notin B} d_Y(y, z) < \delta \right\}.$$

We also define the neighborhood of the boundary of a countable partition $\mathcal{P}$ by $\partial_\delta \mathcal{P} \overset{\text{def}}{=} \bigcup_{P \in \mathcal{P}} \partial_\delta P$. We deal with the entropy with respect to a^{-1}, and thus for a given partition (or a σ-algebra) $\mathcal{P}$ of Y,

$$\mathcal{P}_\ell^{\ell'} \overset{\text{def}}{=} \bigvee_{k=\ell}^{\ell'} a^k \mathcal{P}$$

for any extended integers $\ell \leq \ell'$ in $\mathbb{Z} \cup \{\pm\infty\}$. We first construct a finite partition which has small measures on neighborhoods of the boundary. The following lemma is the main ingredient of the effectivization in this section. A key feature is that the measure estimate below is independent of μ.

Lemma 2.3. *There exists a constant $0 < c < 1/10$ depending only on G such that the following holds. Let μ be a probability measure on Y. For any $0 < r < 1$ and any measurable subset $\Omega \subset Y(2r)$, there exist a measurable subset $K \subset Y$ and a partition $\mathcal{P} = \{P_1, \ldots, P_N\}$ of K such that:*

(1) $\Omega \subseteq K \subseteq B_{(11/10)r}^G \Omega$;

(2) *for each $1 \leq i \leq N$, there exists $z_i \in B_{r/10}^G \Omega$ such that*

$$B_{r/5}^G \cdot z_i \subseteq P_i \subseteq B_r^G \cdot z_i, \quad K = \bigcup_{i=1}^N B_r^G \cdot z_i;$$

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(3) *for any* $0 < \delta < cr$,

$$\mu(\partial_\delta \mathcal{P}) \leq \left(\frac{\delta}{r}\right)^{1/2} \mu(B^G_{(12/10)r}\Omega).$$

Proof. Choose a maximal $(9/10)r$-separated set $\{y_1, \ldots, y_N\}$ of Ω.

CLAIM. *There exist a constant $0 < c < 1/10$ depending only on G, and $\{g_i\}_{i=1}^N \subset B^G_{r/10}$ such that for $z_i = g_i y_i$ and for any $0 < \delta < cr$,*

$$\sum_i (\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{r/2} \cdot z_i))) \leq \left(\frac{\delta}{r}\right)^{1/2} \mu(B^G_{(12/10)r}\Omega). \tag{2.3}$$

Proof of the claim. To prove this claim, we randomly choose each g_i with the independent uniform distribution on $B^G_{r/10}$. For $0 < \delta < r/10$ fixed, we have

$$\begin{aligned}
&\mathbb{E}\bigg(\sum_i \mu(\partial_\delta(B^G_r \cdot z_i))\bigg) \\
&\quad = \sum_i \frac{1}{m_G(B^G_{r/10})} \int_{B^G_{r/10}} \int_Y \mathbb{1}_{B^G_{r+\delta}\cdot g_i y_i \setminus B^G_{r-\delta}\cdot g_i y_i}(y)\, d\mu(y)\, dm_G(g_i) \\
&\quad \asymp \sum_i \frac{1}{r^{\dim G}} \int_Y m_G(\{g_i \in B^G_{r/10} : r - \delta \leq d(g_i y_i, y) < r + \delta\})\, d\mu(y) \\
&\quad \ll \sum_i \frac{1}{r^{\dim G}} \int_{B^G_{(11/10)r+\delta}\cdot y_i} \delta r^{\dim G - 1}\, d\mu \leq \frac{\delta}{r} \int_{B^G_{(12/10)r}\Omega} \sum_i \mathbb{1}_{B_{(12/10)r}\cdot y_i}(y)\, d\mu(y).
\end{aligned}$$

For any $y \in B^G_{(12/10)r}\Omega$, the number of y_i terms contained in $B^G_{(12/10)r} \cdot y$ is at most $(33/9)^{\dim G}$ since $B^G_{(9/20)r} \cdot y_i$ terms are disjoint and contained in $B^G_{(33/20)r} \cdot y$. It implies that $\sum_i \mathbb{1}_{B_{(12/10)r}\cdot y_i}(y) \leq 4^{\dim G}$ for any $y \in B^G_{(12/10)r}\Omega$. It follows that

$$\mathbb{E}\bigg(\sum_i \mu(\partial_\delta(B^G_r \cdot z_i))\bigg) \ll \frac{\delta}{r} \int_{B^G_{(12/10)r}\Omega} 4^{\dim G}\, d\mu(y) \ll \frac{\delta}{r} \mu(B^G_{(12/10)r}\Omega),$$

where the implied constant depends only on G.

Applying the same argument for $\partial_\delta(B^G_{r/2} \cdot z_i)$ instead of $\partial_\delta(B^G_r \cdot z_i)$,

$$\mathbb{E}\bigg(\sum_i (\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{r/2} \cdot z_i)))\bigg) \ll \frac{\delta}{r} \mu(B^G_{(12/10)r}\Omega).$$

It follows from Chebyshev's inequality that

$$\mathbb{P}\bigg(\sum_i (\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{r/2} \cdot z_i))) \geq \frac{1}{2}\left(\frac{\delta}{r}\right)^{1/2} \mu(B^G_{(12/10)r}\Omega)\bigg) \ll \left(\frac{\delta}{r}\right)^{1/2}.$$

Hence, we have

$$\begin{aligned}
&\mathbb{P}\bigg(\bigcap_{k\geq 0} \bigg\{\sum_i (\mu(\partial_{2^{-k}\delta}(B^G_r \cdot z_i)) + \mu(\partial_{2^{-k}\delta}(B^G_{r/2} \cdot z_i))) < \frac{1}{2}\left(\frac{2^{-k}\delta}{r}\right)^{1/2} \mu(B^G_{(12/10)r}\Omega)\bigg\}\bigg) \\
&\quad > 1 - O\bigg(\left(\frac{\delta}{r}\right)^{1/2}\bigg).
\end{aligned} \tag{2.4}$$

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Thus, there exists $0 < c < 1/10$ so that the right-hand side of equation (2.4) is positive for any $\delta < cr$. It follows that we can find $\{g_i\}_{i=1}^N$ such that the $z_i = g_i y_i$ terms satisfy equation (2.3) for any $0 < \delta < cr$. □

Let $c > 0$ and $\{g_i\}_{i=1}^N \subset B^G_{r/10}$ be as in the above claim. The set $\{z_i = g_i y_i\}_{i=1}^N$ is $(7/10)r$-separated since $\{y_i\}_{i=1}^N$ is $(9/10)r$-separated. Let $K \overset{\text{def}}{=} \bigcup_{i=1}^N B^G_r \cdot z_i$. Since $B^G_{(9/10)r} \cdot y_i \subseteq B^G_r \cdot z_i \subseteq B^G_{(11/10)r} \cdot y_i$, we have

$$\Omega \subseteq \bigcup_{i=1}^N B^G_{(9/10)r} \cdot y_i \subseteq K \subseteq \bigcup_{i=1}^N B^G_{(11/10)r} \cdot y_i \subseteq B^G_{(11/10)r}\Omega.$$

Now we define a partition $\mathcal{P}$ of K inductively as follows:

$$P_i \overset{\text{def}}{=} B^G_r \cdot z_i \setminus \left(\bigcup_{j=1}^{i-1} P_j \cup \bigcup_{j=i+1}^{N} B^G_{r/2} \cdot z_j \right)$$

for $1 \le i \le N$. By definition, we have $B^G_{r/5} \cdot z_i \subseteq P_i \subseteq B^G_r \cdot z_i$ and $z_i \in B^G_{r/10}\Omega$ for $1 \le i \le N$. We also observe that the δ-neighborhood of $\mathcal{P}$ is contained in $\bigcup_{i=1}^N (\partial_\delta(B^G_r \cdot z_i) \cup \partial_\delta(B^G_{r/2} \cdot z_i))$. Hence, it follows from the above claim that

$$\mu(\partial_\delta \mathcal{P}) \le \sum_i (\mu(\partial_\delta(B^G_r \cdot z_i)) + \mu(\partial_\delta(B^G_{r/2} \cdot z_i))) \le \left(\frac{\delta}{r}\right)^{1/2} \mu(B^G_{(12/10)r}\Omega)$$

for any $0 < \delta < cr$. □

Remark 2.4. In Lemma 2.3, if $y \in \Omega$ is given and we let $y_1 = y \in \Omega$ in the proof, then $y \in B^G_{r/10} \cdot z_1$, and thus $y \notin \partial\mathcal{P}$, which will be used in the proofs of Propositions 4.1 and 5.4.

We need the following thickening properties. It can easily be checked that for any $r > \delta > 0$, we have

$$B^G_\delta Y(r) \subset Y(r-\delta) \quad \text{and} \quad B^G_\delta Y(r)^c \subset Y(r+\delta)^c. \tag{2.5}$$

Using Lemma 2.3 inductively, we have the following partition of Y with its subpartition having small boundary measures. Recall that $Y(r) = \varnothing$ for any $r > 1$ by equation (2.1).

LEMMA 2.5. *Let $0 < r_0 \le 1$ be given and μ be a probability measure on Y. There exists a partition $\{K_k\}_{k=1}^\infty$ of Y such that for each $k \ge 1$, the following statements hold:*

(1) $K_k \subseteq Y(2^{-k}) \setminus Y(2^{-k+2})$;

(2) *there exist a partition $\mathcal{P}_k = \{P_{k1}, \ldots, P_{kN_k}\}$ of K_k and a point $z_i \in B^G_{(1/10)r_0 2^{-k-1}} K_k$ for each $1 \le i \le N_k$ satisfying*

$$B^G_{(1/5)r_0 2^{-k-1}} \cdot z_i \subseteq P_{ki} \subseteq B^G_{r_0 2^{-k-1}} \cdot z_i;$$

(3) $\mu(\partial_\delta \mathcal{P}_k) \le (r_0^{-1} 2^{k+4}\delta)^{1/2} \mu(Y(2^{-k-1}) \setminus Y(2^{-k+3}))$ *for any* $0 < \delta < c r_0 2^{-k-2}$, *where $c > 0$ is the constant in Lemma 2.3.*

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Proof. We will construct $\{K_k\}_{k\geq 1}$ and $\{\mathcal{P}_k\}_{k\geq 1}$ using Lemma 2.3 inductively. For each $k \geq 1$, let us say that K_k *and* $\mathcal{P}_k$ *satisfy* ($\spadesuit_k$) if they satisfy the three conditions in the statement. We will also need auxiliary bounded sets $K'_k \subset Y$ and corresponding partitions $\mathcal{P}'_k$ of K'_k during the inductive process. Let us say that K'_k *and* $\mathcal{P}'_k$ *satisfy* ($\clubsuit_k$) if they satisfy the following three conditions:

(1) $Y(2^{-k+1}) \setminus \bigcup_{j=1}^{k-1} K_j \subseteq K'_k \subseteq B^G_{(11/10)r_0 2^{-k-1}}(Y(2^{-k+1}) \setminus \bigcup_{j=1}^{k-1} K_j)$;

(2) for each $1 \leq i \leq N_k$, there exists $z_{ki} \in B^G_{(1/10)r_0 2^{-k-1}} K'_k$ such that

$$B^G_{(1/5)r_0 2^{-k-1}} \cdot z_{ki} \subseteq P'_{ki} \subseteq B^G_{r_0 2^{-k-1}} \cdot z_{ki} \quad \text{and} \quad K'_k = \bigcup_{i=1}^{N} B^G_{r_0 2^{-k-1}} \cdot z_{ki};$$

(3) $\mu(\partial_\delta \mathcal{P}'_k) \leq (r_0^{-1} 2^{k+1} \delta)^{1/2} \mu(Y(2^{-k}) \setminus Y(2^{-k+3}))$ for any $0 < \delta < c r_0 2^{-k-1}$.

Here, $\bigcup_{j=1}^{0} K_j \overset{\text{def}}{=} \emptyset$.

We first choose $\Omega_1 = Y(1)$ and apply Lemma 2.3 with $r = r_0 2^{-2}$ and $\Omega = \Omega_1 \subset Y(r_0/2)$. Then we have a subset $K'_1 \subset Y$ and a partition $\mathcal{P}'_1$ of K'_1 satisfying conditions (1), (2) of ($\clubsuit_1$), and

$$\mu(\partial_\delta \mathcal{P}'_1) \leq (r_0^{-1} 2^2 \delta)^{1/2} \mu(B^G_{(12/10)r_0 2^{-2}} \Omega_1)$$

for any $0 < \delta < c r_0 2^{-2}$. It follows from equation (2.5) that $B^G_{(12/10)r_0 2^{-2}} Y(1) \subset Y(\frac{1}{2})$, which implies condition (3) of ($\clubsuit_1$) since $Y(4) = \varnothing$. Note that $K'_1 \subset B^G_{(11/10)r_0 2^{-2}} Y(1) \subset Y(\frac{1}{2})$.

Now let $\Omega_2 = Y(\frac{1}{2}) \setminus K'_1$ and apply Lemma 2.3 again with $r = r_0 2^{-3}$ and $\Omega = \Omega_2 \subset Y(r_0/4)$. We have a subset $K'_2 \subset Y$ and a partition $\mathcal{P}'_2$ of K'_2 satisfying $\Omega_2 \subset K'_2 \subset B^G_{(11/10)r_0 2^{-3}} \Omega_2$, (2) of ($\clubsuit_2$), and $\mu(\partial_\delta \mathcal{P}'_2) \leq (r_0^{-1} 2^3 \delta)^{1/2} \mu(B^G_{(12/10)r_0 2^{-3}} \Omega_2)$ for any $0 < \delta < c r_0 2^{-3}$. Setting $K_1 = K'_1 \setminus K'_2$, condition (1) of ($\clubsuit_2$) and condition (1) of ($\spadesuit_1$) follow since $Y(2) = \varnothing$. Since $K'_1 \supset Y(1)$, it follows from equation (2.5) that $B^G_{(12/10)r_0 2^{-3}} \Omega_2 \subset Y(\frac{1}{4}) \setminus Y(2)$, which implies condition (3) of ($\clubsuit_2$).

Define a partition $\mathcal{P}_1 = \{P_{11}, \ldots, P_{1N_1}\}$ from $\mathcal{P}'_1 = \{P'_{11}, \ldots, P'_{1N_1}\}$ by $P_{1i} = P'_{1i} \setminus K'_2$ for each $1 \leq i \leq N_1$. For each $1 \leq i \leq N_1$ and $y \in B^G_{(1/5)r_0 2^{-2}} \cdot z_{1i}$, observe that $y \notin K'_2$ since $B^G_{r_0 2^{-2}} \cdot z_{1i} \subset K'_1$ and $K'_2 \subset B^G_{(11/10)r_0 2^{-3}} \Omega_2 \subset B^G_{(11/10)r_0 2^{-3}}(Y \setminus K'_1)$. Hence, $B^G_{(1/5)r_0 2^{-2}} \cdot z_{1i} \subset P_{1i}$ holds, so condition (2) of ($\spadesuit_1$) follows. Since $P_{1i} = P'_{1i} \setminus K'_2$ for each $1 \leq i \leq N_1$, we have

$$\begin{aligned}
\mu(\partial_\delta \mathcal{P}_1) &\leq \mu(\partial_\delta \mathcal{P}'_1) + \mu(\partial_\delta \mathcal{P}'_2) \\
&\leq (r_0^{-1} 2^2 \delta)^{1/2} \mu(Y(2^{-1}) \setminus Y(2^2)) + (r_0^{-1} 2^3 \delta)^{1/2} \mu(Y(2^{-2}) \setminus Y(2)) \\
&\leq (r_0^{-1} 2^5 \delta)^{1/2} \mu(Y(2^{-2}) \setminus Y(2^2))
\end{aligned}$$

for any $0 < \delta < c r_0 2^{-3}$. Hence, condition (3) of ($\spadesuit_1$) follows.

Our desired disjoint sets $\{K_k\}_{k\geq 1}$ and partitions $\{\mathcal{P}_k\}_{k\geq 1}$ will be obtained by applying this process repeatedly.

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CLAIM. *For $k \geq 2$, suppose that we have disjoint bounded sets K_j of Y and corresponding partitions $\mathcal{P}_j$ satisfying ($\spadesuit_j$) for $j = 1, \ldots, k-1$, and a subset $K_k' \subset Y$ and a partition $\mathcal{P}_k'$ satisfying ($\clubsuit_k$). Then we can find $K_k \subseteq K_k'$ and a partition $\mathcal{P}_k$ of K_k satisfying ($\spadesuit_k$), and $K_{k+1}' \subset Y$ and a partition $\mathcal{P}_{k+1}'$ of K_{k+1}' satisfying ($\clubsuit_{k+1}$).*

Proof of the claim. Note that $K_k' \subset B^G_{(11/10)r_0 2^{-k-1}} Y(2^{-k+1}) \subset Y(2^{-k})$ and $K_j \subset Y(2^{-j}) \subset Y(2^{-k})$ for each $j = 1, \ldots, k-1$. Let $\Omega_{k+1} = Y(2^{-k}) \setminus (\bigcup_{j=1}^{k-1} K_j \cup K_k')$ and apply Lemma 2.3 with $r = r_0 2^{-k-2}$ and $\Omega = \Omega_{k+1} \subset Y(r_0 2^{-k-1})$. There exist $K_{k+1}' \subset Y$ and a partition $\mathcal{P}_{k+1}' = \{P_{(k+1)1}', \ldots, P_{(k+1)N_{k+1}}'\}$ of K_{k+1}' satisfying $\Omega_{k+1} \subset K_{k+1}' \subset B^G_{(11/10)r_0 2^{-k-2}} \Omega_{k+1}$, condition (2) of ($\clubsuit_{k+1}$), and $\mu(\partial_\delta \mathcal{P}_{k+1}') \leq (r_0^{-1} 2^{k+2} \delta)^{1/2} \mu(B^G_{(12/10)r_0 2^{-k-2}} \Omega_{k+1})$ for any $0 < \delta < c r_0 2^{-k-2}$. Setting $K_k = K_k' \setminus K_{k+1}'$, condition (1) of ($\clubsuit_{k+1}$) follows. Since $\bigcup_{j=1}^{k-1} K_j \supset Y(2^{-k+2})$ and $K_k \subset K_k' \subset Y(2^{-k}) \setminus \bigcup_{j=1}^{k-1} K_j$, condition (1) of ($\spadesuit_k$) follows. It follows from $\bigcup_{j=1}^{k-1} K_j \cup K_k' \supset Y(2^{-k+1})$ and equation (2.5) that

$$B^G_{(12/10)r_0 2^{-k-2}} \Omega_{k+1} \subset Y(2^{-k-1}) \setminus Y(2^{-k+2}),$$

which implies condition (3) of ($\clubsuit_{k+1}$). Define a partition $\mathcal{P}_k = \{P_{k1}, \ldots, P_{kN_k}\}$ from $\mathcal{P}_k' = \{P_{k1}', \ldots, P_{kN_k}'\}$ by $P_{ki} = P_{ki}' \setminus K_{k+1}'$ for any $1 \leq i \leq N_k$. For each $1 \leq i \leq N_k$ and $y \in B^G_{(1/5)r_0 2^{-k-1}} \cdot z_{ki}$, observe that $y \notin K_{k+1}'$ since $B^G_{r_0 2^{-k-1}} \cdot z_{ki} \subseteq K_k'$ and $K_{k+1}' \subseteq B^G_{(11/10)r_0 2^{-k-2}} \Omega_{k+1} \subset B^G_{(11/10)r_0 2^{-k-2}} (Y \setminus K_k')$. Hence, $B^G_{(1/5)r_0 2^{-k-1}} \cdot z_{ki} \subset P_{ki}$ holds, so condition (2) of ($\spadesuit_k$) follows. Since $P_{ki} = P_{ki}' \setminus K_{k+1}'$ for each $1 \leq i \leq N_k$, we have

$$\begin{aligned}
\mu(\partial_\delta \mathcal{P}_k) &\leq \mu(\partial_\delta \mathcal{P}_k') + \mu(\partial_\delta \mathcal{P}_{k+1}') \\
&\leq (r_0^{-1} 2^{k+1} \delta)^{1/2} \mu(Y(2^{-k}) \setminus Y(2^{-k+3})) + (r_0^{-1} 2^{k+2} \delta)^{1/2} \mu(Y(2^{-k-1}) \setminus Y(2^{-k+2})) \\
&\leq (r_0^{-1} 2^{k+4} \delta)^{1/2} \mu(Y(2^{-k-1}) \setminus Y(2^{-k+3}))
\end{aligned}$$

for any $0 < \delta < c r_0 2^{-k-2}$. Hence, condition (3) of ($\spadesuit_k$) follows. □

The claim concludes the proof of Lemma 2.5. □

By [**EL10**, Lemmas 7.29 and 7.45], there are constants $\alpha > 0$ and $d_0 > 0$ depending on a and G such that for every $r \in (0, 1]$,

$$a^{-k} B_r^{G^+} a^k \subset B^G_{d_0 e^{-k\alpha} r} \tag{2.6}$$

for any $k \in \mathbb{Z}$. It implies that $a^k B_r^G a^{-k} \subset B^G_{d_0 e^{k\alpha} r}$ for $k \geq 0$.

The following lemma is a quantitative strengthening of [**EL10**, Lemma 7.31]. We remark that the constants below are independent of μ and $\mathcal{P}$ while the 'dynamical δ-boundary' E_δ depends on μ.

Definition 2.6. We define the *dynamical δ-boundary of the partition $\mathcal{P}$* by

$$E_\delta = \bigcup_{k=0}^{\infty} a^k \partial_{d_0 e^{-k\alpha} \delta} \mathcal{P}.$$

LEMMA 2.7. *Given $0 < r_0 \leq 1$ and an a-invariant probability measure μ on Y, let $\{K_j\}_{j\geq 1}$ and $\{\mathcal{P}_j\}_{j\geq 1}$ be the sets and the partitions in Lemma 2.5. Let $c > 0$ and $d_0 > 0$ be the constants in Lemma 2.3 and equation (2.6), respectively. Let $\mathcal{P}$ be the countable partition $\mathcal{P} \overset{\text{def}}{=} \bigcup_{j=1}^{\infty} \mathcal{P}_j$ of Y.*

There exist $C_1, C_2 > 0$, depending only on r_0, a, and G, such that for any $0 < \delta < \min((cr_0/16d_0)^2, 1)$, the dynamical δ-boundary $E_\delta \subset Y$ satisfies

$$\mu(E_\delta) < \mu(Y \setminus Y(C_1\delta^{1/2})) + C_2\delta^{1/4}$$

and $B_\delta^{G^+} \cdot y \subset [y]_{\mathcal{P}_0^\infty}$ for any $y \in Y \setminus E_\delta$.

Proof. We split E_δ into two subsets

$$E'_\delta = \bigcup_{k=0}^{\infty} a^k \Bigg(\bigcup_{i=2+\lceil(\alpha/\log 2)k-(\log\delta)/(2\log 2)\rceil}^{\infty} \partial_{d_0e^{-k\alpha}\delta}\mathcal{P}_i \Bigg) \quad \text{and}$$

$$E''_\delta = \bigcup_{k=0}^{\infty} a^k \Bigg(\bigcup_{i=1}^{1+\lceil(\alpha/\log 2)k-(\log\delta)/(2\log 2)\rceil} \partial_{d_0e^{-k\alpha}\delta}\mathcal{P}_i \Bigg).$$

We first claim that $E'_\delta \subset Y \setminus Y((d_0 + d_0^2)\delta^{1/2})$. Let $y \in E'_\delta$, that is, $y \in a^k\partial_{d_0e^{-k\alpha}\delta}P$ for some $k \geq 0$ and $P \in \mathcal{P}_i$ for some $i \geq 2 + \lceil(\alpha/\log 2)k - (\log\delta)/(2\log 2)\rceil$. By Lemma 2.5,

$$P \subset K_i \subset Y(2^{-i}) \setminus Y(2^{-i+2}) \subset Y(2^{-i+2})^c.$$

It follows from equation (2.5) that

$$\partial_{d_0e^{-k\alpha}\delta}P \subset B^G_{d_0e^{-k\alpha}\delta}P \subset B^G_{d_0e^{-k\alpha}\delta}Y(2^{-i+2})^c \subset Y(2^{-i+2} + d_0e^{-k\alpha}\delta)^c. \tag{2.7}$$

Using equation (2.6), for any $0 < r < 1$, $a^kY(r)^c \subset Y(d_0e^{k\alpha}r)^c$. Since $e^{k\alpha}2^{-i+2} \leq \delta^{1/2}$, combining with equation (2.7),

$$a^k\partial_{d_0e^{-k\alpha}\delta}P \subset a^kY(2^{-i+2} + d_0e^{-k\alpha}\delta)^c \subset Y((d_0 + d_0^2)\delta^{1/2})^c.$$

This proves the claim. It follows that

$$\mu(E'_\delta) \leq \mu(Y \setminus Y(C_1\delta^{1/2})), \tag{2.8}$$

where $C_1 = d_0 + d_0^2$ is a constant depending only on a and G.

Next we estimate $\mu(E''_\delta)$. It follows from the a-invariance of μ that

$$\mu(E''_\delta) \leq \sum_{k=0}^{\infty} \sum_{i=1}^{1+\lceil(\alpha/\log 2)k-(\log\delta)/(2\log 2)\rceil} \mu(\partial_{d_0e^{-k\alpha}\delta}\mathcal{P}_i) = \sum_{i=1}^{\infty}\sum_{k=k_i}^{\infty} \mu(\partial_{d_0e^{-k\alpha}\delta}\mathcal{P}_i), \tag{2.9}$$

where $k_i \in \mathbb{N}$ denotes the smallest number of k such that $1 + \lceil(\alpha/\log 2)k - (\log\delta)/(2\log 2)\rceil \geq i$. Note that $k_i \geq (\log 2/\alpha)(i-2) + (\log\delta)/2\alpha$.

However, by Lemma 2.5, we have

$$\mu(\partial_{d_0e^{-k\alpha}\delta}\mathcal{P}_i) \leq (r_0^{-1}2^{i+4}d_0e^{-k\alpha}\delta)^{1/2}\mu(Y(2^{-i-1}) \setminus Y(2^{-i+3})) \tag{2.10}$$

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for any $k \geq k_i$, since $d_0 e^{-k\alpha}\delta \leq d_0 2^{-i+2}\delta^{1/2} < cr_0 2^{-i-2}$. By equations (2.9) and (2.10), we have

$$\begin{aligned}\mu(E''_\delta) &\leq \sum_{i=1}^{\infty}\sum_{k=k_i}^{\infty}\mu(\partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}_i) \leq \sum_{i=1}^{\infty}\sum_{k=k_i}^{\infty}(r_0^{-1}2^{i+4}d_0 e^{-k\alpha}\delta)^{1/2}\mu(Y(2^{-i-1})\setminus Y(2^{-i+3}))\\ &= \sum_{i=1}^{\infty}(r_0^{-1}2^{i+4}e^{-k_i\alpha}\delta)^{1/2}(1-e^{-\alpha/2})^{-1}\mu(Y(2^{-i-1})\setminus Y(2^{-i+3}))\\ &\leq r_0^{-1/2}2^3\delta^{1/4}(1-e^{-\alpha/2})^{-1}\sum_{i=1}^{\infty}\mu(Y(2^{-i-1})\setminus Y(2^{-i+3})) \leq C_2\delta^{1/4}, \end{aligned} \tag{2.11}$$

where $C_2 = 2^5 r_0^{-1/2}(1-e^{-\alpha/2})^{-1}$ is a constant depending only on r_0, a, and G. Combining equations (2.8) and (2.11), we finally have

$$\mu(E_\delta) < \mu(Y \setminus Y(C_1\delta^{1/2})) + C_2\delta^{1/4}$$

and the constants $C_1, C_2 > 0$ depend only on r_0, a, and G.

It remains to check that $B_\delta^{G^+}\cdot y \subset [y]_{\mathcal{P}_0^\infty}$ for any $y \in Y \setminus E_\delta$. Let $h \in B_\delta^{G^+}$ and suppose $[hy]_{\mathcal{P}_0^\infty} \neq [y]_{\mathcal{P}_0^\infty}$. There is some $k \geq 0$ such that $a^{-k}hy$ and $a^{-k}y$ belong to different elements of the partition $\mathcal{P}$. Since $a^{-k}ha^k \in a^{-k}B_\delta^{G^+}a^k \subset B^G_{d_0e^{-k\alpha}\delta}$ by equation (2.6), we have

$$d_Y(a^{-k}hy, a^{-k}y) \leq d_G(a^{-k}ha^k, \mathrm{id}) \leq d_0 e^{-k\alpha}\delta.$$

It follows that both $a^{-k}hy$ and $a^{-k}y$ belong to $\partial_{d_0e^{-k\alpha}\delta}\mathcal{P}$, and hence $y \in E_\delta$. It concludes that $B_\delta^{G^+}\cdot y \subset [y]_{\mathcal{P}_0^\infty}$ for any $y \in Y \setminus E_\delta$. □

The following proposition is a quantitative version of [**EL10**, Proposition 7.37]. Given a-invariant measure μ, the proposition provides a σ-algebra which is a^{-1}-descending and subordinate to L in the following quantitative sense.

PROPOSITION 2.8. *Let $0 < r_0 \leq 1$ be given, μ be an a-invariant probability measure on Y, and $L < G^+$ be a closed subgroup normalized by a. There exists a countably generated sub-σ-algebra $\mathcal{A}^L$ of Borel σ-algebra of Y satisfying:*

(1) *$a\mathcal{A}^L \subset \mathcal{A}^L$, that is, $\mathcal{A}^L$ is a^{-1}-descending;*

(2) *$[y]_{\mathcal{A}^L} \subset B^L_{r_0 2^{-k+1}}\cdot y$ for any $y \in Y(2^{-k})\setminus Y(2^{-k+2})$ with $k \geq 1$;*

(3) *if $0 < \delta < \min((cr_0/16d_0)^2, 1)$, then*

$$B_\delta^L\cdot y \subset [y]_{\mathcal{A}^L} \quad \text{for any } y \in Y(\delta)\setminus E_\delta,$$

where $c, d_0 > 0$ are the constants in Lemma 2.3 and equation (2.6), and E_δ is the dynamical δ-boundary defined in Lemma 2.7.

In particular, the σ-algebra $\mathcal{A}^L$ is L-subordinate modulo μ.

Proof. For a given a-invariant probability measure μ on Y, let $\mathcal{P}$ be the countable partition of Y constructed in Lemma 2.7. We will construct a countably generated σ-algebra $\mathcal{P}^L$ by

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taking L-plaques in each $P \in \mathcal{P}$ as atoms of $\mathcal{P}^L$. Then, $\mathcal{A}^L \stackrel{\text{def}}{=} (\mathcal{P}^L)_0^\infty$ will be the desired σ-algebra.

For each $P \in \mathcal{P}$, by Lemma 2.5, there exist $j \geq 1$ and $z \in P$ such that $P \in Y(2^{-j}) \setminus Y(2^{-j+2})$ and $B^G_{(1/5)r_0 2^{-j-1}} \cdot z \subseteq P \subseteq B^G_{r_0 2^{-j-1}} \cdot z$. We can find $B_P \subset G$ with $\operatorname{diam}(B_P) \leq r_0 2^{-j}$ such that $P = \pi_Y(B_P)$, where $\pi_Y : G \to Y$ is the natural quotient map. Let $\mathcal{B}_{G/L}$ be the Borel σ-algebra of the quotient G/L. Note that since L is closed, $\mathcal{B}_{G/L}$ is countably generated. Define the σ-algebra

$$\mathcal{P}^L = \sigma(\{\pi_Y(B_P \cap S) : P \in \mathcal{P},\ S \in \mathcal{B}_{G/L}\}).$$

Then, $\mathcal{P}^L$ is a refinement of $\mathcal{P}$ such that atoms of $\mathcal{P}^L$ are open L-plaques, that is, for any $y \in P \in \mathcal{P}$, $[y]_{\mathcal{P}^L} = [y]_{\mathcal{P}} \cap B^L_{r_0 2^{-j}} \cdot y = V_y \cdot y$, where $V_y \subset B^L_{r_0 2^{-j}}$ is an open bounded set.

It is clear that $\mathcal{P}^L$ is countably generated, and hence $\mathcal{A}^L = (\mathcal{P}^L)_0^\infty$ is also countably generated. By construction, we have $a\mathcal{A}^L = (\mathcal{P}^L)_1^\infty \subset \mathcal{A}^L$, which proves the assertion (1).

For any $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$ with $k \geq 1$, take $P \in \mathcal{P}$ such that $y \in P$. By Lemma 2.5, there exist $j \geq 1$ and $z \in P$ such that $P \in Y(2^{-j}) \setminus Y(2^{-j+2})$ and $P \subseteq B^G_{r_0 2^{-j-1}} \cdot z$. Observe that $2^{-j+2} > 2^{-k}$ and $2^{-j} < 2^{-k+2}$, that is, $j - 2 < k < j + 2$. Hence, we have

$$[y]_{\mathcal{A}^L} \subset [y]_{\mathcal{P}^L} = V_y \cdot y \subset B^L_{r_0 2^{-j}} \cdot y \subset B^L_{r_0 2^{-k+1}} \cdot y,$$

which proves the assertion (2).

For a given $0 < \delta < \min((cr_0/16d_0)^2, 1)$ and $y \in Y(\delta) \setminus E_\delta$, assume that $z = hy$ with $h \in B^L_\delta$. By Lemma 2.7, $B^{G^+}_\delta \cdot y \subset [y]_{\mathcal{P}_0^\infty}$. Hence, it follows that for any $k \geq 0$, $a^{-k}y$ and $a^{-k}z$ belong to the same atom $P_k \subset \mathcal{P}$. Then, we have

$$a^{-k}y,\ a^{-k}z = a^{-k}ha^k \cdot (a^{-k}y)\ \in P_k.$$

Note that for any $y \in Y(\delta)$, the map $B^{G^+}_\delta \ni g \mapsto gy$ is injective, and hence the map $a^{-k}B^{G^+}_\delta a^k \ni g \mapsto ga^{-k}y$ is injective. Since $a^{-k}ha^k \in a^{-k}B^L_\delta a^k$, $a^{-k}y$ and $a^{-k}z$ belong to the same atom of $\mathcal{P}^L$. This proves the assertion (3). $\square$

As in [**LSS19**, Lemma 3.4], we need to compare the dynamical entropy and the static entropy. In [**LSS19**], the σ-algebra $\pi^{-1}(\mathcal{B}_X)$ is used to deal with the entropy relative to X, where $\mathcal{B}_X$ is the Borel σ-algebra of X. To deal with the entropy relative to the general closed subgroup $L < G^+$ normalized by a, we consider the following tail σ-algebra with respect to $\mathcal{A}^L$ in Proposition 2.8. Denote by

$$\mathcal{A}^L_\infty \stackrel{\text{def}}{=} \bigcap_{k=1}^\infty a^k\mathcal{A}^L = \bigcap_{k=1}^\infty (\mathcal{P}^L)_k^\infty. \tag{2.12}$$

This tail σ-algebra may not be countably generated but it satisfies strict a-invariance, that is, $a\mathcal{A}^L_\infty = \mathcal{A}^L_\infty = a^{-1}\mathcal{A}^L_\infty$.

LEMMA 2.9. *Let $0 < r_0 \leq 1$ be given, μ be an a-invariant probability measure on Y, $L < G^+$ be a closed subgroup normalized by a, and $\mathcal{A}^L$ be as in Proposition 2.8. Then, the σ-algebra $(\mathcal{A}^L)_{-\infty}^\infty$ is the Borel σ-algebra of Y modulo μ.*

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Proof. Let $\mathcal{P}^L$ be as in the proof of Proposition 2.8. Since $(\mathcal{A}^L)_{-\infty}^{\infty} = (\mathcal{P}^L)_{-\infty}^{\infty}$ and $Y = \bigcup_{k\geq 1} Y(2^{-k}) \setminus Y(2^{-k+2})$, it is enough to show that for each $k \geq 1$ and for μ-a.e. $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$, we have $[y]_{(\mathcal{P}^L)_{-\infty}^{\infty}} = \{y\}$.

For fixed $k \geq 1$, it follows from Poincaré recurrence (e.g. see [EW11, Theorem 2.11]) that for μ-a.e. $y \in Y(2^{-k}) \setminus Y(2^{-k+2})$, there exists an increasing sequence $(k_i)_{i\geq 1} \subset \mathbb{N}$ such that

$$a^{k_i} y \in Y(2^{-k}) \setminus Y(2^{-k+2}) \quad \text{and} \quad k_i \to \infty \text{ as } i \to \infty.$$

By Proposition 2.8(2), it follows that for each $i \geq 1$,

$$[a^{k_i} y]_{\mathcal{A}^L} = [a^{k_i} y]_{(\mathcal{P}^L)_0^{\infty}} \subset B^L_{r_0 2^{-k+1}} \cdot a^{k_i} y.$$

Since $[a^{k_i} y]_{(\mathcal{P}^L)_0^{\infty}} = a^{k_i}[y]_{a^{-k_i}(\mathcal{P}^L)_0^{\infty}} = a^{k_i}[y]_{(\mathcal{P}^L)_{-k_i}^{\infty}}$, using equation (2.6), we have

$$[y]_{(\mathcal{P}^L)_{-k_i}^{\infty}} \subset a^{-k_i} B^L_{r_0 2^{-k+1}} \cdot a^{k_i} y = a^{-k_i} B^L_{r_0 2^{-k+1}} a^{k_i} \cdot y \subset B^L_{d_0 e^{-\alpha k_i} r_0 2^{-k+1}} \cdot y.$$

Taking $i \to \infty$, we conclude that $[y]_{(\mathcal{P}^L)_{-\infty}^{\infty}} = \{y\}$. ☐

PROPOSITION 2.10. *Let $0 < r_0 \leq 1$ be given, μ be an a-invariant probability measure on Y, $L < G^+$ be a closed subgroup normalized by a, $\mathcal{A}^L$ be as in Proposition 2.8, and $\mathcal{A}^L_{\infty}$ be as in equation (2.12). Then, we have*

$$h_\mu(a|\mathcal{A}^L_\infty) = h_\mu(a^{-1}|\mathcal{A}^L_\infty) = H_\mu(\mathcal{A}^L|a\mathcal{A}^L). \tag{2.13}$$

Moreover, equation (2.13) holds for almost every ergodic component of μ.

Proof. Let $\mathcal{P}^L$ be as in the proof of Proposition 2.8. Since $\mathcal{P}^L$ is countably generated, we can take an increasing sequence of finite partitions $(\mathcal{P}^L_k)_{k\geq 1}$ of Y such that $\mathcal{P}^L_k \nearrow \mathcal{P}^L$. By Lemma 2.9, we have $\mathcal{B}_Y = (\mathcal{P}^L)_{-\infty}^{\infty} = \bigvee_{k=1}^{\infty} (\mathcal{P}^L_k)_{-\infty}^{\infty}$ modulo μ, where $\mathcal{B}_Y$ is the Borel σ-algebra of Y. It is clear that $(\mathcal{P}^L_k)_{-\infty}^{\infty} \subseteq (\mathcal{P}^L_{k+1})_{-\infty}^{\infty}$ for all $k \in \mathbb{N}$. Hence, it follow from Kolmogorov–Sinaı˘ theorem [ELW, Proposition 2.20] that

$$h_\mu(a^{-1}|\mathcal{A}^L_\infty) = \lim_{k\to\infty} h_\mu(a^{-1}, \mathcal{P}^L_k|\mathcal{A}^L_\infty).$$

Using the future formula [ELW, Proposition 2.19(8)], we have

$$\lim_{k\to\infty} h_\mu(a^{-1}, \mathcal{P}^L_k|\mathcal{A}^L_\infty) = \lim_{k\to\infty} H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)_1^{\infty} \vee \mathcal{A}^L_\infty).$$

It follows from monotonicity and continuity of entropy [ELW, Propositions 2.10, 2.12, and 2.13] that for any fixed $k \geq 1$,

$$\lim_{\ell\to\infty} H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_\ell)_1^{\infty} \vee \mathcal{A}^L_\infty) \leq H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)_1^{\infty} \vee \mathcal{A}^L_\infty) \leq \lim_{\ell\to\infty} H_\mu(\mathcal{P}^L_\ell|(\mathcal{P}^L_k)_1^{\infty} \vee \mathcal{A}^L_\infty),$$

and hence, we have

$$H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L)_1^{\infty} \vee \mathcal{A}^L_\infty) \leq H_\mu(\mathcal{P}^L_k|(\mathcal{P}^L_k)_1^{\infty} \vee \mathcal{A}^L_\infty) \leq H_\mu(\mathcal{P}^L|(\mathcal{P}^L_k)_1^{\infty} \vee \mathcal{A}^L_\infty).$$

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Taking $k \to \infty$, it follows that

$$\lim_{k\to\infty} H_\mu(\mathcal{P}_k^L|(\mathcal{P}_k^L)_1^\infty \vee \mathcal{A}_\infty^L) = H_\mu(\mathcal{P}^L|(\mathcal{P}^L)_1^\infty \vee \mathcal{A}_\infty^L) = H_\mu(\mathcal{A}^L|a\mathcal{A}^L),$$

which concludes equation (2.13).

Note that $\mathcal{B}_Y = (\mathcal{P}^L)_{-\infty}^\infty = \bigvee_{k=1}^\infty (\mathcal{P}_k^L)_{-\infty}^\infty$ modulo almost every ergodic component of μ. Thus, following the same argument as above, we can conclude equation (2.13) for almost every ergodic component of μ. $\square$

The quantity $H_\mu(\mathcal{A}^L|a\mathcal{A}^L)$ is called *empirical entropy* and is the average of the *conditional information function*

$$I_\mu(\mathcal{A}^L|a\mathcal{A}^L)(x) = -\log \mu_x^{a\mathcal{A}^L}([x]_{\mathcal{A}}),$$

and indeed the *entropy contribution* of L (see [**EL10**, 7.8] for definition).

2.4. *Effective variational principle.* This subsection is to effectivize the variational principle. We first recall the following ineffective variational principle. Combining [**EL10**, Proposition 7.34] and [**EL10**, Theorem 7.9], we have the following upper bound of an empirical entropy (or entropy contribution), and the entropy rigidity.

THEOREM 2.11. [**EL10**] *Let $L < G^+$ be a closed subgroup normalized by a and let $\mathfrak{l}$ denote the Lie algebra of L. Let μ be an a-invariant ergodic probability measure on Y. If $\mathcal{A}$ is a countably generated sub-σ-algebra of the Borel σ-algebra which is a^{-1}-descending and L-subordinate, then*

$$H_\mu(\mathcal{A}|a\mathcal{A}) \le \log|\det(Ad_a|_{\mathfrak{l}})|$$

and equality holds if and only if μ is L-invariant.

Let $L < G^+$ be a closed subgroup normalized by a, m_L be the Haar measure on L, and μ be an a-invariant probability measure on Y. Let $\mathcal{A}$ be a countably generated sub-σ-algebra of Borel σ-algebra which is a^{-1}-descending and L-subordinate modulo μ. Note that for any $j \in \mathbb{Z}_{\ge 0}$, the sub-σ-algebra $a^j\mathcal{A}$ is also countably generated, a^{-1}-descending, and L-subordinate modulo μ.

For $y \in Y$, denote by $V_y \subset L$ the shape of the $\mathcal{A}$-atom at $y \in Y$ so that $V_y \cdot y = [y]_{\mathcal{A}}$. It has positive m_L-measure for μ-a.e. $y \in Y$ since $\mathcal{A}$ is L-subordinate modulo μ. Note that for any $j \in \mathbb{Z}_{\ge 0}$, we have $[y]_{a^j\mathcal{A}} = a^j V_{a^{-j}y} a^{-j} \cdot y$.

As in [**EL10**, 7.55] which is the proof of [**EL10**, Theorem 7.9], let us define $\tau_y^{a^j\mathcal{A}}$ for μ-a.e $y \in Y$ to be the normalized push forward of $m_L|_{a^j V_{a^{-j}y} a^{-j}}$ under the orbit map, that is,

$$\tau_y^{a^j\mathcal{A}} = \frac{1}{m_L(a^j V_{a^{-j}y} a^{-j})} m_L|_{a^j V_{a^{-j}y} a^{-j}} \cdot y,$$

which is a probability measure on $[y]_{a^j\mathcal{A}}$.

The following proposition is an effective version of Theorem 2.11.

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THEOREM 2.12. *Let $L < G^+$ be a closed subgroup normalized by a and μ be an a-invariant ergodic probability measure on Y. Fix $j \in \mathbb{N}$ and denote by $J \geq 0$ the maximal entropy contribution of L for a^j, that is,*

$$J = \log |\det(Ad_{a^j}|_{\mathfrak{l}})|.$$

Let $\mathcal{A}$ be a countably generated sub-σ-algebra of Borel σ-algebra which is a^{-1}-descending and L-subordinate. Suppose there exist a measurable subset $K \subset Y$ and a symmetric measurable subset $B \subset L$ such that $[y]_{\mathcal{A}} \subset B \cdot y$ for any $y \in K$. Then, we have

$$H_\mu(\mathcal{A}|a^j\mathcal{A}) \leq J + \int_Y \log \tau_y^{a^j\mathcal{A}}\big((Y \setminus K) \cup B \operatorname{Supp} \mu\big)\, d\mu(y).$$

Proof. By for instance [**EL10**, Theorem 5.9], for μ-a.e. $y \in Y$, $\mu_y^{a^j\mathcal{A}}$ is a probability measure on $[y]_{a^j\mathcal{A}} = a^j V_{a^{-j}y} a^{-j} \cdot y$, and $H_\mu(\mathcal{A}|a^j\mathcal{A})$ can be written as

$$H_\mu(\mathcal{A}|a^j\mathcal{A}) = -\int_Y \log \mu_y^{a^j\mathcal{A}}([y]_{\mathcal{A}})\, d\mu(y).$$

Note that $m_L(a^j B a^{-j}) = e^J m_L(B)$ for any measurable $B \subset L$. Let

$$p(y) \stackrel{\text{def}}{=} \mu_y^{a^j\mathcal{A}}([y]_{\mathcal{A}}) \quad \text{and} \quad p^{\text{Haar}}(y) \stackrel{\text{def}}{=} \tau_y^{a^j\mathcal{A}}([y]_{\mathcal{A}}).$$

Then, we have

$$p^{\text{Haar}}(y) = \frac{m_L(V_y)}{m_L(a^j V_{a^{-j}y} a^{-j})} = \frac{m_L(V_y)}{m_L(V_{a^{-j}y})} e^{-J},$$

and hence, applying the ergodic theorem, we have $-\int_Y \log p^{\text{Haar}}(y)\, d\mu(y) = J$.

Now we estimate an upper bound of $H_\mu(\mathcal{A}|a^j\mathcal{A}) - J$ following the computation in [**EL10**, 7.55]. Following [**EL10**, 7.55], we can partition $[y]_{a^j\mathcal{A}}$ into a countable union of $\mathcal{A}$-atoms as follows:

$$[y]_{a^j\mathcal{A}} = \bigcup_{i=1}^{\infty} [x_i]_{\mathcal{A}} \cup N_y,$$

where N_y is a null set with respect to $\mu_y^{a^j\mathcal{A}}$. Note that $\mu_y^{a^j\mathcal{A}}$ is supported on $\operatorname{Supp} \mu$ for μ-a.e y. Since $B \subset L$ is symmetric, if $x_i \in K \setminus B \operatorname{Supp} \mu$, then $[x_i]_{\mathcal{A}} \subset B \cdot x_i \subset K \setminus \operatorname{Supp} \mu$, and hence we have $\mu_y^{a^j\mathcal{A}}([x_i]_{\mathcal{A}}) = 0$. If $x_i \in (Y \setminus K) \cup B \operatorname{Supp} \mu$ and $[x_i]_{\mathcal{A}} \not\subset (Y \setminus K) \cup B \operatorname{Supp} \mu$, then there exists $x_i' \in [x_i]_{\mathcal{A}}$ such that $x_i' \in K \setminus B \operatorname{Supp} \mu$, and hence $\mu_y^{a^j\mathcal{A}}([x_i]_{\mathcal{A}}) = \mu_y^{a^j\mathcal{A}}([x_i']_{\mathcal{A}}) = 0$. Thus, we denote by Z the set of x_i terms in $(Y \setminus K) \cup B \operatorname{Supp} \mu$ such that $[x_i]_{\mathcal{A}} \subset (Y \setminus K) \cup B \operatorname{Supp} \mu$. It follows that

$$\begin{aligned} H_\mu(\mathcal{A}|a^j\mathcal{A}) - J &= -\int_Y (\log p(z) - \log p^{\text{Haar}}(z))\, d\mu(z) \\ &= \int_Y \int_Y (\log p^{\text{Haar}}(z) - \log p(z))\, d\mu_y^{a^j\mathcal{A}}(z)\, d\mu(y) \\ &= \int_Y \sum_{x_i \in Z} \int_{z \in [x_i]_{\mathcal{A}}} (\log p^{\text{Haar}}(z) - \log p(z))\, d\mu_y^{a^j\mathcal{A}}(z)\, d\mu(y) \end{aligned}$$

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$$
\begin{aligned}
&= \int_Y \sum_{x_i \in Z} \log \left(\frac{\tau_y^{a^j \mathcal{A}}([x_i]_{\mathcal{A}})}{\mu_y^{a^j \mathcal{A}}([x_i]_{\mathcal{A}})} \right) \mu_y^{a^j \mathcal{A}}([x_i]_{\mathcal{A}})\, d\mu(y) \\
&\leq \int_Y \log \left(\sum_{x_i \in Z} \tau_y^{a^j \mathcal{A}}([x_i]_{\mathcal{A}}) \right) d\mu(y) \\
&\leq \int_Y \log \tau_y^{a^j \mathcal{A}}((Y \setminus K) \cup B \operatorname{Supp} \mu)\, d\mu(y).
\end{aligned}
$$

The second last inequality follows from the convexity of the logarithm. This proves the proposition. □

In particular, if $\mathcal{A}$ is of the form $a^k \mathcal{A}^L$ for $k \in \mathbb{Z}$, then Theorem 2.12 still holds without assuming the ergodicity of μ.

COROLLARY 2.13. *Let $0 < r_0 \leq 1$ be given, μ be an a-invariant probability measure on Y, $L < G^+$ be a closed subgroup normalized by a, and $\mathcal{A}^L$ be as in Proposition 2.8. Then, Theorem 2.12 holds for $\mathcal{A}$ of the form $a^k \mathcal{A}^L$ for $k \in \mathbb{Z}$.*

Proof. Writing the ergodic decomposition $\mu = \int \mu_z^{\mathcal{E}}\, d\mu(z)$, we have

$$
h_\mu(a^j | \mathcal{A}_\infty^L) = \int h_{\mu_z^{\mathcal{E}}}(a^j | \mathcal{A}_\infty^L)\, d\mu(z),
$$

where $\mathcal{A}_\infty^L$ is the σ-algebra as in equation (2.12). By Proposition 2.10, we also have

$$
H_\mu(\mathcal{A}^L | a^j \mathcal{A}^L) = \int H_{\mu_z^{\mathcal{E}}}(\mathcal{A}^L | a^j \mathcal{A}^L)\, d\mu(z).
$$

It follows from the a-invariance of μ and $\mu_z^{\mathcal{E}}$ that

$$
H_\mu(\mathcal{A} | a^j \mathcal{A}) = \int H_{\mu_z^{\mathcal{E}}}(\mathcal{A} | a^j \mathcal{A})\, d\mu(z).
$$

Applying Theorem 2.12 for each $\mu_z^{\mathcal{E}}$, we obtain

$$
\begin{aligned}
H_\mu(\mathcal{A} | a^j \mathcal{A}) &= \int H_{\mu_z^{\mathcal{E}}}(\mathcal{A} | a^j \mathcal{A})\, d\mu(z) \\
&\leq J + \int_Y \int_Y \log \tau_y^{a^j \mathcal{A}}(B^2 \operatorname{Supp} \mu_z^{\mathcal{E}})\, d\mu_z^{\mathcal{E}}(y)\, d\mu(z) \\
&\leq J + \int_Y \log \tau_y^{a^j \mathcal{A}}(B^2 \operatorname{Supp} \mu)\, d\mu(y).
\end{aligned}
$$

□

3. *Preliminaries for the upper bound*

From now on, we fix the following notation:

$$
d = m + n, \quad G = \mathrm{ASL}_d(\mathbb{R}), \quad \Gamma = \mathrm{ASL}_d(\mathbb{Z}), \quad \text{and} \quad Y = G/\Gamma.
$$

We use all notation in §2.2 with this setting. In particular, we choose a right invariant metric d_G on G so that $r_{max} \leq 1$. Denote by d_∞ the metric on G induced from the max

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norm on $M_{d+1,d+1}(\mathbb{R})$. Since d_G and d_∞ are locally bi-Lipschitz, there are constants $0 < r_0 < 1$ and $C_0 \geq 1$ such that for any $x, y \in B^G_{r_0}$,

$$\frac{1}{C_0} d_\infty(x, y) \leq d_G(x, y) \leq C_0 d_\infty(x, y). \tag{3.1}$$

Note that r_0 and C_0 depend only on G. In the rest of the article, all the statements from Lemma 2.5 to Proposition 2.10 will be applied to this r_0.

Recall the notation a_t, $a = a_1$, U, and W in the introduction. The subgroups U and W are closed subgroups in G^+ normalized by a, where G^+ is the unstable horospherical subgroup associated to a. Denote by $\mathfrak{u}$ and $\mathfrak{w}$ the Lie algebras of U and W, respectively. We now consider the following quasinorms on $\mathfrak{u} = \mathbb{R}^{mn} = M_{m,n}(\mathbb{R})$ and $\mathfrak{w} = \mathbb{R}^m$: For $A \in M_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$, define

$$\|A\|_{\mathbf{r}\otimes\mathbf{s}} = \max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} |A_{ij}|^{1/(r_i+s_j)} \quad \text{and} \quad \|b\|_{\mathbf{r}} = \max_{1\leq i\leq m} |b_i|^{1/r_i}.$$

We call these quasinorms $\mathbf{r}\otimes\mathbf{s}$-quasinorm and $\mathbf{r}$-quasinorm, respectively.

We remark that for $A, A' \in M_{m,n}(\mathbb{R})$ and $b, b' \in \mathbb{R}^m$, using the convexity of functions $s \mapsto s^{1/(r_i+s_j)}$ and $s \mapsto s^{1/r_i}$,

$$\begin{aligned} \|A + A'\|_{\mathbf{r}\otimes\mathbf{s}} &\leq 2^{(1-(r_m+s_n))/(r_m+s_n)}(\|A\|_{\mathbf{r}\otimes\mathbf{s}} + \|A'\|_{\mathbf{r}\otimes\mathbf{s}}); \\ \|b + b'\|_{\mathbf{r}} &\leq 2^{(1-r_m)/r_m}(\|b\|_{\mathbf{r}} + \|b'\|_{\mathbf{r}}). \end{aligned} \tag{3.2}$$

It also holds that

$$\|\mathrm{Ad}_{a_t} A\|_{\mathbf{r}\otimes\mathbf{s}} = e^t\|A\|_{\mathbf{r}\otimes\mathbf{s}} \quad \text{and} \quad \|\mathrm{Ad}_{a_t} b\|_{\mathbf{r}} = e^t\|b\|_{\mathbf{r}}$$

for any $A \in M_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$.

By a *quasi-metric* on a space Z, we mean a map $d_Z : Z \times Z \to \mathbb{R}_{\geq 0}$ which is a symmetric, positive definite map such that for some constant C, for all $x, y \in Z$, $d_Z(x, y) \leq C(d_Z(x, z) + d_Z(z, y))$. The $\mathbf{r}\otimes\mathbf{s}$-quasinorm (respectively $\mathbf{r}$-quasinorm) induces the quasi-metric $d_{\mathbf{r}\otimes\mathbf{s}}$ (respectively $d_{\mathbf{r}}$) on $\mathfrak{u}$ (respectively $\mathfrak{w}$). Note that the logarithm map is defined on U and W, and hence the quasi-metric $d_{\mathbf{r}\otimes\mathbf{s}}$ (respectively $d_{\mathbf{r}}$) induces the quasi-metric on U (respectively W) via the logarithm map. For simplicity, we keep the notation $d_{\mathbf{r}\otimes\mathbf{s}}$ and $d_{\mathbf{r}}$ for the quasi-metrics on U and W, respectively. We similarly denote by $B^{U,\mathbf{r}\otimes\mathbf{s}}_r$ (respectively $B^{W,\mathbf{r}}_r$) the open r-ball around the identity in U (respectively W) with respect to the quasi-metric $d_{\mathbf{r}\otimes\mathbf{s}}$ (respectively $d_{\mathbf{r}}$). For any $y \in Y$, we also denote by $d_{\mathbf{r}\otimes\mathbf{s}}$ (respectively $d_{\mathbf{r}}$) the induced quasi-metric on the fiber $B^U_{r_y} \cdot y$ (respectively $B^W_{r_y} \cdot y$).

As in Theorem 2.11, we can explicitly compute the maximum entropy contributions for $L = U$ and $L = W$. For $L = U$, the restricted adjoint map is the expansion $\mathrm{Ad}_a : (A_{ij}) \mapsto (e^{r_i+s_j} A_{ij})$ of $A \in M_{m,n}(\mathbb{R})$, and hence

$$\log|\det(Ad_a|_{\mathfrak{u}})| = \sum_{i=1}^{m}\sum_{j=1}^{n}(r_i + s_j) = m + n.$$

For $L = W$, the restricted adjoint map is the expansion $Ad_a : (b_i) \mapsto (e^{r_i} b_i)$ of $b \in \mathbb{R}^m$, and hence

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$$\log|\det(Ad_a|_{\mathfrak{w}})| = \sum_{i=1}^{m} r_i = 1.$$

Denote by $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ and by $\pi : Y \to X$ the natural projection sending a translated lattice $x + v$ to the lattice x. Equivalently, it is defined by $\pi((\begin{smallmatrix} g & v \\ 0 & 1 \end{smallmatrix})\Gamma) = g\,\mathrm{SL}_d(\mathbb{Z})$ for $g \in \mathrm{SL}_d(\mathbb{R})$ and $v \in \mathbb{R}^d$. We also use the following notation: $w(v) = (\begin{smallmatrix} I_d & v \\ 0 & 1 \end{smallmatrix})$ for $v \in \mathbb{R}^d$.

3.1. *Dimensions.* Let Z be a space endowed with a quasi-metric d_Z. For a bounded subset $S \subset Z$, the lower Minkowski dimension $\underline{\dim}_{d_Z} S$ with respect to the quasi-metric d_Z is defined by

$$\underline{\dim}_{d_Z} S \overset{\text{def}}{=} \liminf_{\delta \to 0} \frac{\log N_\delta(S)}{\log 1/\delta},$$

where $N_\delta(S)$ is the maximal cardinality of a δ-separated subset of S for d_Z.

Now, for subsets $S \subset \mathfrak{u} = \mathbb{R}^{mn}$ and $S' \subset \mathfrak{w} = \mathbb{R}^m$ in the Lie algebras $\mathfrak{u}$ and $\mathfrak{w}$, we denote the lower Minkowski dimensions of these subsets as follows:

$$\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}} S \overset{\text{def}}{=} \underline{\dim}_{d_{\mathbf{r}\otimes\mathbf{s}}} S, \quad \underline{\dim}_{\mathbf{r}} S' \overset{\text{def}}{=} \underline{\dim}_{d_{\mathbf{r}}} S'.$$

We will also consider Hausdorff dimensions $\dim_H S$ and $\dim_H S'$, always defined with respect to the standard metric.

LEMMA 3.1. [LSS19, Lemma 2.2] *For subsets $S \subset \mathfrak{u}$ and $S' \subset \mathfrak{w}$:*

(1) $\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}} \mathfrak{u} = \sum_{i,j}(r_i + s_j) = m + n$ *and* $\underline{\dim}_{\mathbf{r}} \mathfrak{w} = \sum_i r_i = 1$*;*

(2) $\underline{\dim}_{\mathbf{r}\otimes\mathbf{s}} S \ge (m+n) - (r_1 + s_1)(mn - \dim_H S)$*;*

(3) $\underline{\dim}_{\mathbf{r}} S' \ge 1 - r_1(m - \dim_H S')$.

3.2. *Correspondence with dynamics.* For $y = (\begin{smallmatrix} g & v \\ 0 & 1 \end{smallmatrix})\Gamma \in Y$ with $g \in \mathrm{SL}_d(\mathbb{R})$ and $v \in \mathbb{R}^d$, denote by Λ_y the corresponding unimodular grid $g\mathbb{Z}^d + v$ in $\mathbb{R}^d$. We denote the $(\mathbf{r}, \mathbf{s})$-quasinorm of $v = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n$ by $\|v\|_{\mathbf{r},\mathbf{s}} = \max\{\|\mathbf{x}\|_{\mathbf{r}}^{d/m}, \|\mathbf{y}\|_{\mathbf{s}}^{d/n}\}$. Let

$$\mathcal{L}_\epsilon \overset{\text{def}}{=} \{y \in Y : \text{ for all } v \in \Lambda_y, \|v\|_{\mathbf{r},\mathbf{s}} \ge \epsilon\},$$

which is a (non-compact) closed subset of Y. Following [Kle99, §1.3], we say that the pair $(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$ is *rational* if there exists some $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$ such that $Aq - b + p = 0$, and *irrational* otherwise.

PROPOSITION 3.2. *For any irrational pair $(A, b) \in M_{m,n}(\mathbb{R}) \times \mathbb{R}^m$, $(A, b) \in$ **Bad**(ϵ) if and only if the a_t-orbit of the point $y_{A,b}$ is eventually in $\mathcal{L}_\epsilon$, that is, there exists $T \ge 0$ such that $a_t y_{A,b} \in \mathcal{L}_\epsilon$ for all $t \ge T$.*

Proof. Suppose that there exist arbitrarily large t terms satisfying $a_t y_{A,b} \notin \mathcal{L}_\epsilon$. Denote $e^{\mathbf{r}t} \overset{\text{def}}{=} \mathrm{diag}(e^{r_1 t}, \ldots, e^{r_m t}) \in M_{m,m}(\mathbb{R})$ and $e^{\mathbf{s}t} \overset{\text{def}}{=} \mathrm{diag}(e^{s_1 t}, \ldots, e^{s_n t}) \in M_{n,n}(\mathbb{R})$. Then, the vectors in the grid $\Lambda_{a_t y_{A,b}}$ can be represented as

$$a_t\left(\begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} -b \\ 0 \end{pmatrix}\right) = \begin{pmatrix} e^{\mathbf{r}t}(Aq + p - b) \\ e^{-\mathbf{s}t} q \end{pmatrix}$$

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for $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n$. Therefore, $a_t x_{A,b} \notin \mathcal{L}_\epsilon$ implies that for some $q \in \mathbb{Z}^n$,

$$e^t \langle Aq - b\rangle_{\mathbf{r}} < \epsilon^{m/d} \quad \text{and} \quad e^{-t}\|q\|_{\mathbf{s}} < \epsilon^{n/d}, \tag{3.3}$$

and thus $\|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} < \epsilon$. Since $\langle Aq - b\rangle_{\mathbf{r}} \neq 0$ for all q, we use the condition $\langle Aq - b\rangle_{\mathbf{r}} < e^{-t}\epsilon^{m/d}$ for arbitrarily large t to conclude that $\|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} < \epsilon$ holds for infinitely many q terms. This is a contradiction to the assumption that $(A, b) \in \mathbf{Bad}(\epsilon)$.

However, if $(A, b) \notin \mathbf{Bad}(\epsilon)$, then since (A, b) is irrational, there are infinitely many $q \in \mathbb{Z}^n$ such that $\|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} < \epsilon$. Thus, we can choose arbitrarily large t so that equation (3.3) holds, which contradicts the assumption that the a_t-orbit of the point $y_{A,b}$ is eventually in $\mathcal{L}_\epsilon$. □

Remark 3.3. We claim that for a fixed $b \in \mathbb{R}^m$, the subset $\mathbf{Bad}^b_0(\epsilon)$ of rational (A, b) terms in $\mathbf{Bad}^b(\epsilon)$ is a subset of $\mathbf{Bad}^0(\epsilon)$. Indeed, if $A \in \mathbf{Bad}^b(\epsilon)$ for some b and (A, b) is rational, then $\langle Aq_0 - b\rangle_{\mathbf{r}} = 0$ for some $q_0 \in \mathbb{Z}^m$ and $\liminf_{\|q\|_{\mathbf{s}}\to\infty} \|q\|_{\mathbf{s}}\langle Aq - b\rangle_{\mathbf{r}} \geq \epsilon$, and thus $\liminf_{\|q\|_{\mathbf{s}}\to\infty} \|q\|_{\mathbf{s}}\langle A(q - q_0)\rangle_{\mathbf{r}} \geq \epsilon$. Therefore, we have

$$\dim_H \mathbf{Bad}^b_0(\epsilon) \leq \dim_H \mathbf{Bad}^0(\epsilon) = mn - c_{m,n}\frac{\epsilon}{\log 1/\epsilon} < mn$$

for some constant $c_{m,n} > 0$ [**KM19**]. For a fixed $A \in M_{m,n}(\mathbb{R})$, the subset of $\mathbf{Bad}_A(\epsilon)$ such that (A, b) is rational is of the form $Aq + p$ for some $q, p \in \mathbb{Z}^m$ and thus has Hausdorff dimension zero.

In the rest of the article, we will focus on the elements $y_{A,b}$ that are eventually in $\mathcal{L}_\epsilon$.

3.3. *Covering counting lemma.* To construct measures of large entropy in Proposition 4.1 and 5.4, we will need the following counting lemma, which is a generalization of [**LSS19**, Lemma 2.4].

Here, we consider two cases: $L = U$ and $L = W$. Denote by $\mathbf{c} = (c_1, \ldots, c_{\dim \mathfrak{l}})$ either $\mathbf{r} \otimes \mathbf{s}$ (for $L = U$) or $\mathbf{r}$ (for $L = U$), and denote by $\|\cdot\|_{\mathbf{c}}$ either $\|\cdot\|_{\mathbf{r}\otimes\mathbf{s}}$ (for $L = U$) or $\|\cdot\|_{\mathbf{r}}$ (for $L = W$). Let J_L be the maximal entropy contribution for L. Recall that $J_U = m + n$ and $J_W = 1$.

Before stating the main result of this subsection, we fix the following notation. Fix a 'cusp part' $Q^0_\infty \subset X$ that is a connected subset such that $X \setminus Q^0_\infty$ has compact closure. Set $Q_\infty = \pi^{-1}(Q^0_\infty)$ and denote by $r(Q_\infty) > 0$ the infimum of injectivity radius on $Y \setminus Q_\infty$. For any $D > J_L$, choose large enough $T_D \in \mathbb{N}$ such that for all $i = 1, \ldots, \dim \mathfrak{l}$,

$$\lceil e^{c_i T_D}\rceil \leq e^{c_i T_D} e^{(D - J_L)/\dim \mathfrak{l}}. \tag{3.4}$$

For $r_0 > 0$ and $C_0 \geq 1$ from equation (3.1), fix $0 < r_D = r_D(Q^0_\infty) < \min(r_0, 1/2)$ small enough so that

$$B^{L,\mathbf{c}}_{2^{1/\min \mathbf{c}} C_0 r_D^{1/\max \mathbf{c}} T_D} \subset B^L_{\min(r_0,(1/2)r(Q_\infty))} \quad \text{and} \quad B^G_{r_D}(Y \setminus Q_\infty) \subset Y\big(\tfrac{1}{2}r(Q_\infty)\big). \tag{3.5}$$

LEMMA 3.4. *For any $Q^0_\infty \subset X$ and $D > J_L$, we fix the above notation. Let $y \in Y \setminus Q_\infty$ and $I = \{t \in \mathbb{N} \mid a_t y \in Q_\infty\}$. For any non-negative integer T, let*

$$E_{y,T} = \{z \in B^L_{r_D} \cdot y \mid \textit{for all } t \in \{1, \ldots, T\} \setminus I,\ d_Y(a_t y, a_t z) \leq r_D\}.$$

The set $E_{y,T}$ can be covered by $Ce^{D|I\cap\{1,\ldots,T\}|}$ $d_{\mathbf{c}}$-balls of radius $r_D^{1/\max\mathbf{c}}e^{-T}$, where C is a constant depending on Q_∞^0 and D, but independent of T.

Proof. For $s\in\{0,\ldots,T_D-1\}$ and $k\in\mathbb{Z}_{\geq 0}$, let us denote $I_{s,k}(T_D)=\{s,s+T_D,\ldots,s+kT_D\}$ and

$$E^s_{y,k}=\{z\in B^L_{r_D}\cdot y:\ \text{for all } t\in I_{s,k}(T_D)\setminus I,\, d_Y(a_ty,a_tz)\leq r_D\}.$$

Following the proof of [**LSS19**, Lemma 2.4] with $E^s_{y,k}$ instead of $E_{y,T}$, we obtain the following claim.

CLAIM. *The set $E^s_{y,k}$ can be covered by $C_se^{(J_L(T_D-1)+D)|I\cap I_{s,k}(T_D)|}d_{\mathbf{c}}$-balls of radius $C_0r_D^{1/\max\mathbf{c}}e^{-(s+kT_D)}$, where C_s is a constant depending on Q^0_∞, D, and s, but independent of k.*

Proof of the claim. We prove the claim by induction on k. Since the number of $d_{\mathbf{c}}$-balls of radius $C_0r_D^{1/\max\mathbf{c}}e^{-s}$ needed to cover $B^L_{r_D}\cdot y$ is bounded by a constant C_s depending on Q^0_∞, D, and s, the claim holds for $k=0$.

Suppose that $E^s_{y,k-1}$ can be covered by $N_{k-1}=C_se^{(J_L(T_D-1)+D)|I\cap I_{s,k-1}(T_D)|}$ $d_{\mathbf{c}}$-balls $\{B_j : j=1,\ldots,N_{k-1}\}$ of radius $C_0r_D^{1/\max\mathbf{c}}e^{-(s+(k-1)T_D)}$. By the inequality in equation (3.4), any $d_{\mathbf{c}}$-ball of radius $C_0r_D^{1/\max\mathbf{c}}e^{-(s+(k-1)T_D)}$ can be covered by

$$\begin{aligned}\prod_{i=1}^{\dim\mathfrak{l}}\left\lceil\frac{e^{-(s+(k-1)T_D)c_i}}{e^{-(s+kT_D)c_i}}\right\rceil=\prod_{i=1}^{\dim\mathfrak{l}}\lceil e^{T_Dc_i}\rceil&\leq\prod_{i=1}^{\dim\mathfrak{l}}e^{c_iT_D}e^{(D-J_L)/\dim\mathfrak{l}}\\&=e^{J_LT_D}e^{D-J_L}=e^{J_L(T_D-1)+D}\end{aligned}$$

$d_{\mathbf{c}}$-balls of radius $C_0r_D^{1/\max\mathbf{c}}e^{-(s+kT_D)}$. Thus, if $s+kT_D\in I$, then $E^s_{y,k}$ can be covered by $N_k=e^{J_L(T_D-1)+D}N_{k-1}d_{\mathbf{c}}$-balls of radius $C_0r_D^{1/\max\mathbf{c}}e^{-(s+kT_D)}$.

Suppose that $s+kT_D\notin I$. Since $E^s_{y,k}\subset E^s_{y,k-1}$, the sets $E^s_{y,k}\cap B_j$ with $j=1,\ldots,N_{k-1}$ cover $E^s_{y,k}$. We now claim that for any $1\leq j\leq N_{k-1}$ and $x_1,x_2\in E^s_{y,k}\cap B_j$, we have

$$d_{L,\mathbf{c}}(x_1,x_2)\leq 2^{1/\min\mathbf{c}}C_0r_D^{1/\max\mathbf{c}}e^{-(s+kT_D)}.$$

Indeed, since B_j is a $d_{L,\mathbf{c}}$-ball of radius $C_0r_D^{1/\max\mathbf{c}}e^{-(s+(k-1)T_D)}$ and $x_1,x_2\in B_j\subset B^L_{r_D}\cdot y$, there are $h\in B^L_{r_D}$ and $h_1,h_2\in B^{L,\mathbf{c}}_{C_0r_D^{1/\max\mathbf{c}}e^{-(s+(k-1)T_D)}}$ such that $x_1=h_1hy$ and $x_2=h_2hy$. It follows from $s+kT_d\notin I$ and $x_1,x_2\in E^s_{y,k}$ that $a^{s+kT_d}y\subset Y\setminus Q_\infty$ and $d_Y(a^{s+kT_D}y,a^{s+kT_D}x_\ell)\leq r_D$ for $\ell=1,2$, and hence by equation (3.5), we have $a^{s+kT_D}x_1\in B^G_{r_D}(Y\setminus Q_\infty)\subset Y(\frac{1}{2}r(Q_\infty))$ and $d_Y(a^{s+kT_D}x_1,a^{s+kT_D}x_2)\leq 2r_D$. Observe that by equation (3.5),

$$\begin{aligned}a^{s+kT_D}h_1h_2^{-1}a^{-(s+kT_D)}&\subset a^{s+kT_D}B^{L,\mathbf{c}}_{2^{1/\min\mathbf{c}}C_0r_D^{1/\max\mathbf{c}}e^{-(s+(k-1)T_D)}}a^{-(s+kT_D)}\\&=B^{L,\mathbf{c}}_{2^{1/\min\mathbf{c}}C_0r_D^{1/\max\mathbf{c}}e^{T_D}}\subset B^L_{\min(r_0,(1/2)r(Q_\infty))}.\end{aligned}$$

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Thus, it follows from equation (3.1) and the above observations that

$$\begin{aligned}2r_D \geq d_Y(a^{s+kT_D}x_1, a^{s+kT_D}x_2) &= d_L(a^{s+kT_D}h_1h_2^{-1}a^{-(s+kT_D)}, \mathrm{id})\\ &\geq \frac{1}{C_0}d_\infty(a^{s+kT_D}h_1h_2^{-1}a^{-(s+kT_D)}, \mathrm{id})\\ &= \frac{1}{C_0}\max_{i=1,\ldots,\dim \mathfrak{l}} e^{c_i(s+kT_D)}|(\log h_1h_2^{-1})_i|,\end{aligned}$$

where $(\log h_1h_2^{-1})_i$ is the ith coordinate of $\log h_1h_2^{-1}$ with respect to the standard basis $\{e_i : 1 \leq i \leq \dim \mathfrak{l}\}$ of $\mathfrak{l}$. Since $L = U$ or $L = W$, that is, a commutative subgroup of G, for each $i = 1, \ldots, \dim \mathfrak{l}$, we have

$$|(\log h_1h_2^{-1})_i| = |(\log h_1 - \log h_2)_i| \leq 2r_DC_0e^{-c_i(s+kT_D)}.$$

Note that

$$d_{L,\mathbf{c}}(x_1, x_2) = d_{L,\mathbf{c}}(h_1, h_2) = \max_{i=1,\ldots,\dim \mathfrak{l}} |(\log h_1 - \log h_2)_i|^{1/c_i}.$$

Therefore, we have

$$d_{L,\mathbf{c}}(x_1, x_2) \leq \max_{i=1,\ldots,\dim \mathfrak{l}} (2r_DC_0)^{1/c_i}e^{-(s+kT_D)} \leq 2^{1/\min \mathbf{c}}C_0r_D^{1/\max \mathbf{c}}e^{-(s+kT_D)}.$$

It follows from the claim that $E^s_{y,k} \cap B_j$ is contained in a single $d_{L,\mathbf{c}}$-ball of radius $C_0r_D^{1/\max \mathbf{c}}e^{-(s+kT_D)}$ for each $j = 1, \ldots, N_{k-1}$. Hence, $E^s_{y,k}$ can be covered by $N_k = N_{k-1}d_{L,\mathbf{c}}$-balls of radius $C_0r_D^{1/\max \mathbf{c}}e^{-(s+kT_D)}$. □

Now, for any non-negative integer T, we can find $s \in \{0, \ldots, T_D - 1\}$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$T_D|I \cap I_{s,k}(T_D)| \leq |I \cap \{1, \ldots, T\}| \quad \text{and} \quad T - T_D < s + kT_D \leq T$$

from the pigeon hole principle. By the above observation, $E_{y,T} \subset E^s_{y,k}$ can be covered by $C_se^{(J_L(T_D-1)+D)|I\cap I_{s,k}(T_D)|}d_{\mathbf{c}}$-balls of radius $C_0r_D^{1/\max \mathbf{c}}e^{-(s+kT_D)}$. Since $T - T_D < s + kT_D \leq T$ and $D > J_L$, $E_{y,T}$ can be covered by $(\max_{0\leq s\leq T_D-1} C_s)e^{D|I\cap\{1,\ldots,T\}|}d_{\mathbf{c}}$-balls of radius $C_0e^{T_D}r_D^{1/\max \mathbf{c}}e^{-T}$. Hence, there exists a constant $C > 0$ depending on Q^0_∞, r, and D, but independent of T such that $E_{y,T}$ can be covered by $Ce^{D|I\cap\{1,\ldots,T\}|}d_{\mathbf{c}}$-balls of radius $r_D^{1/\max \mathbf{c}}e^{-T}$. □

4. *Upper bound for Hausdorff dimension of* $\mathbf{Bad}_A(\epsilon)$

In this section, we will prove Theorem 1.2 by constructing an a-invariant probability measure on Y with large entropy. Here and in the next section, we will consider the dynamical entropy of a instead of a^{-1} in contrast to §2. Hence, let us use the following notation. For a given partition Q of Y and a integer $q \geq 1$, we denote

$$Q^{(q)} = \bigvee_{i=0}^{q-1} a^{-i}Q.$$

4.1. *Constructing measure with entropy lower bound.* Let us denote by $\overline{X}$ and $\overline{Y}$ the one-point compactifications of X and Y, respectively. Let $\mathcal{A}$ be a given countably generated σ-algebra of X or Y. We denote by $\overline{\mathcal{A}}$ the σ-algebra generated by $\mathcal{A}$ and $\{\infty\}$. The diagonal action a_t is extended to the action on $\overline{X}$ and $\overline{Y}$ by $a_t(\infty) = \infty$ for $t \in \mathbb{R}$. For a finite partition $Q = \{Q_1, \ldots, Q_N, Q_\infty\}$ of Y which has only one non-compact element Q_∞, denote by $\overline{Q}$ the finite partition $\{Q_1, \ldots, Q_N, \overline{Q_\infty} \stackrel{\text{def}}{=} Q_\infty \cup \{\infty\}\}$ of $\overline{Y}$. Note that $\overline{Q^{(q)}} = \overline{Q}^{(q)}$ for any $q \in \mathbb{N}$. Denote by $\mathscr{P}(X)$ the space of probability measures on X, and use similar notation for Y, $\overline{X}$, and $\overline{Y}$.

In this subsection, we construct an a-invariant measure on $\overline{Y}$ with a lower bound on the conditional entropy for the proof of Theorem 1.2. Here, the conditional entropy will be computed with respect to the σ-algebras constructed in §2. If x_A has no escape of mass, such measure was constructed in [**LSS19**, Proposition 2.3]. The following proposition generalizes the measure construction for x_A terms with some escape of mass.

PROPOSITION 4.1. *For $A \in M_{m,n}(\mathbb{R})$ fixed, let*

$$\eta_A = \sup\{\eta : x_A \text{ has } \eta\text{-escape of mass on average}\}.$$

Then, there exists $\mu_A \in \mathscr{P}(\overline{X})$ with $\mu_A(X) = 1 - \eta_A$ such that for any $\epsilon > 0$, there exists an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ satisfying:

(1) *$\operatorname{Supp} \overline{\mu} \subset \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y)$;*
(2) *$\pi_* \overline{\mu} = \mu_A$, in particular, there exists an a-invariant measure $\mu \in \mathscr{P}(Y)$ such that*

$$\overline{\mu} = (1 - \eta_A)\mu + \eta_A \delta_\infty,$$

where δ_∞ is the dirac delta measure on $\overline{Y} \setminus Y$;
(3) *let $\mathcal{A}^W$ be as in Proposition 2.8 for μ, r_0, and $L = W$, and let $\mathcal{A}^W_\infty$ be as in equation (2.12). Then, we have*

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^W_\infty}) \geq 1 - \eta_A - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)).$$

Remark 4.2.

(1) Note that if $\eta_A > 0$, then x_A has η_A-escape of mass on average;
(2) one can check that $\eta_A = 0$ if and only if x_A is *heavy*, which is defined in [**LSS19**, Definition 1.1].

Proof. Since x_A has η_A-escape of mass on average but no more than η_A, we may fix an increasing sequence of integers $\{k_i\}_{i \geq 1}$ such that

$$\frac{1}{k_i} \sum_{k=0}^{k_i - 1} \delta_{a^k x_A} \xrightarrow{\text{w}^*} \mu_A \in \mathscr{P}(\overline{X})$$

with $\mu_A(X) = 1 - \eta_A$.

Let us denote by $\mathbb{T}^m = [0, 1]^m/\sim$ the torus in $\mathbb{R}^m$, where the equivalence relation is modulo 1. Consider the increasing family of sets

$$R^{A,T} \stackrel{\text{def}}{=} \{b \in \mathbb{T}^m | \text{ for all } t \geq T, a_t y_{A,b} \in \mathcal{L}_\epsilon\} \cap \mathbf{Bad}_A(\epsilon).$$

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By Proposition 3.2 and Remark 3.3, $\bigcup_{T=1}^{\infty} R^{A,T}$ has Hausdorff dimension equal to $\dim_H \mathbf{Bad}_A(\epsilon)$. For any $\gamma > 0$, it follows that there exists $T_\gamma \in \mathbb{N}$ satisfying $\dim_H R^{A,T_\gamma} \geq \dim_H \mathbf{Bad}_A(\epsilon) - \gamma$.

Let $\phi_A : \mathbb{T}^m \to Y$ be the map defined by $\phi_A(b) = y_{A,b}$. Note that ϕ_A is a one-to-one Lipschitz map between $\mathbb{T}^m$ and $\phi_A(\mathbb{T}^m)$, so we may consider a quasinorm on $\phi_A(\mathbb{T}^m)$ induced from the $\mathbf{r}$-quasinorm on $\mathbb{R}^m$ and denote it again by $\|\cdot\|_{\mathbf{r}}$.

For each $k_i \geq T_\gamma$, let S_i be a maximal e^{-k_i}-separated subset of R^{A,T_γ} with respect to the $\mathbf{r}$-quasinorm. By Lemma 3.1(3),

$$\liminf_{i\to\infty} \frac{\log |S_i|}{k_i} \geq \underline{\dim}_{\mathbf{r}}(R^{A,T_\gamma}) \geq 1 - r_1(m + \gamma - \dim_H \mathbf{Bad}_A(\epsilon)).$$

Let $\nu_i \stackrel{\text{def}}{=} (1/|S_i|) \sum_{b\in S_i} \delta_{y_{A,b}}$ be the normalized counting measure on the set $D_i \stackrel{\text{def}}{=} \{y_{A,b} : b \in S_i\} \subset Y$. Extracting a subsequence if necessary, we may assume that

$$\mu_i \stackrel{\text{def}}{=} \frac{1}{k_i} \sum_{k=0}^{k_i-1} a_*^k \nu_i \xrightarrow{\text{w}^*} \mu^\gamma \in \mathscr{P}(\overline{Y}).$$

The measure μ^γ is a-invariant since $a_*\mu_i - \mu_i$ goes to zero measure.

Choose any sequence of positive real numbers $(\gamma_j)_{j\geq 1}$ converging to zero and let $\{\mu^{\gamma_j}\}$ be a family of a-invariant probability measures on $\overline{Y}$ obtained from the above construction for each γ_j. Extracting a subsequence again if necessary, we may take a weak*-limit measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ of $\{\mu^{\gamma_j}\}$. We prove that $\overline{\mu}$ is the desired measure. The measure $\overline{\mu}$ is clearly a-invariant.

(1) We show that for all $\gamma > 0$, $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$. For any $b \in S_i \subseteq R^{A,T_\gamma}$, $a_T y_{A,b} \in \mathcal{L}_\epsilon$ holds for $T > T_\gamma$. Thus, we have

$$\begin{aligned}\mu_i(Y \setminus \mathcal{L}_\epsilon) &= \frac{1}{k_i}\sum_{k=0}^{k_i-1} a_*^k \nu_i(Y \setminus \mathcal{L}_\epsilon) = \frac{1}{k_i}\sum_{k=0}^{T_\gamma} a_*^k \nu_i(Y \setminus \mathcal{L}_\epsilon)\\ &= \frac{1}{k_i|S_i|}\sum_{y\in D_i, 0\leq k\leq T_\gamma} \delta_{a^k y}(Y \setminus \mathcal{L}_\epsilon) \leq \frac{T_\gamma}{k_i}.\end{aligned}$$

By taking $k_i \to \infty$, we have $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$ for arbitrary $\gamma > 0$, and hence

$$\overline{\mu}(Y \setminus \mathcal{L}_\epsilon) = \lim_{j\to\infty} \mu^{\gamma_j}(Y \setminus \mathcal{L}_\epsilon) = 0.$$

(2) For all $\gamma > 0$, $\pi_*\mu^\gamma = \mu_A$ since $\pi_*\nu_i = \delta_{x_A}$ for all $i \geq 1$. It follows that $\pi_*\overline{\mu} = \mu_A$. Hence,

$$\overline{\mu}(\overline{Y} \setminus Y) = \lim_{j\to\infty} \mu^{\gamma_j}(\overline{Y} \setminus Y) = \mu_A(\overline{X} \setminus X) = \eta_A,$$

so we have a decomposition $\overline{\mu} = (1 - \eta_A)\mu + \eta_A\delta_\infty$ for some a-invariant $\mu \in \mathscr{P}(Y)$.

(3) We first fix any $D > J_W = 1$ and $Q_\infty^0 \subset X$ such that $X \setminus Q_\infty^0$ has compact closure. As in [LSS19, Proof of Theorem 4.2, Claim 2], we can construct a finite partition Q of Y satisfying:

- $\mathcal{Q}$ contains an atom Q_∞ of the form $\pi^{-1}(Q_\infty^0)$;
- for all $Q \in \mathcal{Q} \setminus \{Q_\infty\}$, diam $Q < r_D = r_D(Q_\infty^0)$, where r_D is from equation (3.5);
- for all $Q \in \mathcal{Q}$, for all $j \geq 1$, $\mu^{\gamma_j}(\partial Q) = 0$.

Remark that for all $i \geq 1$, $D_i \subset \phi_A(\mathbb{T}^m)$, which is a compact set in Y; therefore, we can choose Q_∞^0 so that

$$Q_\infty \cap D_i = \varnothing. \tag{4.1}$$

We claim that it suffices to show the following statement. For all $q \geq 1$,

$$\frac{1}{q} H_{\overline{\mu}}(\overline{\mathcal{Q}}^{(q)} | \overline{\mathcal{A}_\infty^W}) \geq 1 - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)) - D\overline{\mu}(\overline{Q_\infty}). \tag{4.2}$$

Indeed, by taking $q \to \infty$, we have

$$h_{\overline{\mu}}(a | \overline{\mathcal{A}^W}) \geq 1 - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)) - D\overline{\mu}(\overline{Q_\infty}).$$

Taking $D \to 1$ and $Q_\infty^0 \subset X$ such that $\overline{\mu}(\overline{Q_\infty}) \to \overline{\mu}(\overline{Y} \setminus Y) = \eta_A$ and $D \to 1$, we conclude equation (3).

In the rest of the proof, we show the inequality in equation (4.2). It is clear if $\overline{\mu}(Q_\infty) = 1$, so assume that $\overline{\mu}(Q_\infty) < 1$, and hence for all large enough $j \geq 1$, $\mu^{\gamma_j}(Q_\infty) < 1$. Now, we fix such $j \geq 1$ and write temporarily $\gamma = \gamma_j$.

Choose $\beta > 0$ such that $\mu^\gamma(Q_\infty) < \beta < 1$. For large enough $i \geq 1$, we have

$$\mu_i(Q_\infty) = \frac{1}{k_i |S_i|} \sum_{y \in D_i, 0 \leq k < k_i} \delta_{a^k y}(Q_\infty) = \frac{1}{k_i} \sum_{0 \leq k < k_i} \delta_{a^k x_A}(Q_\infty^0) < \beta.$$

In other words, there exist at most βk_i number of $a^k x_A$ terms in Q_∞^0, and thus for any $y \in D_i$, we have

$$|\{k \in \{0, \ldots, k_i - 1\} : a^k y \in Q_\infty\}| < \beta k_i.$$

From Lemma 3.4 with $L = W$ and equation (4.1), if Q is any non-empty atom of $\mathcal{Q}^{(k_i)}$, fixing any $y \in D_i \cap Q$, the set

$$D_i \cap Q = D_i \cap [y]_{\mathcal{Q}^{(k_i)}} \subset E_{y, k_i - 1}$$

can be covered by $Ce^{D\beta k_i}$ many $r_D^{1/r_1} e^{-k_i}$-balls for $d_{\mathbf{r}}$, where C is a constant depending on Q_∞^0 and D, but not on k_i. Since D_i is e^{-k_i}-separated with respect to $d_{\mathbf{r}}$ and $r_D^{1/r_1} < \frac{1}{2}$, we get

$$\mathrm{Card}(D_i \cap Q) \leq Ce^{D\beta k_i}. \tag{4.3}$$

Now let $\mathcal{A}^W = (\mathcal{P}^W)_0^\infty = \bigvee_{i=0}^\infty a^i \mathcal{P}^W$ be as in Proposition 2.8 for μ, r_0, and $L = W$, and let $\mathcal{A}_\infty^W$ be as in equation (2.12).

Claim. $H_{\nu_i}(\mathcal{Q}^{(k_i)} | \mathcal{A}_\infty^W) = H_{\nu_i}(\mathcal{Q}^{(k_i)})$.

Proof of the claim. Using the continuity of entropy, we have

$$H_{\nu_i}(\mathcal{Q}^{(k_i)} | \mathcal{A}_\infty^W) = \lim_{\ell \to \infty} H_{\nu_i}(\mathcal{Q}^{(k_i)} | (\mathcal{P}^W)_\ell^\infty).$$

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Now we show $H_{\nu_i}(Q^{(k_i)}|(\mathcal{P}^W)_\ell^\infty) = H_{\nu_i}(Q^{(k_i)})$ for all large enough $\ell \geq 1$. Let E_δ be the dynamical δ-boundary of $\mathcal{P}$ as in Lemma 2.7 for μ and r_0. As mentioned in Remark 2.4, we may assume that there exists $y \in \phi_A(\mathbb{T}^m)$ such that $y \notin \partial\mathcal{P}$. Since $E_\delta = \bigcup_{k=0}^\infty a^k \partial_{d_0 e^{-k\alpha}\delta}\mathcal{P}$, there exists $\delta > 0$ such that $y \in Y \setminus E_\delta$. For any $\ell \geq 1$, we have $a^{-\ell} y \in Y \setminus a^{-\ell} E_\delta \subset Y \setminus E_\delta$. Hence, it follows from equation (2.6) and Proposition 2.8 that

$$[y]_{(\mathcal{P}^W)_\ell^\infty} = a^\ell [a^{-\ell} y]_{(\mathcal{P}^W)_0^\infty} = a^\ell [a^{-\ell} y]_{\mathcal{A}^W} \supset a^\ell B_\delta^W a^{-\ell} y \supset B_{d_0 e^{\alpha\ell}\delta}^W y.$$

Since the support of ν_i is a set of finite points on a single compact W-orbit $\phi_A(\mathbb{T}^m)$, ν_i is supported on a single atom of $(\mathcal{P}^W)_\ell^\infty$ for all large enough $\ell \geq 1$. This proves the claim. $\square$

Combining equation (4.3) and the above claim, it follows that

$$H_{\nu_i}(Q^{(k_i)}|\mathcal{A}_\infty^W) = H_{\nu_i}(Q^{(k_i)}) \geq \log |S_i| - D\beta k_i - \log C. \tag{4.4}$$

For any $q \geq 1$, write the Euclidean division of large enough $k_i - 1$ by q as

$$k_i - 1 = qk' + s \text{ with } s \in \{0, \ldots, q-1\}.$$

By subadditivity of the entropy with respect to the partition, for each $p \in \{0, \ldots, q-1\}$,

$$H_{\nu_i}(Q^{(k_i)}|\mathcal{A}_\infty^W) \leq H_{a^p\nu_i}(Q^{(q)}|\mathcal{A}_\infty^W) + \cdots + H_{a^{p+qk'}\nu_i}(Q^{(q)}|\mathcal{A}_\infty^W) + 2q \log |Q|.$$

Summing those inequalities for $p = 0, \ldots, q-1$, and using the concave property of entropy with respect to the measure, we obtain

$$\begin{aligned} q H_{\nu_i}(Q^{(k_i)}|\mathcal{A}_\infty^W) &\leq \sum_{k=0}^{k_i-1} H_{a^k\nu_i}(Q^{(q)}|\mathcal{A}_\infty^W)_0^M + 2q^2 \log |Q| \\ &\leq k_i H_{\mu_i}(Q^{(q)}|\mathcal{A}_\infty^W) + 2q^2 \log |Q|, \end{aligned} \tag{4.5}$$

and it follows from equation (4.4) that

$$\begin{aligned} \frac{1}{q} H_{\mu_i}(Q^{(q)}|\mathcal{A}_\infty^W) &\geq \frac{1}{k_i} H_{\nu_i}(Q^{(k_i)}|\mathcal{A}_\infty^W) - \frac{2q \log |Q|}{k_i} \\ &\geq \frac{1}{k_i}(\log |S_i| - D\beta k_i - \log C - 2q \log |Q|). \end{aligned}$$

Now we can take $i \to \infty$ because the atoms Q of $\overline{Q}$ and hence of $\overline{Q}^{(q)}$ satisfy $\mu^\gamma(\partial Q) = 0$. Also, the constants C and $|Q|$ are independent to k_i. Thus, we obtain

$$\frac{1}{q} H_{\mu^\gamma}(\overline{Q}^{(q)}|\overline{\mathcal{A}_\infty^W}) \geq 1 - r_1(m + \gamma - \dim_H \mathbf{Bad}_A(\epsilon)) - D\beta.$$

By taking $\beta \to \overline{\mu}(\overline{Q_\infty})$ and $\gamma = \gamma_j \to 0$, the inequality in equation (4.2) follows. $\square$

4.2. *The proof of Theorem 1.2.* In this subsection, we will estimate the dimension upper bound in Theorem 1.2 using the a-invariant measure with large relative entropy constructed in Proposition 4.1 and the effective variational principle in Theorem 2.12. To use the effective variational principle, we need the following lemma.

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For $x \in X$ and $H \geq 1$, we set

$$\mathrm{ht}(x) \overset{\text{def}}{=} \sup\{\|gv\|^{-1} : x = gSL_d(\mathbb{Z}), v \in \mathbb{Z}^d \setminus \{0\}\},$$
$$X_{\leq H} \overset{\text{def}}{=} \{x \in X : \mathrm{ht}(x) \leq H\}, \quad Y_{\leq H} \overset{\text{def}}{=} \pi^{-1}(X_{\leq H}).$$

Note that $\mathrm{ht}(x) \geq 1$ for any $x \in X$ by Minkowski's theorem, and $X_{\leq H}$ and $Y_{\leq H}$ are compact sets for all $H \geq 1$ by Mahler's compact criterion.

LEMMA 4.3. *Let $\mathcal{A}$ be a countably generated sub-σ-algebra of Borel σ-algebra which is a^{-1}-descending and W-subordinate. Let us fix $y \in Y_{\leq H}$ and suppose that $B_\delta^{W,\mathbf{r}} \cdot y \subset [y]_{\mathcal{A}} \subset B_r^{W,\mathbf{r}} \cdot y$ for some $0 < \delta < r$. For any $0 < \epsilon < 1$, if $j_1 \geq \log((2dH^{d-1})^{1/r_m}\delta^{-1})$ and $j_2 \geq \log((dH^{d-1})^{1/s_n}\epsilon^{-n/d})$, then $\tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_\epsilon) \leq 1 - e^{-j_1 - j_2} r^{-1}\epsilon^{m/d}$, where $\tau_y^{a^{j_1}\mathcal{A}}$ is as in §2.4.*

Proof. For $x = \pi(y) \in X_{\leq H}$, there exists $g \in SL_d(\mathbb{R})$ such that $x = gSL_d(\mathbb{Z})$ and $\inf_{v \in \mathbb{Z}^d \setminus \{0\}} \|gv\| \geq H^{-1}$. By Minkowski's second theorem with a convex body $[-1, 1]^d$, we can choose vectors $gv_1, \ldots, gv_d$ in $g\mathbb{Z}^d$ so that $\prod_{i=1}^d \|gv_i\| \leq 1$. Then, for any $1 \leq i \leq d$,

$$\|gv_i\| \leq \prod_{j \neq i} \|gv_j\|^{-1} \leq H^{d-1}.$$

Let $\Delta \subset \mathbb{R}^d$ be the parallelepiped generated by $gv_1, \ldots, gv_d$, then $\|b\| \leq dH^{d-1}$ for any $b \in \Delta$. It follows that $\|b^+\|_\mathbf{r} \leq (dH^{d-1})^{1/r_m}$ and $\|b^-\|_\mathbf{s} \leq (dH^{d-1})^{1/s_n}$ for any $b = (b^+, b^-) \in \Delta$, where $b^+ \in \mathbb{R}^m$ and $b^- \in \mathbb{R}^n$. Note that the set $\pi^{-1}(x) \subset Y$ is parameterized as follows:

$$\pi^{-1}(x) = \{w(b)g\Gamma \in Y : b \in \Delta\}.$$

Write $y = w(b_0)g\Gamma$ for some $b_0 = (b_0^+, b_0^-) \in \Delta$. Denote by $V_y \subset W$ the shape of $\mathcal{A}$-atom so that $V_y \cdot y = [y]_{a^{j_1}\mathcal{A}}$, and $\Xi \subset \mathbb{R}^m$ the corresponding set to V_y containing 0 given by the canonical bijection between W and $\mathbb{R}^m$. Since a^{j_1} expands the $\mathbf{r}$-quasinorm with the ratio e^{j_1}, we have $B_{e^{j_1}\delta}^{W,\mathbf{r}} \cdot y \subset [y]_{a^{j_1}\mathcal{A}} \subset B_{e^{j_1}r}^{W,\mathbf{r}} \cdot y$, that is, $B_{e^{j_1}\delta}^{\mathbb{R}^m,\mathbf{r}} \subset \Xi \subset B_{e^{j_1}r}^{\mathbb{R}^m,\mathbf{r}}$. Then the atom $[y]_{a^{j_1}\mathcal{A}}$ is parameterized as follows:

$$[y]_{a^{j_1}\mathcal{A}} = \{w(b)g\Gamma : b = (b^+, b_0^-), b^+ \in b_0^+ + \Xi\},$$

and $\tau_y^{a^{j_1}\mathcal{A}}$ can be considered as the normalized Lebesgue measure on the set $b_0^+ + \Xi \subset \mathbb{R}^m$.

Let us consider the following sets:

$$\Theta^+ \overset{\text{def}}{=} \{b^+ \in \mathbb{R}^m : \|b^+\|_\mathbf{r} \leq e^{-j_2}\epsilon^{m/d}\} \quad \text{and} \quad \Theta^- \overset{\text{def}}{=} \{b^- \in \mathbb{R}^n : \|b^-\|_\mathbf{s} \leq e^{j_2}\epsilon^{n/d}\}.$$

If $b = (b^+, b^-) \in \Theta^+ \times \Theta^-$, then $\|e^{\mathbf{r}j_2}b^+\|_\mathbf{r} \leq \epsilon^{m/d}$ and $\|e^{-\mathbf{s}j_2}b^-\|_\mathbf{s} \leq \epsilon^{n/d}$, where $e^{\mathbf{r}j_2}b^+$ and $e^{-\mathbf{s}j_2}b^-$ denote the vectors such that $a^{j_2}b = (e^{\mathbf{r}j_2}b^+, e^{-\mathbf{s}j_2}b^-)$. It follows that $w(b)g\Gamma \notin a^{-j_2}\mathcal{L}_\epsilon$ since

$$a^{j_2}w(b^+, b^-)g\Gamma = w(e^{\mathbf{r}j_2}b^+, e^{-\mathbf{s}j_2}b^-)a^{j_2}g\Gamma \notin \mathcal{L}_\epsilon$$

by the definition of $\mathcal{L}_\epsilon$.

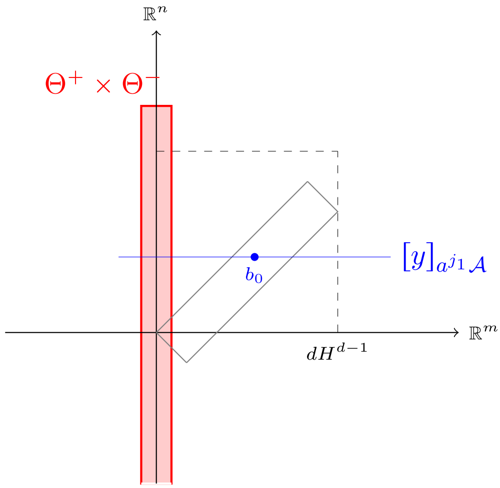

FIGURE 1. Intersection of $\Theta^+ \times \Theta^-$ and $[y]_{a^{j_1}\mathcal{A}}$.

Now we claim that the set $\Theta^+ \times \{b_0^-\}$ is contained in the intersection of $(b_0^+ + \Xi) \times \{b_0^-\}$ and $\Theta^+ \times \Theta^-$. See Figure 1. It is enough to show that $\Theta^+ \subset b_0^+ + \Xi$ and $b_0^- \in \Theta^-$. Since $\|b_0^-\|_s \le (dH^{d-1})^{1/s_n}$, the latter assertion follows from the assumption $j_2 \ge \log((dH^{d-1})^{1/s_n}\epsilon^{-n/d})$. To show the former assertion, fix any $b^+ \in \Theta^+$. By the quasi-metric property of $\|\cdot\|_{\mathbf{r}}$ as in equation (3.2), it follows from the assumptions $j_1 \ge \log((2dH^{d-1})^{1/r_m}\delta^{-1})$ and $j_2 \ge \log((dH^{d-1})^{1/s_n}\epsilon^{-n/d})$ that

$$\begin{aligned}\|b^+ - b_0^+\|_{\mathbf{r}} &\le 2^{(1-r_m)/r_m}(\|b^+\|_{\mathbf{r}} + \|b_0^+\|_{\mathbf{r}}) \le 2^{(1-r_m)/r_m}(e^{-j_2}\epsilon^{m/d} + (dH^{d-1})^{1/r_m})\\ &\le 2^{(1-r_m)/r_m}((dH^{d-1})^{-1/s_n}\epsilon + (dH^{d-1})^{1/r_m}) \le 2^{(1-r_m)/r_m+1}(dH^{d-1})^{1/r_m}\\ &\le e^{j_1}\delta.\end{aligned}$$

Thus, we have $b^+ \in b_0^+ + B^{\mathbb{R}^m,\mathbf{r}}_{e^{j_1}\delta} \subset b_0^+ + \Xi$, which concludes the former assertion.

By the above claim, we obtain

$$\begin{aligned}1 - \tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_\epsilon) &= \tau_y^{a^{j_1}\mathcal{A}}(Y \setminus a^{-j_2}\mathcal{L}_\epsilon)\\ &\ge \frac{m_{\mathbb{R}^m}(\Theta^+)}{m_{\mathbb{R}^m}(b_0^+ + \Xi)} \ge \frac{m_{\mathbb{R}^m}(B^{\mathbb{R}^m,\mathbf{r}}_{e^{-j_2}\epsilon^{m/d}})}{m_{\mathbb{R}^m}(B^{\mathbb{R}^m,\mathbf{r}}_{e^{j_1}r})} = \frac{e^{-j_2}\epsilon^{m/d}}{e^{j_1}r}.\end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.2. Suppose that $A \in M_{m,n}(\mathbb{R})$ is not singular on average, and let

$$\eta_A = \sup\{\eta : x_A \text{ has } \eta\text{-escape of mass}\} < 1.$$

By Proposition 4.1, there is an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ such that

$$\operatorname{Supp}\overline{\mu} \subset \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y),\ \ \pi_*\overline{\mu} = \mu_A \in \mathscr{P}(\overline{X}) \text{ and } \overline{\mu}(\overline{Y} \setminus Y) = \mu_A(\overline{X} \setminus X) = \eta_A.$$

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This measure can be represented by the linear combination

$$\overline{\mu} = (1 - \eta_A)\mu + \eta_A \delta_\infty,$$

where δ_∞ is the dirac delta measure on $\overline{Y} \setminus Y$ and $\mu \in \mathscr{P}(Y)$ is a-invariant. There is a compact set $K \subset X$ such that $\mu_A(K) > 0.99\mu_A(X)$. We can choose $0 < r < 1$ such that $Y(r) \supset \pi^{-1}(K)$ and $\mu(Y(r)) > 0.99$. Note that the choice of r is independent of ϵ since μ_A is only determined by fixed A.

Let $\mathcal{A}^W$ be as in Proposition 2.8 for μ, r_0, and $L = W$, and let $\mathcal{A}^W_\infty$ be as in equation (2.12). It follows from equation (3) of Proposition 4.1 that

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^W_\infty}) \geq (1 - \eta_A) - r_1(m - \dim_H \mathbf{Bad}_A(\epsilon)).$$

Since the entropy function is linear with respect to the measure, it follows that

$$h_\mu(a|\mathcal{A}^W_\infty) = \frac{1}{1-\eta_A} h_{\overline{\mu}}(a|\overline{\mathcal{A}^W_\infty}) \geq 1 - \frac{r_1}{1-\eta_A}(m - \dim_H \mathbf{Bad}_A(\epsilon)).$$

By Proposition 2.10, we obtain

$$H_\mu(\mathcal{A}^W | a\mathcal{A}^W) \geq 1 - \frac{r_1}{1-\eta_A}(m - \dim_H \mathbf{Bad}_A(\epsilon)). \tag{4.6}$$

By Lemma 2.7, there exists $0 < \delta < \min((cr_0/16d_0)^2, r)$ such that the dynamical δ-boundary has measure $\mu(E_\delta) < 0.01$. Note that since r_0 depends only on G, the constants $C_1, C_2 > 0$ in Lemma 2.7 depend only on a and G, and hence δ is independent of ϵ even if the set E_δ might depend on ϵ. We write $Z = Y(r) \setminus E_\delta$ for simplicity. Note that $\mu(Z) \geq \mu(Y(r)) - \mu(E_\delta) > 0.98$.

To apply Lemma 4.3, choose $H \geq 1$ such that

$$Y(r) \subset Y_{\leq H}. \tag{4.7}$$

Note that the constant H depends only on r. Set

$$j_1 = \lceil \log((2dH^{d-1})^{1/r_m}\delta'^{-1})\rceil \quad \text{and} \quad j_2 = \lceil \log((dH^{d-1})^{1/s_n}\epsilon^{-n/d})\rceil,$$

where $\delta' > 0$ will be determined below.

Let $\mathcal{A} = a^{-k}\mathcal{A}^W$ for $k = \lceil \log(2^{1/r_m}\epsilon^{-m/d})\rceil + j_2$. By Proposition 2.8, $[y]_{\mathcal{A}^W} \subset B^W_{r_0} \cdot y$ for all $y \in Y$, and $B^W_\delta \cdot y \subset [y]_{\mathcal{A}^W}$ for all $y \in Z$ since $\delta < r$. It follows from equation (3.1) that

$$\text{for all } y \in Y,\ [y]_{\mathcal{A}^W} \subset B^{W,d_\infty}_{C_0 r_0} \cdot y \quad \text{and} \quad \text{for all } y \in Z,\ B^{W,d_\infty}_{\delta/C_0} \cdot y \subset [y]_{\mathcal{A}^W},$$

where B^{W,d_∞}_r is the d_∞-ball of radius r around the identity in W. For simplicity, we may assume that $r_0 < 1/C_0$ by choosing r_0 small enough. This implies that

$$\text{for all } y \in Y,\ [y]_{\mathcal{A}^W} \subset B^{W,\mathbf{r}}_1 \cdot y \quad \text{and} \quad \text{for all } y \in Z,\ B^{W,\mathbf{r}}_{(\delta/C_0)^{1/r_m}} \cdot y \subset [y]_{\mathcal{A}^W}.$$

Thus, for any $y \in Y$,

$$[y]_{\mathcal{A}} = a^{-k}[a^k y]_{\mathcal{A}^W} \subset a^{-k} B^{W,\mathbf{r}}_1 a^k \cdot y = B^{W,\mathbf{r}}_{e^{-k}} \cdot y \subset B^{W,\mathbf{r}}_{r'} \cdot y, \tag{4.8}$$

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where $r' = 2^{-1/r_m} e^{-j_2} \epsilon^{m/d}$. Similarly, it follows that for any $y \in a^{-k} Z$,

$$B_{\delta'}^{W,\mathbf{r}} \cdot y \subset [y]_{\mathcal{A}} \subset B_{r'}^{W,\mathbf{r}} \cdot y, \tag{4.9}$$

where $\delta' = e^{-1}(\delta/C_0)^{1/r_m} r'$.

Now we will use Corollary 2.13 with $L = W$, $K = Y$, and $B = B_{r'}^{W,\mathbf{r}}$. Note that the maximal entropy contribution of W for a^{j_1} is j_1, and μ is supported on $a^{-j_2}\mathcal{L}_\epsilon$ since $\operatorname{Supp}\mu \subseteq \mathcal{L}_\epsilon$ and μ is a-invariant. Thus, we have

$$B_{r'}^{W,\mathbf{r}} \operatorname{Supp}\mu \subset B_{r'}^{W,\mathbf{r}} a^{-j_2}\mathcal{L}_\epsilon = a^{-j_2} B_{e^{j_2}r'}^{W,\mathbf{r}}\mathcal{L}_\epsilon = a^{-j_2} B_{2^{-1/r_m}\epsilon^{m/d}}^{W,\mathbf{r}}\mathcal{L}_\epsilon \subset a^{-j_2}\mathcal{L}_{2^{-d/mr_m}\epsilon} \tag{4.10}$$

by using the triangular inequality of $\mathbf{r}$-quasinorm as in equation (3.2) and the definition of $\mathcal{L}_\epsilon$ for the last inclusion. Using equation (4.8), it follows from equation (4.10) and Corollary 2.13 with $L = W$, $K = Y$, and $B = B_{r'}^{W,\mathbf{r}}$ that

$$H_\mu(\mathcal{A}|a^{j_1}\mathcal{A}) \le j_1 + \int_Y \log \tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_{2^{-d/mr_m}\epsilon})\, d\mu(y). \tag{4.11}$$

Using equation (4.9), it follows from Lemma 4.3 with $\delta = \delta'$ and $r = r'$ that for any $y \in a^{-k}Z \cap Y_{\le H}$,

$$\tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_{2^{-d/mr_m}\epsilon}) \le 1 - 2^{-1/r_m} e^{-j_1-j_2} r'^{-1}\epsilon^{m/d} = 1 - e^{-j_1},$$

and hence $-\log \tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_{2^{-d/mr_m}\epsilon}) \ge e^{-j_1}$. Since $\mu(a^{-k}Z \cap Y_{\le H}) \ge \frac{1}{2}$, it follows from equation (4.11) that

$$\begin{aligned} 1 - H_\mu(\mathcal{A}^W|a\mathcal{A}^W) &= 1 - \frac{1}{j_1} H_\mu(\mathcal{A}^W|a^{j_1}\mathcal{A}^W) = 1 - \frac{1}{j_1} H_\mu(\mathcal{A}|a^{j_1}\mathcal{A}) \\ &\ge -\frac{1}{j_1}\int_{a^{-k}Z\cap Y_{\le H}} \log \tau_y^{a^{j_1}\mathcal{A}}(a^{-j_2}\mathcal{L}_{2^{-d/mr_m}\epsilon})\, d\mu(y) \ge \frac{e^{-j_1}}{2j_1}. \end{aligned} \tag{4.12}$$

Recall that j_1 is chosen by

$$\begin{aligned} j_1 &= \lceil \log((2dH^{d-1})^{1/r_m} e(\delta/C_0)^{-1/r_m} 2^{1/r_m} e^{j_2}\epsilon^{-m/d})\rceil \\ &\le \lceil \log((2dH^{d-1})^{1/r_m+1/s_n} e^2(\delta/C_0)^{-1/r_m} 2^{1/r_m}\epsilon^{-n/d}\epsilon^{-m/d})\rceil \\ &\le \log((2dH^{d-1})^{1/r_m+1/s_n} e^3(\delta/C_0)^{-1/r_m} 2^{1/r_m}) - \log\epsilon. \end{aligned}$$

Here, the constants H and δ depend on fixed $A \in M_{m,n}(\mathbb{R})$, not on ϵ. Combining equations (4.6) and (4.12), we obtain

$$m - \dim_H \mathbf{Bad}_A(\epsilon) \ge c(A)\frac{\epsilon}{\log(1/\epsilon)},$$

where the constant $c(A) > 0$ depends only on d, $\mathbf{r}$, $\mathbf{s}$, and $A \in M_{m,n}(\mathbb{R})$. It completes the proof. □

5. *Upper bound for Hausdorff dimension of* $\mathbf{Bad}^b(\epsilon)$

In this section, as explained in the introduction, the target vector b is fixed and we only consider the unweighted setting, that is,

$$\mathbf{r} = (1/m, \ldots, 1/m) \quad \text{and} \quad \mathbf{s} = (1/n, \ldots, 1/n).$$

5.1. *Constructing measure with entropy lower bound.* Similar to §4.1, we will construct an a-invariant measure on Y with a lower bound on the conditional entropy to the σ-algebra $\mathcal{A}_\infty^U$ obtained in equation (2.12) and Proposition 2.8 with $L = U$. To control the amount of escape of mass for the desired measure, we need a modification of [**KKLM17**, Theorem 1.1] as Proposition 5.3 below.

For any compact set $\mathfrak{S} \subset X$ and positive integer $k > 0$, and any $0 < \eta < 1$, let

$$F_{\eta,\mathfrak{S}}^k \overset{\text{def}}{=} \left\{ A \in \mathbb{T}^{mn} \subset M_{m,n}(\mathbb{R}) : \frac{1}{k} \sum_{i=0}^{k-1} \delta_{a^i x_A}(X \setminus \mathfrak{S}) < \eta \right\}.$$

Given a compact set $\mathfrak{S}$ of X, $k \in \mathbb{N}$, $\eta \in (0, 1)$, and $t \in \mathbb{N}$, define the set

$$Z(\mathfrak{S}, k, t, \eta) \overset{\text{def}}{=} \left\{ A \in \mathbb{T}^{mn} : \frac{1}{k} \sum_{i=0}^{k-1} \delta_{a^{ti} x_A}(X \setminus \mathfrak{S}) \geq \eta \right\}.$$

In other words, it is the set of $A \in \mathbb{T}^{mn}$ such that among $0, t, 2t, \ldots, (k-1)t$, the proportion of times i for which the orbit point $a^{ti} x_A$ is in the complement of $\mathfrak{S}$ is at least η. The following theorem is one of the main results in [**KKLM17**].

THEOREM 5.1. [**KKLM17**, Theorem 1.5] *There exist $t_0 > 0$ and $C > 0$ such that the following holds. For any $t > t_0$, there exists a compact set $\mathfrak{S} = \mathfrak{S}(t)$ of X such that for any $k \in \mathbb{N}$ and $\eta \in (0, 1)$, the set $Z(\mathfrak{S}, k, t, \eta)$ can be covered with $Ct^{3k} e^{(m+n-\eta)mntk}$ balls in $\mathbb{T}^{mn}$ of radius $e^{-(m+n)tk}$.*

Remark 5.2. Note that we can take $\mathfrak{S}(t)$ to be increasing in t, that is, $\mathfrak{S}(t) \subseteq \mathfrak{S}(t')$ for any $t_0 < t \leq t'$.

The following proposition is a slightly stronger variant of [**KKLM17**, Theorem 1.1] which will be needed later. We prove this using Theorem 5.1.

PROPOSITION 5.3. *There exists a family of compact sets $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ of X such that the following is true. For any $0 < \eta \leq 1$,*

$$\dim_H \left(\mathbb{T}^{mn} \setminus \limsup_{k \to \infty} \bigcap_{\eta' \geq \eta} F_{\eta',\mathfrak{S}_{\eta'}}^k \right) \leq mn - \frac{\eta mn}{2(m+n)}. \tag{5.1}$$

Proof. For $\eta \in (0, 1)$, let $t_\eta \geq 4$ be the smallest integer such that $(3 \log t_\eta)/t_\eta \leq (\eta mn/10)$, and $\mathfrak{S}'_\eta$ be the set $\mathfrak{S}(t_\eta)$ of Theorem 5.1. For $l \geq 4$, denote by $\eta_l > 0$ the smallest real number such that $t_{\eta_l} = l$. Then, $\eta_l \geq (3\eta_{l-1}/4)$ for any $l \geq 5$. For $\eta' \in [\eta_l, \eta_{l-1})$, let us define $\mathfrak{S}''_{\eta'} = \mathfrak{S}'_{\eta_l}$. For any $\eta \in (0, 1)$, we set $\mathfrak{S}_\eta = \bigcup_{-t_\eta \leq t \leq t_\eta} a^t \mathfrak{S}''_\eta$ so that for any $-t_\eta \leq t \leq t_\eta$ and $x \in \mathfrak{S}''_\eta$, $a^t x \in \mathfrak{S}_\eta$.

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Now we prove that this family of compact sets $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ satisfies equation (5.1). Suppose $A \notin F^k_{\eta,\mathfrak{S}_\eta}$, which implies $(1/k)\sum_{i=0}^{k-1}\delta_{a^i x_A}(X\setminus\mathfrak{S}_\eta)\geq\eta$. For sufficiently large k,

$$\frac{1}{\lceil k/t_\eta\rceil}\sum_{i=0}^{\lceil k/t_\eta\rceil-1}\delta_{a^{t_\eta i}x_A}(X\setminus\mathfrak{S}''_\eta)\geq\frac{1}{t_\eta\lceil k/t_\eta\rceil}\sum_{i=0}^{t_\eta(\lceil k/t_\eta\rceil-1)}\delta_{a^i x_A}(X\setminus\mathfrak{S}_\eta)\geq\frac{9}{10}\eta.$$

Hence, $\mathbb{T}^{mn}\setminus F^k_{\eta,\mathfrak{S}_\eta}\subseteq Z(\mathfrak{S}''_\eta,\lceil k/t_\eta\rceil,t_\eta,(9/10)\eta)$ for any $0<\eta<1$ and sufficiently large $k\in\mathbb{N}$.

For any $\eta_l<\eta'\leq\eta_{l-1}$, we have $t_{\eta'}=l$ and the set $Z(\mathfrak{S}''_{\eta'},\lceil k/t_{\eta'}\rceil,t_{\eta'},(9/10)\eta')$ is contained in $Z(\mathfrak{S}'_{\eta_l},\lceil k/t_{\eta_l}\rceil,l,(9/10)\eta_l)$. It follows that for any $0<\eta<1$,

$$\mathbb{T}^{mn}\setminus\bigcap_{\eta'\geq\eta}F^k_{\eta',\mathfrak{S}^k_{\eta'}}\subseteq\bigcup_{\eta'\geq\eta}Z\bigg(\mathfrak{S}''_{\eta'},\bigg\lceil\frac{k}{t_{\eta'}}\bigg\rceil,t_{\eta'},\frac{9}{10}\eta'\bigg)\subseteq\bigcup_{l=4}^{t_\eta}Z\bigg(\mathfrak{S}'_{\eta_l},\bigg\lceil\frac{k}{l}\bigg\rceil,l,\frac{9}{10}\eta_l\bigg),$$

and hence

$$\mathbb{T}^{mn}\setminus\limsup_{k\to\infty}\bigcap_{\eta'\geq\eta}F^k_{\eta',\mathfrak{S}_{\eta'}}\subseteq\bigcup_{k_0\geq1}\bigcap_{k=k_0}^{\infty}\bigcup_{l=4}^{t_\eta}Z\bigg(\mathfrak{S}'_{\eta_l},\bigg\lceil\frac{k}{l}\bigg\rceil,l,\frac{9}{10}\eta_l\bigg).$$

By Theorem 5.1, the set $\bigcup_{l=4}^{t_\eta}Z(\mathfrak{S}'_{\eta_l},\lceil k/l\rceil,l,(9/10)\eta_l)$ can be covered with

$$\begin{aligned}\sum_{l=4}^{t_\eta}Cl^{3\lceil k/l\rceil}e^{(m+n-(9/10)\eta_l)mn\lceil k/l\rceil l}&\leq\sum_{l=4}^{t_\eta}Ct_\eta^3e^{(3\log l)/lk}e^{(m+n-(9/10)\eta_l)mn(k+t_\eta)}\\&\leq\sum_{l=4}^{t_\eta}Ct_\eta^3e^{(m+n)mnt_\eta}e^{(m+n-(8/10)\eta_l)mnk}\\&\leq Ct_\eta^4e^{(m+n)mnt_\eta}e^{(m+n-\eta/2)mnk}\end{aligned}$$

balls in $\mathbb{T}^{mn}$ of radius $e^{-(m+n)k}$. Here, we used $\eta_{t_\eta}\geq(3\eta/4)$ which follows from $\eta_l\geq(3\eta_{l-1}/4)$ for any $l\geq5$. Thus, for any sufficiently large $k_0\in\mathbb{N}$,

$$\begin{aligned}\dim_H\bigg(\bigcap_{k=k_0}^{\infty}\bigcup_{l=4}^{t_\eta}Z\bigg(\mathfrak{S}'_{\eta_l},\bigg\lceil\frac{k}{l}\bigg\rceil,l,\eta_l\bigg)\bigg)&\leq\limsup_{k\to\infty}\frac{\log(Ct_\eta^4e^{(m+n)mnt_\eta}e^{(m+n-\eta/2)mnk})}{-\log(e^{-(m+n)k})}\\&=\limsup_{k\to\infty}\frac{\log(Ct_\eta^4e^{(m+n)mnt_\eta})+(m+n-\eta/2)mnk}{(m+n)k}=mn-\eta mn/2(m+n),\end{aligned}$$

and hence we get $\dim_H(\mathbb{T}^{mn}\setminus\limsup_{k\to\infty}\bigcap_{\eta'\geq\eta}F^k_{\eta',\mathfrak{S}_{\eta'}})\leq mn-\eta mn/2(m+n)$. □

In the rest of this subsection, we will prove the following proposition which gives the bound of $\dim_H\mathbf{Bad}^b(\epsilon)$. The construction of the a-invariant measure with large relative entropy roughly follows the construction in Proposition 4.1. However, the situation is significantly different, as fixing b does not determine the amount of excursion in the cusp. The additional step using Proposition 5.3 is necessary to control the measure near the cusp allowing a small amount of escape of mass.

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PROPOSITION 5.4. *Let $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ be the family of compact sets of X as in Proposition 5.3. For fixed $b \in \mathbb{R}^m$ and $\epsilon > 0$, assume that $\dim_H \mathbf{Bad}^b(\epsilon) > \dim_H \mathbf{Bad}^0(\epsilon)$. Let $\eta_0 \overset{\text{def}}{=} 2(m+n)(1-(\dim_H \mathbf{Bad}^b(\epsilon))/mn)$. Then, there exists an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ such that:*

(1) *$\operatorname{Supp} \overline{\mu} \subseteq \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y)$;*

(2) *$\pi_* \overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \le \eta'$ for any $\eta_0 \le \eta' < 1$, in particular, there exist $\mu \in \mathscr{P}(Y)$ and $0 \le \widehat{\eta} \le \eta_0$ such that*

$$\overline{\mu} = (1-\widehat{\eta})\mu + \widehat{\eta}\delta_\infty,$$

where δ_∞ is the dirac delta measure on $\overline{Y} \setminus Y$;

(3) *let $\mathcal{A}^U$ be as in Proposition 2.8 for μ, r_0, and $L = U$, and let $\mathcal{A}^U_\infty$ be as in equation (2.12). Then, we have*

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^U_\infty}) \ge (1-\widehat{\eta}^{1/2})\big(d - \tfrac{1}{2}\eta_0 - d\widehat{\eta}^{1/2}\big).$$

Remark 5.5. We remark that this proposition is valid for the weighted setting except for the construction of $\{\mathfrak{S}_\eta\}_{0<\eta<1}$ since it depends on the unweighted result (Theorem 5.1) in [**KKLM17**]. So, we keep the notation $\mathbf{r}$ and $\mathbf{s}$ for weights in the following proof.

Proof. For $\epsilon > 0$, denote by R the set $\mathbf{Bad}^b(\epsilon) \setminus \mathbf{Bad}^b_0(\epsilon)$, and let

$$R^T \overset{\text{def}}{=} \{A \in R \cap \mathbb{T}^{mn} \subset M_{m,n}(\mathbb{R}) |\ \text{for all } t \ge T,\ a_t x_{A,b} \in \mathcal{L}_\epsilon\}.$$

The sequence $\{R^T\}_{T\ge 1}$ is increasing, and $R = \bigcup_{T=1}^\infty R^T$ by Proposition 3.2. Since $\dim_H \mathbf{Bad}^b(\epsilon) > \dim_H \mathbf{Bad}^0(\epsilon) \ge \dim_H \mathbf{Bad}^b_0(\epsilon)$, it follows that $\dim_H R = \dim_H \mathbf{Bad}^b(\epsilon)$. Thus, for any $\gamma > 0$, there exists $T_\gamma \ge 1$ satisfying

$$\dim_H R^{T_\gamma} > \dim_H \mathbf{Bad}^b(\epsilon) - \gamma. \tag{5.2}$$

Let $\eta = 2(m+n)(1-(\dim_H \mathbf{Bad}^b(\epsilon) - \gamma)/mn)$. If $0 < \gamma < mn/2(m+n) - (mn - \dim_H \mathbf{Bad}^b(\epsilon))$, then $0 < \eta < 1$. For $k \in \mathbb{N}$, write $\widetilde{F}^k_\eta \overset{\text{def}}{=} \bigcap_{\eta' \ge \eta} F^k_{\eta', \mathfrak{S}_{\eta'}}$ for simplicity. Recall that we have

$$\dim_H(\mathbb{T}^{mn} \setminus \limsup_{k\to\infty} \widetilde{F}^k_\eta) \le mn - \frac{\eta mn}{2(m+n)} = \dim_H \mathbf{Bad}^b(\epsilon) - \gamma \tag{5.3}$$

by Theorem 5.3. It follows from equations (5.2) and (5.3) that

$$\dim_H \left(R^{T_\gamma} \cap \limsup_{k\to\infty} \widetilde{F}^k_\eta\right) > \dim_H \mathbf{Bad}^b(\epsilon) - \gamma.$$

Thus, there is an increasing sequence of positive integers $\{k_i\} \to \infty$ such that

$$\dim_H(R^{T_\gamma} \cap \widetilde{F}^{k_i}_\eta) > \dim_H \mathbf{Bad}^b(\epsilon) - \gamma.$$

For each $k_i \ge T_\gamma$, let S_i be a maximal e^{-k_i}-separated subset of $R^{T_\gamma} \cap \widetilde{F}^{k_i}_\eta$ with respect to the quasi-distance $d_{\mathbf{r}\otimes\mathbf{s}}$. By Lemma 3.1,

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$$\liminf_{i\to\infty} \frac{\log|S_i|}{k_i} \geq \underline{\dim}_{\mathbf{r}\otimes\mathbf{s}}(R^{T_\gamma} \cap \widetilde{F}_\eta^{k_i}) > m+n-(r_1+s_1)(mn-\dim_H \mathbf{Bad}^b(\epsilon)+\gamma)$$
$$= m+n-\frac{m+n}{mn}(mn-\dim_H \mathbf{Bad}^b(\epsilon)+\gamma)$$
$$= \frac{m+n}{mn}(\dim_H \mathbf{Bad}^b(\epsilon)-\gamma). \tag{5.4}$$

Let $\nu_i \stackrel{\text{def}}{=} (1/|S_i|)\sum_{y\in D_i}\delta_y = (1/|S_i|)\sum_{A\in S_i}\delta_{y_{A,b}}$ be the normalized counting measure on the set $D_i \stackrel{\text{def}}{=} \{y_{A,b} : A \in S_i\} \subset Y$ and let μ^γ be a weak*-limit of μ_i:

$$\mu_i \stackrel{\text{def}}{=} \frac{1}{k_i}\sum_{k=0}^{k_i-1} a_*^k \nu_i \xrightarrow{\text{w}^*} \mu^\gamma \in \mathscr{P}(\overline{Y}).$$

By extracting a subsequence if necessary, we may assume that μ^γ is a weak*-accumulation point of $\{\mu_i\}$. The measure μ^γ is clearly an a-invariant measure since $a_*\mu_i - \mu_i$ goes to zero measure.

Choose any sequence of positive real numbers $(\gamma_j)_{j\geq 1}$ converging to zero and $(\eta_j)_{j\geq 1}$ be the corresponding sequence such that

$$\eta_j = 2(m+n)\bigg(1-\frac{\dim_H \mathbf{Bad}^b(\epsilon)-\gamma_j}{mn}\bigg).$$

Let $\{\mu^{\gamma_j}\}$ be a family of a-invariant probability measures on $\overline{Y}$ obtained from the above construction for each γ_j. Extracting a subsequence again if necessary, we may take a weak*-limit measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ of $\{\mu^{\gamma_j}\}$. We prove that $\overline{\mu}$ is the desired measure. The measure $\overline{\mu}$ is clearly a-invariant.

(1) We show that for any γ, $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$. For any $A \in S_i \subseteq R^{T_\gamma}$, $a^T y_{A,b} \in \mathcal{L}_\epsilon$ holds for $T > T_\gamma$. Thus,

$$\mu_i(Y\setminus\mathcal{L}_\epsilon) = \frac{1}{k_i}\sum_{k=0}^{k_i-1}(a^k)_*\nu_i(Y\setminus\mathcal{L}_\epsilon) = \frac{1}{k_i}\sum_{k=0}^{T_\gamma}(a^k)_*\nu_i(Y\setminus\mathcal{L}_\epsilon) \leq \frac{T_\gamma}{k_i}.$$

By taking the limit for $k_i \to \infty$, we have $\mu^\gamma(Y \setminus \mathcal{L}_\epsilon) = 0$ for arbitrary γ, and hence,

$$\overline{\mu}(Y\setminus\mathcal{L}_\epsilon) = \lim_{j\to\infty}\mu^{\gamma_j}(Y\setminus\mathcal{L}_\epsilon) = 0.$$

(2) For any $\gamma = \gamma_j$, if $A \in S_i \subset \widetilde{F}_{\eta_j}^{k_i} = \bigcap_{\eta'\geq\eta_j} F_{\eta',\mathfrak{S}_{\eta'}}^{k_i}$, then for all $i \in \mathbb{N}$ and $\eta_j \leq \eta' \leq 1$, $(1/k_i)\sum_{k=0}^{k_i-1}\delta_{a^k x_A}(X\setminus\mathfrak{S}_{\eta'}) < \eta'$. Therefore, for all $i \in \mathbb{N}$ and $\eta_j \leq \eta' \leq 1$,

$$\pi_*\mu_i(X\setminus\mathfrak{S}_{\eta'}) = \frac{1}{|S_i|}\sum_{A\in S_i}\frac{1}{k_i}\sum_{k=0}^{k_i-1}\delta_{a^k x_A}(X\setminus\mathfrak{S}_{\eta'}) < \eta',$$

and hence $\pi_*\mu^{\gamma_j}(\overline{X}\setminus\mathfrak{S}_{\eta'}) = \lim_{i\to\infty}\pi_*\mu_i(X\setminus\mathfrak{S}_{\eta'}) \leq \eta'$. Since η_j converges to η_0 as $j \to \infty$, we have

$$\pi_*\overline{\mu}(\overline{X}\setminus\mathfrak{S}_{\eta'}) \leq \eta'$$

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for any $\eta' \geq \eta_0$. Hence,

$$\overline{\mu}(\overline{Y} \setminus Y) \leq \lim_{\eta' \to \eta_0} \pi_* \overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \leq \eta_0,$$

so we have a decomposition $\overline{\mu} = (1 - \widehat{\eta})\mu + \widehat{\eta}\delta_\infty$ for some $\mu \in \mathscr{P}(Y)$ and $0 \leq \widehat{\eta} \leq \eta_0$.

For the rest of the proof, let us check the condition (3).

(3) We first fix any $D > J_U = m + n$. As in the proof of Proposition 4.1, there exists a finite partition $\mathcal{Q}$ of Y satisfying:

- $\mathcal{Q}$ contains an atom Q_∞ of the form $\pi^{-1}(Q_\infty^0)$, where $X \setminus Q_\infty^0$ has compact closure;
- for all $Q \in \mathcal{Q} \setminus \{Q_\infty\}$, diam $Q < r_D = r_D(Q_\infty^0)$, where r_D is as in §3.3;
- for all $Q \in \mathcal{Q}$, for all $j \geq 1$, $\mu^{\gamma_j}(\partial Q) = 0$.

Remark that for all $i \geq 1$, $D_i \subset \{y_{A,b} : A \in [0, 1]^{mn}, b \in [0, 1]^m\}$, which is a compact set in Y; therefore we can choose Q_∞^0 so that

$$Q_\infty \cap D_i = \varnothing. \tag{5.5}$$

To prove condition (3), it suffices to prove that for all $q \geq 1$,

$$\frac{1}{q} H_{\overline{\mu}}(\overline{\mathcal{Q}}^{(q)} | \overline{\mathcal{A}_\infty^U}) \geq (1 - \overline{\mu}(\overline{Q_\infty})^{1/2}) \bigg(\frac{m+n}{mn} \dim_H \mathbf{Bad}^b(\epsilon) - D\overline{\mu}(\overline{Q_\infty})^{1/2} \bigg). \tag{5.6}$$

Indeed, taking $D \to m + n$ and $Q_\infty^0 \subset X$ such that $\overline{\mu}(\overline{Q_\infty}) \to \widehat{\eta}$, it follows that

$$\begin{aligned} h_{\overline{\mu}}(a | \overline{\mathcal{A}_\infty^U}) &\geq (m+n)(1 - \widehat{\eta}^{1/2}) \bigg(\frac{1}{mn} \dim_H \mathbf{Bad}^b(\epsilon) - \widehat{\eta}^{1/2} \bigg) \\ &= (1 - \widehat{\eta}^{1/2}) \bigg(d - \frac{1}{2}\eta_0 - d\widehat{\eta}^{1/2} \bigg). \end{aligned}$$

It remains to prove equation (5.6). It is trivial if $\overline{\mu}(\overline{Q_\infty}) = 1$, so assume that $\overline{\mu}(\overline{Q_\infty}) < 1$, and hence for all large enough $j \geq 1$, $\mu^{\gamma_j}(\overline{Q_\infty}) < 1$. Now we fix such $j \geq 1$ and write temporarily $\gamma = \gamma_j$.

Choose $\beta > 0$ such that $\mu^\gamma(\overline{Q_\infty}) < \beta < 1$. Then, for large enough i,

$$\mu_i(Q_\infty) = \frac{1}{k_i |S_i|} \sum_{y \in D_i, 0 \leq k < k_i} \delta_{a^k y}(Q_\infty) < \beta.$$

In other words, there exist at most $\beta k_i |S_i|$ number of $a^k y$ terms in Q_∞ with $y \in D_i$ and $0 \leq k < k_i$.

Let $S_i' \subset S_i$ be the set of $A \in S_i$ terms such that

$$|\{0 \leq k < k_i : a^k y_{A,b} \in Q_\infty\}| \leq \beta^{1/2} k_i. \tag{5.7}$$

Thus, we have $|S_i \setminus S_i'| \leq \beta^{1/2}|S_i|$, and hence

$$|S_i'| \geq (1 - \beta^{1/2})|S_i|. \tag{5.8}$$

Let $\nu_i' \stackrel{\text{def}}{=} (1/|S_i'|) \sum_{y \in S_i'} \delta_y$ be the normalized counting measure on D_i', where $D_i' \stackrel{\text{def}}{=} \{y_{A,b} : A \in S_i'\} \subset Y$. By definition, $\nu_i(Q) \geq |S_i'|/|S_i| \nu_i'(Q)$ for all measurable set $Q \subseteq Y$. Thus,

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$$\begin{aligned}
H_{\nu_i}(\mathcal{Q}) &= -\sum_{\nu_i(Q)\le 1/e} \log(\nu_i(Q))\nu_i(Q) - \sum_{\nu_i(Q)>1/e} \log(\nu_i(Q))\nu_i(Q)\\
&\ge -\sum_{\nu_i(Q)\le 1/e} \log\left(\frac{|S_i'|}{|S_i|}\nu_i'(Q)\right)\frac{|S_i'|}{|S_i|}\nu_i'(Q)\\
&= -\frac{|S_i'|}{|S_i|}\sum_{\nu_i(Q)\le 1/e} \log(\nu_i'(Q))\nu_i'(Q) - \frac{|S_i'|}{|S_i|}\log\frac{|S_i'|}{|S_i|}\sum_{\nu_i(Q)\le 1/e}\nu_i'(Q)\\
&\ge \frac{|S_i'|}{|S_i|}\left\{H_{\nu_i'}(\mathcal{Q}) + \sum_{\nu_i(Q)>1/e}\log(\nu_i'(Q))\nu_i'(Q)\right\}\\
&\ge (1-\beta^{1/2})\left(H_{\nu_i'}(\mathcal{Q}) - \frac{2}{e}\right).
\end{aligned} \tag{5.9}$$

In the last inequality, we use the fact that ν_i' is a probability measure, and thus there can be at most two elements Q of the partition for which $\nu_i'(Q) > 1/e$.

To compute $H_{\nu_i'}(\mathcal{Q}^{(k_i)})$, note that for any $y \in D_i'$, $y \notin Q_\infty$. From Lemma 3.4 with $L = U$, equations (5.5) and (5.7), if $Q \neq Q_\infty$ is any non-empty atom of $\mathcal{Q}^{(k_i)}$, fixing any $y \in D_i' \cap Q$, the set

$$D_i' \cap Q = D_i' \cap [y]_{\mathcal{Q}^{(k_i)}} \subset E_{y,k_i-1}$$

can be covered by $Ce^{D\sqrt{\beta}k_i}\, d_{\mathbf{r}\otimes\mathbf{s}}$-balls of radius $r_D^{1/(r_1+s_1)}e^{-k_i}$, where C is a constant depending on $\mathcal{Q}_\infty^0$ and D, but not on k_i. Since D_i' is e^{-k_i}-separated with respect to $d_{\mathbf{r}\otimes\mathbf{s}}$ and $r_D^{1/(r_1+s_1)} < \frac{1}{2}$, we get

$$|S_i'|\nu_i'(Q) = \mathrm{Card}(D_i' \cap Q) \le Ce^{D\sqrt{\beta}k_i},$$

and hence we have

$$H_{\nu_i'}(\mathcal{Q}^{(k_i)}) \ge \log|S_i'| - D\beta^{1/2}k_i - \log C. \tag{5.10}$$

Now let $\mathcal{A}^U = (\mathcal{P}^U)_0^\infty = \bigvee_{i=0}^\infty a^i\mathcal{P}^U$ be as in Proposition 2.8 for μ, r_0, and $L = U$, and let $\mathcal{A}_\infty^U$ be as in equation (2.12).

Claim. $H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^U) = H_{\nu_i}(\mathcal{Q}^{(k_i)})$.

Proof of the claim. Using the continuity of entropy, we have

$$H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^U) = \lim_{\ell\to\infty} H_{\nu_i}(\mathcal{Q}^{(k_i)}|(\mathcal{P}^U)_\ell^\infty).$$

Now we show $H_{\nu_i}(\mathcal{Q}^{(k_i)}|(\mathcal{P}^U)_\ell^\infty) = H_{\nu_i}(\mathcal{Q}^{(k_i)})$ for all large enough $\ell \ge 1$. Let $\mathcal{P}$ and E_δ be as in Lemma 2.7 for μ and r_0. As mentioned in Remark 2.4, we may assume that there exists $y \in \{y_{A,b} : A \in \mathbb{T}^{mn} \subset M_{m,n}(\mathbb{R})\}$ such that $y \notin \partial\mathcal{P}$. Since $E_\delta = \bigcup_{k=0}^\infty a^k\partial_{d_0e^{-k\alpha}\delta}\mathcal{P}$, $y \in Y \setminus E_\delta$ for some small enough $\delta > 0$, which implies that $a^{-\ell}y \in Y \setminus a^{-\ell}E_\delta \subset Y \setminus E_\delta$. Hence, it follows from equation (2.6) and Proposition 2.8 that

$$[y]_{(\mathcal{P}^U)_\ell^\infty} = a^\ell[a^{-\ell}y]_{(\mathcal{P}^U)_0^\infty} = a^\ell[a^{-\ell}y]_{\mathcal{A}^U} \supset a^\ell B_\delta^U a^{-\ell}y \supset B_{d_0e^{\alpha\ell}\delta}^U y.$$

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Since the support of ν_i is a set of finite points on a single compact U-orbit, ν_i is supported on a single atom of $(\mathcal{P}^U)_\ell^\infty$ for all large enough $\ell \geq 1$. This proves the claim. □

Combining equations (5.8)–(5.10), and the above claim, we have

$$\begin{aligned} H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^U) = H_{\nu_i}(\mathcal{Q}^{(k_i)}) &\geq (1-\beta^{1/2})\bigg(H_{\nu_i'}(\mathcal{Q}^{(k_i)}) - \frac{2}{e}\bigg) \\ &\geq (1-\beta^{1/2})\bigg(\log|S_i| - D\beta^{1/2}k_i - \log C - \frac{2}{e} + \log(1-\beta^{1/2})\bigg). \end{aligned} \tag{5.11}$$

As in equation (4.5), it follows from equation (5.11) that

$$\begin{aligned} &\frac{1}{q}H_{\mu_i}(\mathcal{Q}^{(q)}|\mathcal{A}_\infty^U) \geq \frac{1}{k_i}H_{\nu_i}(\mathcal{Q}^{(k_i)}|\mathcal{A}_\infty^U) - \frac{2q\log|\mathcal{Q}|}{k_i} \\ &\quad\geq \frac{1}{k_i}\bigg((1-\beta^{1/2})\bigg(\log|S_i| - D\beta^{1/2}k_i - \log C - \frac{2}{e} + \log(1-\beta^{1/2})\bigg) - 2q\log|\mathcal{Q}|\bigg). \end{aligned}$$

Now we can take $i \to \infty$ because the atoms Q of $\mathcal{Q}$ and hence of $\mathcal{Q}^{(q)}$ satisfy $\mu^\gamma(\partial Q) = 0$. Also, the constants C, β, and $|\mathcal{Q}|$ are independent of k_i. Thus, it follows from the inequality in equation (5.4) that

$$\frac{1}{q}H_{\mu^\gamma}(\overline{\mathcal{Q}}^{(q)}|\overline{\mathcal{A}_\infty^U}) \geq (1-\beta^{1/2})\bigg(\frac{m+n}{mn}(\dim_H \mathbf{Bad}^b(\epsilon) - \gamma) - D\beta^{1/2}\bigg).$$

By taking $\beta \to \overline{\mu}(\overline{Q_\infty})$ and $\gamma = \gamma_j \to 0$, the inequality in equation (5.6) follows. □

5.2. *Effective equidistribution and the proof of Theorem 1.1.* In this subsection, we recall some effective equidistribution results which are necessary for the proof of Theorem 1.1. Let $\mathfrak{g} = \mathrm{Lie}\ G(\mathbb{R})$ and choose an orthonormal basis for $\mathfrak{g}$. Define the (left) differentiation action of $\mathfrak{g}$ on $C_c^\infty(X)$ by $Zf(x) = df(\exp(tZ)x)/dt|_{t=0}$ for $f \in C_c^\infty(X)$ and Z in the orthonormal basis. This also defines for any $l \in \mathbb{N}$, L^2-Sobolev norms $\mathcal{S}_l$ on $C_c^\infty(Y)$:

$$\mathcal{S}_l(f)^2 \overset{\text{def}}{=} \sum_{\mathcal{D}} \|\mathrm{ht}\circ\pi^l\mathcal{D}(f)\|_{L^2}^2,$$

where $\mathcal{D}$ ranges over all the monomials in the chosen basis of degree $\leq l$ and $\mathrm{ht}\circ\pi$ is the function assigning 1 over the smallest length of a vector in the lattice corresponding to the given grid. Let us define the function $\zeta : (\mathbb{T}^d \setminus \mathbb{Q}^d) \times \mathbb{R}^+ \to \mathbb{N}$ measuring the Diophantine property of b:

$$\zeta(b, T) \overset{\text{def}}{=} \min\bigg\{N \in \mathbb{N} : \min_{1\leq q\leq N} \|qb\|_{\mathbb{Z}} \leq \frac{T^2}{N}\bigg\}.$$

Then there exists a sufficiently large $l \in \mathbb{N}$ such that the following equidistribution theorems hold.

THEOREM 5.6. [**Kim**, Theorem 1.3] *Let K be a bounded subset in $\mathrm{SL}_d(\mathbb{R})$ and $V \subset U$ be a fixed neighborhood of the identity in U with smooth boundary and compact closure.*

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Then, for any $t \geq 0$, $f \in C_c^\infty(Y)$, and $y = gw(b)\Gamma$ with $g \in K$ and $b \in \mathbb{T}^d \setminus \mathbb{Q}^d$, there exists a constant $\alpha_1 > 0$ depending only on d and V so that

$$\frac{1}{m_U(V)} \int_V f(a_t u y)\, dm_U(u) = \int_Y f\, dm_Y + O(\mathcal{S}_l(f)\zeta(b, e^{t/2m})^{-\alpha_1}). \tag{5.12}$$

The implied constant in equation (5.12) depends only on d, V, and K.

For $q \in \mathbb{N}$, define

$$\begin{aligned} X_q &\overset{\text{def}}{=} \{gw(\mathbf{p}/q)\Gamma \in Y : g \in \mathrm{SL}_d(\mathbb{R}), \mathbf{p} \in \mathbb{Z}^d, \gcd(\mathbf{p}, q) = 1\}, \\ \Gamma_q &\overset{\text{def}}{=} \{\gamma \in SL_d(\mathbb{Z}) : \gamma e_1 \equiv e_1 \pmod{q}\}. \end{aligned}$$

LEMMA 5.7. *The subspace $X_q \subset Y$ can be identified with the quotient space $\mathrm{SL}_d(\mathbb{R})/\Gamma_q$. In particular, this identification is locally bi-Lipschitz.*

Proof. The action $\mathrm{SL}_d(\mathbb{R})$ on X_q by the left multiplication is transitive and $\mathrm{Stab}_{\mathrm{SL}_d(\mathbb{R})}(w(e_1/q)\Gamma) = \Gamma_q$. To see the transitivity, it is enough to show the transitivity on each fiber, that is,

$$\mathrm{SL}_d(\mathbb{Z})e_1 \equiv \{\mathbf{p} \in \mathbb{Z}^d : \gcd(\mathbf{p}, q) = 1\} \pmod{q}.$$

Write $D = \gcd(\mathbf{p})$ and $\mathbf{p}' = \mathbf{p}/D$. Since $\gcd(D, q) = 1$, there are $a, b \in \mathbb{Z}$ such that $aD + bq = 1$. Take $A \in M_{d,d}(\mathbb{Z})$ such that $\det(A) = D$ and $Ae_1 = \mathbf{p}$. If we set $\mathbf{u} = b\mathbf{p}' + (a-1)Ae_2$, then by direct calculation, we have $\mathbf{p} + q\mathbf{u} = (A + \mathbf{u} \times {}^t(qe_1 + e_2))e_1$ and $A + \mathbf{u} \times {}^t(qe_1 + e_2) \in \mathrm{SL}_d(\mathbb{Z})$, which concludes the transitivity. Bi-Lipshitz property of the identification follows trivially since both X_q and $\mathrm{SL}_d(\mathbb{R})/\Gamma_q$ are locally isometric to $\mathrm{SL}_d(\mathbb{R})$. □

THEOREM 5.8. [KM12, Theorem 2.3] *For $q \in \mathbb{N}$, let $\mathrm{SL}_d(\mathbb{R})/\Gamma_q \simeq X_q \subset Y$. Let K and V be as in Theorem 5.6. Then, for any $t \geq 0$, $f \in C_c^\infty(Y)$, and $y = gw(\mathbf{p}/q)\Gamma$ with $g \in K$ and $\mathbf{p} \in \mathbb{Z}^d$, there exists a constant $\alpha_2 > 0$ depending only on d and V so that*

$$\frac{1}{m_U(V)} \int_V f(a_t u y)\, dm_U(u) = \int_{X_q} f\, dm_{X_q} + O(\mathcal{S}_l(f)[\Gamma_1 : \Gamma_q]^{1/2} e^{-\alpha_2 t}). \tag{5.13}$$

The implied constant in equation (5.13) depends only on d, V, and K.

Proof. This result was obtained in [KM12, Theorem 2.3] in the case $q = 1$. For general q, we refer the reader to [KM23, Theorem 5.4] which gave a sketch of the required modification. [KM23, Theorem 5.4] is actually stated for different congruence subgroups from our Γ_q, but the modification still works. □

Since we assume the unweighted setting, $\mathcal{L}_\epsilon = \{y \in Y : \text{for all } v \in \Lambda_y,\ \|v\| \geq \epsilon^{1/d}\}$.

LEMMA 5.9. *For any small enough $\epsilon > 0$ and $q \in \mathbb{N}$, $m_Y(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \asymp \epsilon$ and $m_{X_q}(Y_{\leq \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \gg q^{-d}\epsilon$.*

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Proof. Using the Siegel integral formula [**MM11**, Lemma 2.1] with $f = \mathbb{1}_{B_{\epsilon^{1/d}}(0)}$, which is the indicator function on $\epsilon^{1/d}$-ball centered at 0 in $\mathbb{R}^d$, we have $m_Y(Y_{\le \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \ll \epsilon$. However, by [**Ath15**, Theorem 1] with $A = B_{\epsilon^{1/d}}(0)$, we have $m_Y(\mathcal{L}_\epsilon) < 1/(1 + 2^d \epsilon)$. It follows from the Siegel integral formula on X that $m_Y(Y_{>\epsilon^{-1}}) = m_X(X_{>\epsilon^{-1}}) \le 2^d \epsilon^d$. Since $d \ge 2$, we have

$$m_Y(Y_{\le \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \ge m_Y(Y \setminus \mathcal{L}_\epsilon) - m_Y(Y_{>\epsilon^{-1}}) > \frac{2^d \epsilon}{1 + 2^d \epsilon} - 2^d \epsilon^d \gg \epsilon$$

for small enough $\epsilon > 0$, which concludes the first assertion.

To prove the second assertion, observe that for any $x \in X_{>\epsilon^{-1/d}}$, there exists $g \in \mathrm{SL}_d(\mathbb{R})$ such that $x = g\,\mathrm{SL}_d(\mathbb{Z})$ and $\|ge_1\| \le \epsilon^{1/d}$. Then, $gw(e_1/q)\Gamma \in \pi_q^{-1}(x) \cap (Y \setminus \mathcal{L}_\epsilon)$, where $\pi_q : X_q \to X$ is the natural projection. Since $|\pi_q^{-1}(x)| \le q^d$ and $m_X(x \in X : \epsilon^{-1/d} < \mathrm{ht}(x) \le \epsilon^{-1}) \asymp \epsilon$, we have

$$m_{X_q}(Y_{\le \epsilon^{-1}} \setminus \mathcal{L}_\epsilon) \ge \frac{|\pi_q^{-1}(x) \cap (Y \setminus \mathcal{L}_\epsilon)|}{|\pi_q^{-1}(x)|} m_X(x \in X : \epsilon^{-1/d} < \mathrm{ht}(x) \le \epsilon^{-1}) \gg q^{-d}\epsilon.$$

□

PROPOSITION 5.10. *Let $\mathcal{A}$ be a countably generated sub-σ-algebra of the Borel σ-algebra which is a^{-1}-descending and U-subordinate. Fix a compact set $K \subset Y$. Let $1 < R' < R$, $k = \lfloor (mn \log R')/4d \rfloor$. Suppose that $y \in a^{4k} K$ satisfies $B_{R'}^{U,d_\infty} \cdot y \subset [y]_{\mathcal{A}} \subset B_R^{U,d_\infty} \cdot y$, where B_r^{U,d_∞} is the d_∞-ball of radius r around the identity in U. For $\epsilon > 0$, let $\Omega \subset Y$ be a set satisfying $\Omega \cup a^{-3k}\Omega \subseteq \mathcal{L}_{\epsilon/2}$. There exist $M, M' > 0$ such that the following holds. If $R' \ge \epsilon^{-M'}$, then*

$$1 - \tau_y^{\mathcal{A}}(\Omega) \gg \left(\frac{R'}{R}\right)^{mn} \epsilon^{dM+1},$$

where the implied constant depends only on K.

Proof. Denote by $V_y \subset U$ the shape of $\mathcal{A}$-atom of y so that $V_y \cdot y = [y]_{\mathcal{A}}$. Set $V = B_1^{U,d_\infty}$. Since $(mn \log R')/d - 4 \le 4k \le (mn \log R')/d$, we have

$$B_{e^{-4d/mn} R'}^{U,d_\infty} \subseteq a^{4k} V a^{-4k} = B_{e^{d/mn 4k}}^{U,d_\infty} \subseteq B_{R'}^{U,d_\infty} \subseteq V_y.$$

It follows that

$$\begin{aligned}
1 - \tau_y^{\mathcal{A}}(\Omega) &= \frac{1}{m_U(V_y)} \int_{V_y} \mathbb{1}_{Y \setminus \Omega}(uy)\, dm_U(u) \\
&\ge \frac{1}{m_U(B_R^{U,d_\infty})} \int_{a^{4k} V a^{-4k}} \mathbb{1}_{Y \setminus \Omega}(uy)\, dm_U(u) \\
&\ge e^{-4d} \left(\frac{R'}{R}\right)^{mn} \left(\frac{1}{m_U(a^{4k} V a^{-4k})} \int_{a^{4k} V a^{-4k}} \mathbb{1}_{Y \setminus \Omega}(uy)\, dm_U(u)\right) \\
&= e^{-4d} \left(\frac{R'}{R}\right)^{mn} \left(\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y \setminus \Omega}(a^{4k} u a^{-4k} y)\, dm_U(u)\right).
\end{aligned}$$

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It remains to show that

$$\frac{1}{m_U(V)}\int_V \mathbb{1}_{Y\setminus\Omega}(a^{4k}ua^{-4k}y)\,dm_U(u) \gg \epsilon^{dM+1}. \tag{5.14}$$

We will approximate the characteristic function in the above integrand by a smooth function ψ and use effective equidistribution results from Theorems 5.6 and 5.8. Since $\pi(K)\subset X$ is compact, we can choose $g_0\in SL_d(\mathbb{R})$ such that $\|g_0\|<C_K$ with a constant $C_K>0$ depending only on K, and $a^{-4k}y=g_0w(b_0)\Gamma$ with $b_0\in\mathbb{R}^d$. For the constants α_1 in Theorem 5.6 and α_2 in Theorem 5.8, let $\alpha=\min(\alpha_1,\alpha_2)$ and $M=(1/\alpha)(2+l+(\dim G)/2d)$. By [**KM96**, Lemma 2.4.7(b)] with $r=C\epsilon^{1/d}<1$, we can take the approximation function $\theta\in C_c^\infty(G)$ of the identity such that $\theta\geq 0$, $\operatorname{Supp}\theta\subseteq B_r^G(\mathrm{id})$, $\int_G\theta=1$, and $\mathcal{S}_l(\theta)\ll\epsilon^{-(1/d)(l+(\dim G)/2)}$. Let $\psi=\theta*\mathbb{1}_{Y_{\leq\epsilon^{-1}}\setminus\mathcal{L}_{\epsilon/4}}$, then we have $\mathbb{1}_{Y_{\leq(2\epsilon)^{-1}}\setminus\mathcal{L}_{\epsilon/8}}\leq\psi\leq\mathbb{1}_{Y_{\leq 2\epsilon^{-1}}\setminus\mathcal{L}_{\epsilon/2}}$. Moreover, using Young's inequality, its Sobolev norm is bounded as follows:

$$\begin{aligned}\mathcal{S}_l(\psi)^2&=\sum_{\mathcal{D}}\|(\mathrm{ht}\circ\pi)^l\mathcal{D}(\psi)\|_{L^2}^2\ll\epsilon^{-l}\sum_{\mathcal{D}}\|\mathcal{D}(\theta)*\mathbb{1}_{Y_{\leq\epsilon^{-1}}\setminus\mathcal{L}_{\epsilon/4}}\|_{L^2}^2\\&\ll\epsilon^{-l}\|\mathbb{1}_{Y_{\leq\epsilon^{-1}}\setminus\mathcal{L}_{\epsilon/4}}\|_{L^1}^2\sum_{\mathcal{D}}\|\mathcal{D}(\theta)\|_{L^2}^2\ll\epsilon^{-l}\mathcal{S}_l(\theta)^2,\end{aligned}$$

and hence $\mathcal{S}_l(\psi)\ll\epsilon^{-l/2}\mathcal{S}_l(\theta)\leq\epsilon^{-(l+(\dim G)/2d)}$.

We will prove equation (5.14) applying Theorems 5.6 and 5.8 to the following two cases, respectively:

$$\textit{Case (i)}\quad \zeta(b_0,e^{2k/m})\geq\frac{r_0}{C_KC_0}\epsilon^{-M}\quad\text{and}\quad\textit{Case (ii)}\quad\zeta(b_0,e^{2k/m})<\frac{r_0}{C_KC_0}\epsilon^{-M}.$$

Case (i): Applying Theorem 5.6, we have

$$\begin{aligned}&\frac{1}{m_U(V)}\int_V\mathbb{1}_{Y\setminus\Omega}(a^{4k}ua^{-4k}y)\,dm_U(u)\geq\frac{1}{m_U(V)}\int_V\psi(a^{4k}ua^{-4k}y)\,dm_U(u)\\&\quad=\frac{1}{m_U(V)}\int_V\psi(a^{4k}ug_0w(b_0)\Gamma)\,dm_U(u)=\int_Y\psi\,dm_Y+O(\mathcal{S}_l(\psi)\zeta(b_0,e^{2k/m})^{-\alpha})\\&\quad\geq m_Y(Y_{\leq(2\epsilon)^{-1}}\setminus\mathcal{L}_{\epsilon/8})+O(\epsilon^{-(l+(\dim G)/2d)}\epsilon^{M\alpha}).\end{aligned}$$

It follows from Lemma 5.9 and $M\alpha=2+(l+(\dim G)/2d)$ that

$$\frac{1}{m_U(V)}\int_V\mathbb{1}_{Y\setminus\Omega}(a^{4k}ua^{-4k}y)\,dm_U(u)\geq m_Y(Y_{\leq(2\epsilon)^{-1}}\setminus\mathcal{L}_{\epsilon/2})+O(\epsilon^2)\asymp\epsilon\geq\epsilon^{dM+1}.$$

Case (ii): The assumption $\zeta(b_0,e^{2k/m})<(r_0/C_KC_0)\epsilon^{-M}$ implies that there exists $q\leq(r_0/C_KC_0)\epsilon^{-M}$ such that $\|qb_0\|_{\mathbb{Z}}\leq q^2e^{-2k/m}$, whence

$$\left\|b_0-\frac{\mathbf{p}}{q}\right\|\leq qe^{-2k/m}\leq\frac{r_0}{C_KC_0}\epsilon^{-M}e^{-2k/m} \tag{5.15}$$

for some $\mathbf{p} \in \mathbb{Z}^d$. Let $y' = a^{4k} g_0 w(\mathbf{p}/q)\Gamma$. Then, for any $u \in V$,

$$\begin{aligned}
&d_Y(a^k u a^{-4k} y, a^k u a^{-4k} y') \\
&\quad \leq d_G\bigg(a^k u g_0 w(b_0), a^k u g_0 w\bigg(\frac{\mathbf{p}}{q}\bigg)\bigg) = d_G\left(\begin{pmatrix} I_d & a^k u g_0\Big(b_0 - \dfrac{\mathbf{p}}{q}\Big) \\ & 1 \end{pmatrix}, \mathrm{id}\right) \\
&\quad \leq C_0 d_\infty\left(\begin{pmatrix} I_d & a^k u g_0\Big(b_0 - \dfrac{p}{q}\Big) \\ & 1 \end{pmatrix}, \mathrm{id}\right) \leq C_0 e^{k/m}\|g_0\| \left\| b_0 - \frac{\mathbf{p}}{q} \right\| \leq r_0 \epsilon^{-M} e^{-k/m}
\end{aligned}$$

by equations (3.1) and (5.15). Hence, we have

$$|\psi(a^k u a^{-4k} y) - \psi(a^k u a^{-4k} y')| \ll \mathcal{S}_l(\psi) d_Y(a^k u a^{-4k} y, a^k u a^{-4k} y') \ll \mathcal{S}_l(\psi)\epsilon^{-M} e^{-k/m}. \tag{5.16}$$

It follows from the assumption $a^{-3k}\Omega \subseteq \mathcal{L}_{\epsilon/2}$, equation (5.16), and Theorem 5.8 that

$$\begin{aligned}
&\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y\setminus\Omega}(a^{4k} u a^{-4k} y)\, dm_U(u) = \frac{1}{m_U(V)} \int_V \mathbb{1}_{Y\setminus a^{-3k}\Omega}(a^k u a^{-4k} y)\, dm_U(u) \\
&\quad \geq \frac{1}{m_U(V)} \int_V \psi(a^k u a^{-4k} y)\, dm_U(u) \\
&\quad = \frac{1}{m_U(V)} \int_V \psi(a^k u a^{-4k} y')\, dm_U(u) + O(\mathcal{S}_l(\psi)\epsilon^{-M} e^{-k/m}) \\
&\quad = \int_{X_q} \psi\, dm_Y + O(\mathcal{S}_l(\psi) q^{d/2} e^{-\alpha k} + \mathcal{S}_l(\psi)\epsilon^{-M} e^{-k/m}) \\
&\quad \geq m_{X_q}(Y_{\leq(2\epsilon)^{-1}} \setminus \mathcal{L}_{\epsilon/8}) + O(\epsilon^{-(l+(\dim G)/2d)-dM/2} e^{-\alpha k} + \epsilon^{-(l+(\dim G)/2d)-M} e^{-k/m}).
\end{aligned}$$

Let $M' = \min(4d/\alpha(l+(\dim G)/2d+3dM/2+2), 4dm(l+(\dim G)/2d+(d+1)M+2))$. If $R' > \epsilon^{-M'}$, then $e^{-4dk} < e^{4d}\epsilon^{M'}$, so $\epsilon^{-(l+(\dim G)/2d)-dM/2} e^{-\alpha k} \ll \epsilon^{dM+2}$ and $\epsilon^{-(l+(\dim G)/2d)-M} e^{-k/m} \ll \epsilon^{dM+2}$. Combining this with Lemma 5.9, it follows that

$$\begin{aligned}
&\frac{1}{m_U(V)} \int_V \mathbb{1}_{Y\setminus\Omega}(a^{4k} u a^{-4k} y)\, dm_U(u) \\
&\quad \gg q^{-d}\epsilon + O(\epsilon^{dM+2}) \gg \epsilon^{dM+1} + O(\epsilon^{dM+2}) \gg \epsilon^{dM+1}.
\end{aligned}$$ □

Proof of Theorem 1.1. For fixed b, let $\eta_0 = 2(m+n)(1 - (\dim_H \mathbf{Bad}^b(\epsilon))/mn)$ as in Proposition 5.4. It is enough to consider the case when $\mathbf{Bad}^b(\epsilon)$ is sufficiently close to the full dimension mn, so we may assume $\dim_H \mathbf{Bad}^b(\epsilon) > \dim_H \mathbf{Bad}^0(\epsilon)$ and $\eta_0 \leq 0.01$. By Proposition 5.4, there is an a-invariant measure $\overline{\mu} \in \mathscr{P}(\overline{Y})$ such that $\operatorname{Supp} \overline{\mu} \subseteq \mathcal{L}_\epsilon \cup (\overline{Y} \setminus Y)$, and $\pi_*\overline{\mu}(\overline{X} \setminus \mathfrak{S}_{\eta'}) \leq \eta'$ for any $\eta_0 \leq \eta' \leq 1$. We also have a-invariant $\mu \in \mathscr{P}(Y)$ and $0 \leq \widehat{\eta} \leq \eta_0$ such that

$$\overline{\mu} = (1-\widehat{\eta})\mu + \widehat{\eta}\delta_\infty.$$

In particular, for $\eta' = 0.01$, we have $\mu(\pi^{-1}(\mathfrak{S}_{0.01})) \geq 0.99$. We can choose $0 < r < 1$ such that $Y(r) \supset \pi^{-1}(\mathfrak{S}_{0.01})$. Note that the choice of r is independent of ϵ and b since $\mathfrak{S}_{0.01}$ is constructed in Proposition 5.3 independent to ϵ and b.

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Let $\mathcal{A}^U$ be as in Proposition 2.8 for μ, r_0, and $L = U$, and let $\mathcal{A}^U_\infty$ be as in equation (2.12). It follows from item (3) of Proposition 5.4 that

$$h_{\overline{\mu}}(a|\overline{\mathcal{A}^U_\infty}) \geq (1 - \widehat{\eta}^{1/2})\big(d - \tfrac{1}{2}\eta_0 - d\widehat{\eta}^{1/2}\big).$$

By the linearity of the entropy function with respect to the measure, we have

$$h_\mu(a|\mathcal{A}^U_\infty) \geq (1 + \widehat{\eta}^{1/2})^{-1}\big(d - \tfrac{1}{2}\eta_0 - d\widehat{\eta}^{1/2}\big) \geq d - 2d\widehat{\eta}^{1/2} - \tfrac{1}{2}\eta_0. \tag{5.17}$$

However, we shall get an upper bound of $h_\mu(a|\mathcal{A}^U_\infty)$ from Proposition 2.10 and Corollary 2.13. By Lemma 2.7, there exists $0 < \delta < \min((cr_0/16d_0)^2, r)$ such that $\mu(E_\delta) < 0.01$. Note that since r_0 depends only on G, the constants $C_1, C_2 > 0$ in Lemma 2.7 depend only on a and G, and hence δ is independent of ϵ even if the set E_δ depends on ϵ. We write $Z = Y(r) \setminus E_\delta$ for simplicity. Note that $\mu(Z) \geq \mu(Y(r)) - \mu(E_\delta) > 0.98$.

By Proposition 2.8, $[y]_{\mathcal{A}^U} \subset B^U_{r_0} \cdot y$ for all $y \in Y$, and $B^U_\delta \cdot y \subset [y]_{\mathcal{A}^U}$ for all $y \in Z$ since $\delta < r$. It follows from equation (3.1) that

$$\text{for all } y \in Y,\ [y]_{\mathcal{A}^U} \subset B^{U,d_\infty}_{C_0 r_0} \cdot y \quad \text{and} \quad \text{for all } y \in Z,\ B^{U,d_\infty}_{\delta/C_0} \cdot y \subset [y]_{\mathcal{A}^U}, \tag{5.18}$$

where B^{U,d_∞}_r is the d_∞-ball of radius r around the identity in U. For simplicity, we may assume that $r_0 < 1/C_0$ by choosing r_0 small enough.

Let M and M' be the constants in Proposition 5.10, $r' = 1 - 1/2^{1/d}$, $R' = \epsilon^{-M'}$, $R = e^{mn/d} C_0/\delta R'$, and $k = \lfloor (mn \log R')/4d \rfloor$. Let $\mathcal{A}_1 = a^{-j_1}\mathcal{A}^U$ and $\mathcal{A}_2 = a^{j_2}\mathcal{A}^U$, where

$$j_1 = \bigg\lceil -\frac{mn}{d}\log r' \bigg\rceil \quad \text{and} \quad j_2 = \bigg\lceil -\frac{mn}{d}\log\frac{\delta}{C_0 R'}\bigg\rceil.$$

By equation (5.18), we have that for any $y \in Y$,

$$[y]_{\mathcal{A}_1} = a^{-j_1}[a^{j_1}y]_{\mathcal{A}^U} \subset a^{-j_1} B^{U,d_\infty}_1 a^{j_1} \cdot y \subset B^{U,d_\infty}_{r'} \cdot y. \tag{5.19}$$

Similarly, it follows from equation (5.18) that $B^{U,d_\infty}_{R'} \cdot y \subset [y]_{\mathcal{A}_2} \subset B^{U,d_\infty}_R \cdot y$ for any $y \in a^{j_2}Z$.

Let $\Omega = B^{U,d_\infty}_{r'} \operatorname{Supp}\mu$. For any $v \in \mathbb{R}^d$ with $\|v\| \geq \epsilon^{1/d}$ and $u \in B^{U,d_\infty}_{r'}$,

$$\|uv\| \geq \|v\| - \|(u - \mathrm{id})v\| \geq (1 - r')\epsilon^{1/d} = (\epsilon/2)^{1/d},$$

and hence $\Omega \subseteq B^{U,d_\infty}_{r'}\mathcal{L}_\epsilon \subseteq \mathcal{L}_{\epsilon/2}$. Since $\operatorname{Supp}\mu$ is an a-invariant set, we also have

$$a^{-3k}\Omega = (a^{-3k}B^{U,d_\infty}_{r'}a^{3k})a^{-3k}\operatorname{Supp}\mu \subseteq (a^{-3k}B^{U,d_\infty}_{r'}a^{3k})\mathcal{L}_\epsilon \subseteq \mathcal{L}_{\epsilon/2}.$$

Applying Proposition 5.10 with $K = Y(r)$, $\mathcal{A} = \mathcal{A}_2$, and the same R', R, Ω as we just defined, for any $\epsilon > 0$ and $y \in a^{4k}Y(r) \cap a^{j_2}Z$,

$$1 - \tau^{\mathcal{A}_2}_y(\Omega) \gg \epsilon^{dM+1} \tag{5.20}$$

since R'/R is bounded below by a constant independent of ϵ.

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By Proposition 2.10, we have

$$(j_1 + j_2)(d - h_\mu(a|\mathcal{A}_\infty^U)) = (j_1 + j_2)(d - H_\mu(\mathcal{A}^U|a\mathcal{A}^U)) = (j_1 + j_2)d - H_\mu(\mathcal{A}_1|\mathcal{A}_2). \tag{5.21}$$

Note that the maximal entropy contribution of U for $a^{j_1+j_2}$ is $(j_1 + j_2)d$. Using equation (5.19), it follows from Corollary 2.13 with $\mathcal{A} = \mathcal{A}_1$, $K = Y$, and $B = B_{r'}^{U,d_\infty}$ that

$$(j_1 + j_2)d - H_\mu(\mathcal{A}_1|\mathcal{A}_2) \geq -\int_Y \log \tau_y^{\mathcal{A}_2}(\Omega)\, d\mu(y). \tag{5.22}$$

Combining equations (5.20), (5.21), and (5.22), since $\mu(a^{4k}Y(r) \cap a^{j_2}Z) \geq \frac{1}{2}$, we have

$$(j_1 + j_2)(d - h_\mu(a|\mathcal{A}_\infty^U)) \geq \int_{a^{4k}Y(r)\cap a^{j_2}Z} (1 - \tau_y^{\mathcal{A}_2}(\Omega))\, d\mu(y) \gg \frac{1}{2}\epsilon^{dM+1}.$$

It follows from equation (5.17) and $j_1 + j_2 \asymp \log(1/\epsilon)$ that

$$\eta_0^{1/2} \gg 2d\widehat{\eta}^{1/2} + \tfrac{1}{2}\eta_0 \geq d - h_\mu(a|\mathcal{A}_\infty^U) \gg \epsilon^{dM+2}.$$

Since $\eta_0 = 2(m+n)(1 - (\dim_H \mathbf{Bad}^b(\epsilon))/mn)$, we have

$$mn - \dim_H \mathbf{Bad}'(\epsilon) \geq c_0\epsilon^{2(dM+2)}$$

for some constant $c_0 > 0$ depending only on d. □

6. *Characterization of singular on average property and dimension estimates*

In this section, we will show (2) $\implies$ (1) in Theorem 1.3. Let $A \in M_{m,n}$ and consider two subgroups

$$G(A) \overset{\text{def}}{=} A\mathbb{Z}^n + \mathbb{Z}^m \subset \mathbb{R}^m \quad \text{and} \quad G({}^tA) \overset{\text{def}}{=} {}^tA\mathbb{Z}^m + \mathbb{Z}^n \subset \mathbb{R}^n.$$

If we view alternatively $G(A)$ as a subgroup of classes modulo $\mathbb{Z}^m$, lying in the m-dimensional torus $\mathbb{T}^m$, Kronecker's theorem asserts that $G(A)$ is dense in $\mathbb{T}^m$ if and only if the group $G({}^tA)$ has maximal rank $m+n$ over $\mathbb{Z}$ (see [**Cas57**, Ch. III, Theorem IV]). Thus, if $\mathrm{rank}_{\mathbb{Z}}(G({}^tA)) < m+n$, then $\mathrm{Bad}_A(\epsilon)$ has full Hausdorff dimension for any $\epsilon > 0$. Hence, throughout this section, we consider only matrices A for which $\mathrm{rank}_{\mathbb{Z}}(G({}^tA)) = m+n$.

6.1. *Best approximations.*

We set up a weighted version of the best approximations following [**CGGMS20**]. (See also [**BKLR21**, **BL05**] and for the unweighted setting.)

Definition 6.1. Given $A \in M_{m,n}$, we denote

$$M(\mathbf{y}) = \inf_{\mathbf{q}\in\mathbb{Z}^n} \|{}^tA\mathbf{y} - \mathbf{q}\|_{\mathbf{s}}.$$

A sequence $(\mathbf{y}_i)_{i\geq 1}$ in $\mathbb{Z}^n$ is called *a sequence of weighted best approximations to* ${}^tA$ if the sequence satisfies the following properties:

(1) setting $Y_i = \|\mathbf{y}_i\|_{\mathbf{r}}$ and $M_i = M(\mathbf{y}_i)$, we have

$$Y_1 < Y_2 < \cdots \quad \text{and} \quad M_1 > M_2 > \cdots;$$

(2) $M(\mathbf{y}) \geq M_i$ for all non-zero $\mathbf{y} \in \mathbb{Z}^m$ with $\|\mathbf{y}\|_{\mathbf{r}} < Y_{i+1}$.

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Our assumption $\mathrm{rank}_{\mathbb{Z}}(G({}^t\!A)) = m+n$ guarantees that $M(\mathbf{y}) > 0$ for all non-zero $\mathbf{y} \in \mathbb{Z}^m$, and hence the existence of a sequence of best approximations to ${}^tA$. Moreover, the following lemma says that $(Y_i)_{i\geq 1}$ has at least geometric growth.

LEMMA 6.2. [CGGMS20, Proof of Lemma 4.3] *There exists a positive integer V such that for all $i \geq 1$,*

$$Y_{i+V} \geq 2Y_i.$$

In particular, there exist $c > 0$ and $\gamma > 1$ such that for all $i \geq 1$, $Y_i \geq c\gamma^i$.

Remark 6.3. From the weighted Dirichlet's theorem (see [Kle98, Theorem 2.2]), one can check that $M_k Y_{k+1} \leq 1$ for all $k \geq 1$.

6.2. *Characterization of singular on average property.* In this section, we will characterize the singular on average property in terms of best approximations. At first, we will show A is singular on average if and only if ${}^t\!A$ is singular on average. To do this, following [Cas57, Ch. V], we prove a transference principle between two homogeneous approximations with weights. See also [GE15, Ger20].

Definition 6.4. Given positive numbers $\lambda_1, \ldots, \lambda_d$, consider the parallelepiped

$$\mathcal{P} = \{\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : |z_i| \leq \lambda_i,\ i = 1, \ldots, d\}.$$

We call the parallelepiped

$$\mathcal{P}^* = \left\{\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : |z_i| \leq \frac{1}{\lambda_i}\prod_{j=1}^{d}\lambda_j,\ i = 1, \ldots, d\right\}$$

the *pseudo-compound* of $\mathcal{P}$.

THEOREM 6.5. [GE15] *Let $\mathcal{P}$ be as in Definition 6.4 and let Λ be a full-rank lattice in $\mathbb{R}^d$. Then,*

$$\mathcal{P}^* \cap \Lambda^* \neq \{\mathbf{0}\} \implies c\mathcal{P} \cap \Lambda \neq \{\mathbf{0}\},$$

where $c = d^{1/2(d-1)}$ and Λ^* is the dual lattice of Λ, that is, $\Lambda^* = \{x \in \mathbb{R}^d : x \cdot y \in \mathbb{Z}$ for all $y \in \Lambda\}$.

COROLLARY 6.6. *For positive integer m, n, let $d = m+n$, and let $A \in M_{m,n}$ and $0 < \epsilon < 1$ be given. For all large enough $X \geq 1$, if there exists a non-zero $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$\langle A\mathbf{q}\rangle_{\mathbf{r}} \leq \epsilon T^{-1} \quad \text{and} \quad \|\mathbf{q}\|_{\mathbf{s}} \leq T, \tag{6.1}$$

then there exists a non-zero $\mathbf{y} \in \mathbb{Z}^m$ such that

$$\langle {}^t\!A\mathbf{y}\rangle_{\mathbf{s}} \leq c^{(1/r_m+1/s_n)}\epsilon^{r_m s_n/(s_n+r_1(1-s_n))}T_1^{-1} \quad \text{and} \quad \|\mathbf{y}\|_{\mathbf{r}} \leq T_1,$$

where c is as in Theorem 6.5 and $T_1 = c^{1/r_m}\epsilon^{-r_m(1-s_n)/(s_n+r_1(1-s_n))}T$.

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Proof. Consider the following two parallelepipeds:

$$Q = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : \begin{array}{ll} |z_i| \leq \epsilon^{r_i} T^{-r_i}, & i = 1, \ldots, m \\ |z_{m+j}| \leq T^{s_j}, & j = 1, \ldots, n \end{array} \right\},$$

$$\mathcal{P} = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : \begin{array}{ll} |z_i| \leq Z^{r_i}, & i = 1, \ldots, m \\ |z_{m+j}| \leq \delta^{s_j} Z^{-s_j}, & j = 1, \ldots, n \end{array} \right\},$$

where

$$\delta = \epsilon^{r_m s_n/(s_n + r_1(1-s_n))} \quad \text{and} \quad Z = \epsilon^{-r_m(1-s_n)/(s_n+r_1(1-s_n))} T.$$

Observe that the pseudo-compound of $\mathcal{P}$ is given by

$$\mathcal{P}^* = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d : \begin{array}{ll} |z_i| \leq \delta Z^{-r_i}, & i = 1, \ldots, m \\ |z_{m+j}| \leq \delta^{1-s_j} Z^{s_j}, & j = 1, \ldots, n \end{array} \right\}$$

and that $Q \subset \mathcal{P}^*$ since $\epsilon^{r_i} T^{-r_i} \leq \delta Z^{-r_i}$ and $T^{s_j} \leq \delta^{1-s_j} Z^{s_j}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Now, the existence of a non-zero solution $\mathbf{q} \in R_v^n$ of the inequalities in equation (6.1) implies that $\left(\begin{smallmatrix} I_m & A \\ & I_n \end{smallmatrix}\right)\mathbb{Z}^d$ intersects Q, and thus $\mathcal{P}^*$. By Theorem 6.5, $\left(\begin{smallmatrix} I_m & \\ -{}^t\!A & I_n \end{smallmatrix}\right)\mathbb{Z}^d$ intersects $c\mathcal{P}$, which concludes the proof of Corollary 6.6. □

COROLLARY 6.7. *Let m, n be positive integers and $A \in M_{m,n}$. Then, A is singular on average if and only if ${}^tA$ is singular on average.*

Proof. It follows from Corollary 6.6. □

Now, we will characterize the singular on average property in terms of best approximation. Let $A \in M_{m,n}$ be a matrix and $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of weighted best approximations to ${}^tA$ and write

$$Y_k = \|\mathbf{y}_k\|_{\mathbf{r}}, \quad M_k = \inf_{\mathbf{q}\in\mathbb{Z}^n} \|{}^t\!A\mathbf{y}_k - \mathbf{q}\|_{\mathbf{s}}.$$

PROPOSITION 6.8. *Let $A \in M_{m,n}$ be a matrix and let $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of best approximations to ${}^tA$. Then, the following are equivalent:*

(1) *${}^tA$ is singular on average;*

(2) *for all $\epsilon > 0$,*

$$\lim_{k\to\infty} \frac{1}{\log Y_k} |\{i \leq k : M_i Y_{i+1} > \epsilon\}| = 0.$$

Proof. (1) $\implies$ (2): Let $0 < \epsilon < 1$. Observe that for each integer X with $Y_k \leq T < Y_{k+1}$, the inequalities

$$\|{}^t\!A\mathbf{p} - \mathbf{q}\|_{\mathbf{s}} \leq \epsilon T^{-1} \quad \text{and} \quad 0 < \|\mathbf{p}\|_{\mathbf{r}} \leq T \tag{6.2}$$

have a solution if and only if $T \leq (\epsilon/M_k)$. Thus, for each integer $\ell \in [\log_2 Y_k, \log_2 Y_{k+1})$ the inequalities in equation (6.2) have no solutions for $T = 2^\ell$ if and only if

$$\log_2 \epsilon - \log_2 M_k < \ell < \log_2 Y_{k+1}. \tag{6.3}$$

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Now we assume that ${}^t\!A$ is singular on average. For given $\delta > 0$, if the set $\{k \in \mathbb{N} : M_k Y_{k+1} > \delta\}$ is finite, then it is done. Suppose the set $\{k \in \mathbb{N} : M_k Y_{k+1} > \delta\}$ is infinite and let

$$\{k \in \mathbb{N} : M_k Y_{k+1} > \delta\} = \{j(1) < j(2) < \cdots < j(k) < \cdots : k \in \mathbb{N}\}.$$

Set $\epsilon = \delta/2$ and fix a positive integer V in Lemma 6.2. For an integer ℓ in $[\log_2 Y_{j(k)+1} - 1, \log_2 Y_{j(k)+1})$, observe that

$$\log_2 \epsilon - \log_2 M_{j(k)} < \log_2 Y_{j(k)+1} - 1.$$

Hence, the inequalities in equation (6.2) have no solutions for $T = 2^\ell$ by equation (6.3). By Lemma 6.2, $\log_2 Y_{j(k)+1+V} - 1 \geq \log_2 Y_{j(k)+1}$. So, we have $\log_2 Y_{j(k+V)+1} - 1 \geq \log_2 Y_{j(k)+1}$. Now fix $i = 0, \ldots, V-1$. Then, the intervals

$$[\log_2 Y_{j(i+sV)+1} - 1, \log_2 Y_{j(i+sV)+1}), \quad s = 1, \ldots, k$$

are disjoint. Thus, for an integer $N \in [\log_2 Y_{j(i+kV)+1}, \log_2 Y_{j(i+(k+1)V)+1})$, the number of ℓ in $\{1, \ldots, N\}$ such that equation (6.2) has no solutions for $T = 2^\ell$ is at least k. Since ${}^t\!A$ is singular on average,

$$\frac{k}{\log_2 Y_{j(i+(k+1)V)+1}} \leq \frac{1}{N} |\{\ell \in \{1, \ldots, N\} : \text{equation (6.2) has no solutions for } T = 2^\ell\}|$$

tends to 0 with k, which gives $(i+1+kV)/\log_2 Y_{j(i+1+kV)}$ tends to 0 with k for all $i = 0, \ldots, V-1$. Thus, we have $k/\log_2 Y_{j(k)}$ tends to 0 with k.

For any $k \geq 1$, there is an unique positive integer s_k such that

$$j(s_k) \leq k < j(s_k + 1),$$

and observe that $s_k = |\{i \leq k : M_i Y_{i+1} > \delta\}|$. Thus, by the monotonicity of Y_k, we have

$$\lim_{k\to\infty} \frac{1}{\log_2 Y_k} |\{i \leq k : M_i Y_{i+1} > \delta\}| \leq \lim_{k\to\infty} \frac{s_k}{\log_2 Y_{j(s_k)}} = 0.$$

(2) $\implies$ (1): Given $0 < \epsilon < 1$, the number of integers ℓ in $[\log_2 Y_k, \log_2 Y_{k+1})$ such that equation (6.2) has no solutions for $T = 2^\ell$ is at most

$$\lceil \log_2 M_k Y_{k+1} - \log_2 \epsilon \rceil \leq \log_2 M_k Y_{k+1} - \log_2 \epsilon + 1.$$

Thus, for an integer N in $[\log_2 Y_k, \log_2 Y_{k+1})$, we have

$$\begin{aligned}
&\frac{1}{N} |\{\ell \in \{1, \ldots, N\} : \text{equation (6.2) has no solutions for } T = 2^\ell\}| \\
&\quad \leq \frac{1}{N} \sum_{i=1}^{k} \max(0, \log_2 M_i Y_{i+1} - \log_2 \epsilon + 1) \\
&\quad \leq \frac{1}{\log_2 Y_k} \sum_{i=1}^{k} \max(0, \log_2 M_i Y_{i+1} - \log_2 \epsilon + 1).
\end{aligned}$$

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Since $M_i Y_{i+1} \leq 1$ for each $i \geq 1$,

$$\frac{1}{\log_2 Y_k} \sum_{i=1}^{k} \max(0, \log_2 M_i Y_{i+1} - \log_2 \epsilon + 1)$$
$$\leq \frac{1}{\log_2 Y_k}(-\log_2 \epsilon + 1)|\{i \leq k : M_i Y_{i+1} > \epsilon/2\}|.$$

Therefore, ${}^t A$ is singular on average. □

6.3. *Modified Bugeaud–Laurent sequence.* In this subsection, we construct the following modified Bugeaud–Laurent sequence assuming the singular on average property. We refer the reader to [**BL05**, §5] for the original version of the Bugeaud–Laurent sequence.

PROPOSITION 6.9. *Let $A \in M_{m,n}$ be such that ${}^t A$ is singular on average and let $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of weighted best approximations to ${}^t A$. For all S and R with $S > R > 1$, there exists an increasing function $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ satisfying the following properties:*

(1) *for any integer $i \geq 1$,*

$$Y_{\varphi(i+1)} \geq R Y_{\varphi(i)} \quad \textit{and} \quad M_{\varphi(i)} Y_{\varphi(i+1)} \leq R; \tag{6.4}$$

(2)

$$\limsup_{k\to\infty} \frac{k}{\log Y_{\varphi(k)}} \leq \frac{1}{\log S}. \tag{6.5}$$

Proof. The function φ is constructed in the following way. Fix a positive integer V in Lemma 6.2 and let $\mathcal{J} = \{j \in \mathbb{Z}_{\geq 1} : M_j Y_{j+1} \leq R/S^3\}$. Since ${}^t A$ is singular on average, by Proposition 6.8 with $\epsilon = R/S^3$, we have

$$\lim_{k\to\infty} \frac{1}{\log Y_k} |\{i \leq k : i \in \mathcal{J}^c\}| = 0. \tag{6.6}$$

If the set $\mathcal{J}$ is finite, then we have $\lim_{k\to\infty} Y_k^{1/k} = \infty$ by equation (6.6), and hence the proof of [**BKLR21**, Theorem 2.2] implies that there exists a function $\varphi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ for which

$$Y_{\varphi(i+1)} \geq R Y_{\varphi(i)} \quad \text{and} \quad Y_{\varphi(i)+1} \geq R^{-1} Y_{\varphi(i+1)}.$$

The fact that $M_i Y_{i+1} \leq 1$ for all $i \geq 1$ implies $M_{\varphi(i)} Y_{\varphi(i+1)} \leq R$. Equation (6.5) follows from $\lim_{k\to\infty} Y_k^{1/k} = \infty$, which concludes the proof of Proposition 6.9.

Now, suppose that $\mathcal{J}$ is infinite. Then there are two possible cases:

(i) $\mathcal{J}$ contains all sufficiently large positive integers;

(ii) there are infinitely many positive integers in $\mathcal{J}^c$.

Case (i). Assume the first case and let $\psi(1) = \min\{j : \mathcal{J} \supset \mathbb{Z}_{\geq j}\}$. Define the auxiliary increasing sequence $(\psi(i))_{i\geq 1}$ by

$$\psi(i+1) = \min\{j \in \mathbb{Z}_{\geq 1} : S Y_{\psi(i)} \leq Y_j\},$$

which is well defined since $(Y_i)_{i\geq 1}$ is increasing. Note that $\psi(i+1) \leq \psi(i) + \lceil \log_2 S\rceil V$ since $Y_{\psi(i)+\lceil \log_2 S\rceil V} \geq SY_{\psi(i)}$ by Lemma 6.2. Let us now define the sequence $(\varphi(i))_{i\geq 1}$ by, for each $i \geq 1$,

$$\varphi(i) = \begin{cases} \psi(i) & \text{if } M_{\psi(i)}Y_{\psi(i+1)} \leq R/S, \\ \psi(i+1) - 1 & \text{otherwise.} \end{cases}$$

Then, the sequence $(\varphi(i))_{i\geq 1}$ is increasing and $\varphi \geq \psi$.

Now we claim that for each $i \geq 1$,

$$Y_{\varphi(i+1)} \geq SY_{\varphi(i)} > RY_{\varphi(i)} \quad \text{and} \quad M_{\varphi(i)}Y_{\varphi(i+1)} \leq R, \tag{6.7}$$

which implies equation (6.5) since $Y_{\varphi(k)} \geq S^{k-1}Y_{\varphi(1)}$ for all $k \geq 1$. Thus, the claim concludes the proof of Proposition 6.9.

Proof of equation (6.7). There are four possible cases on the values of $\varphi(i)$ and $\varphi(i+1)$.

- Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+1)$. By the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+1)} \geq SY_{\psi(i)} = SY_{\varphi(i)}.$$

If $\psi(i) \neq \psi(i+1) - 1$, then by the definition of $\varphi(i)$, we have

$$M_{\varphi(i)}Y_{\varphi(i+1)} = M_{\psi(i)}Y_{\psi(i+1)} \leq R/S \leq R.$$

If $\psi(i) = \psi(i+1) - 1$, then $\varphi(i+1) = \varphi(i) + 1$, and hence

$$M_{\varphi(i)}Y_{\varphi(i+1)} = M_{\varphi(i)}Y_{\varphi(i)+1} \leq 1 \leq R.$$

This proves equation (6.7).

- Assume that $\varphi(i) = \psi(i)$ and $\varphi(i+1) = \psi(i+2) - 1$. By the definition of $\psi(i+1)$, we have

$$Y_{\varphi(i+1)} = Y_{\psi(i+2)-1} \geq Y_{\psi(i+1)} \geq SY_{\psi(i)} = SY_{\varphi(i)}.$$

It follows from the minimality of $\psi(i+2)$ that $SY_{\psi(i+1)} > Y_{\psi(i+2)-1}$. If $\psi(i+1) > \psi(i) + 1$, then $M_{\psi(i)}Y_{\psi(i+1)} \leq R/S$ by the definition of $\varphi(i)$. Hence, we have

$$M_{\varphi(i)}Y_{\varphi(i+1)} = M_{\psi(i)}Y_{\psi(i+2)-1} \leq SM_{\psi(i)}Y_{\psi(i+1)} \leq R.$$

If $\psi(i+1) = \psi(i) + 1$, then $M_{\psi(i)}Y_{\psi(i)+1} \leq R/S^3$ since $\psi(i) \in \mathcal{J}$. Hence,

$$M_{\varphi(i)}Y_{\varphi(i+1)} = M_{\psi(i)}Y_{\psi(i+2)-1} \leq SM_{\psi(i)}Y_{\psi(i)+1} \leq R/S^2 \leq R.$$

This proves equation (6.7).

- Assume that $\varphi(i) = \psi(i+1) - 1$ and $\varphi(i+1) = \psi(i+1)$. Since $\psi(i+1) - 1 \in \mathcal{J}$, we have

$$M_{\varphi(i)}Y_{\varphi(i+1)} = M_{\psi(i+1)-1}Y_{\psi(i+1)} \leq R/S^3 \leq R.$$

If $\psi(i+1) - 1 = \psi(i)$, then by the definition of $\psi(i+1)$, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}} = \frac{Y_{\psi(i+1)}}{Y_{\psi(i)}} \geq S.$$

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If $\psi(i+1)-1>\psi(i)$, then we have $M_{\psi(i)}Y_{\psi(i+1)}>R/S$ by the definition of $\varphi(i)$, and we have $Y_{\psi(i+1)-1}<SY_{\psi(i)}\leq SY_{\psi(i)+1}$ from the minimality of $\psi(i+1)$. We also have $M_{\psi(i)}Y_{\psi(i)+1}\leq R/S^3$ since $\psi(i)\in\mathcal{J}$. Therefore,

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}}=\frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}}=\frac{M_{\psi(i)}Y_{\psi(i+1)}}{M_{\psi(i)}Y_{\psi(i+1)-1}}\geq\frac{R/S}{SM_{\psi(i)}Y_{\psi(i)+1}}\geq\frac{R/S}{R/S^2}=S.$$

This proves equation (6.7).

• Assume that $\varphi(i)=\psi(i+1)-1$ and $\varphi(i+1)=\psi(i+2)-1$. As in the previous case, we have

$$\frac{Y_{\varphi(i+1)}}{Y_{\varphi(i)}}=\frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)-1}}\geq\frac{Y_{\psi(i+1)}}{Y_{\psi(i+1)-1}}\geq S.$$

We have $SY_{\psi(i+1)}>Y_{\psi(i+2)-1}$ from the minimality of $\psi(i+2)$. Thus, since $\psi(i+1)-1\in\mathcal{J}$, we have

$$M_{\varphi(i)}Y_{\varphi(i+1)}=M_{\psi(i+1)-1}Y_{\psi(i+2)-1}=M_{\psi(i+1)-1}Y_{\psi(i+1)}\left(\frac{Y_{\psi(i+2)-1}}{Y_{\psi(i+1)}}\right)\leq R.$$

This proves equation (6.7). □

Case (ii). Now we assume the second case and let $j_0=\min\mathcal{J}$. Partition $\mathbb{Z}_{\geq j_0}$ into disjoint subset

$$\mathbb{Z}_{\geq j_0}=C_1\sqcup D_1\sqcup C_2\sqcup D_2\sqcup\cdots,$$

where $C_i\subset\mathcal{J}$ and $D_j\subset\mathcal{J}^c$ are sets of consecutive integers with

$$\max C_i<\min D_i\leq\max D_i<\min C_{i+1}$$

for all $i\geq 1$. We consider the following two subcases.

Case (ii)-1. If there is $i_0\geq 1$ such that $|C_i|<3\lceil\log_2 S\rceil V$ for all $i\geq i_0$, then we have, for $k_0=\min C_{i_0}$,

$$\frac{k}{\log Y_k}\leq\frac{k_0+(3\lceil\log_2 S\rceil V+1)|\{i\leq k: i\in\mathcal{J}^c\}|}{\log Y_k},$$

since there exists an element of $\mathcal{J}^c$ in any finite sequence of $3\lceil\log_2 S\rceil V+1$ consecutive integers at least k_0. Therefore, $\lim_{k\to\infty}Y_k^{1/k}=\infty$ by equation (6.6) and this concludes the proof of Proposition 6.9 following the proof when $\mathcal{J}$ is finite at the beginning.

Case (ii)-2. The remaining case is that the set

$$\{i:|C_i|\geq 3\lceil\log_2 S\rceil V\}=\{i(1)<i(2)<\cdots<i(k)<\cdots:k\in\mathbb{N}\}$$

is infinite.

For each $k\geq 1$, let us define an increasing finite sequence $(\psi_k(i))_{1\leq i\leq m_k+1}$ of positive integers by setting $\psi_k(1)=\min C_{i(k)}$ and by induction,

$$\psi_k(i+1)=\min\{j\in C_{i(k)}:SY_{\psi_k(i)}\leq Y_j\},$$

as long as this set is non-empty. Since $C_{i(k)}$ is a finite sequence of consecutive positive integers with length at least $3\lceil\log_2 S\rceil V$ and $Y_{i+\lceil\log_2 S\rceil V}\geq SY_i$ for every

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$i \geq 1$ by Lemma 6.2, there exists an integer $m_k \geq 2$ such that $\psi_k(i)$ is defined for $i = 1, \ldots, m_k + 1$. Note that $\psi_k(i)$ belongs to $\mathcal{J}$ since $C_{i(k)} \subset \mathcal{J}$.

As in *Case (i)*, let us define an increasing finite sequence $(\varphi_k(i))_{1 \leq i \leq m_k}$ of positive integers by

$$\varphi_k(i) = \begin{cases} \psi_k(i) & \text{if } M_{\psi_k(i)} Y_{\psi_k(i+1)} \leq R/S, \\ \psi_k(i+1) - 1 & \text{otherwise.} \end{cases}$$

Following the proof of *Case (i)*, we have for each $i = 1, \ldots, m_k - 1$,

$$Y_{\varphi_k(i+1)} \geq S Y_{\varphi_k(i)} \quad \text{and} \quad M_{\varphi_k(i)} Y_{\varphi_k(i+1)} \leq R. \tag{6.8}$$

Note that $\varphi_k(m_k) < \varphi_{k+1}(1)$. Let us define an increasing finite sequence $(\varphi'_k(i))_{1 \leq i \leq n_k+1}$ of positive integers to interpolate between $\varphi_k(m_k)$ and $\varphi_{k+1}(1)$. Let $j_0 = \varphi_{k+1}(1)$. If the set $\{j \in \mathbb{Z}_{\geq \varphi_k(m_k)} : Y_{j_0} \geq R Y_j\}$ is empty, then we set $n_k = 0$ and $\varphi'_k(1) = j_0 = \varphi_{k+1}(1)$. Otherwise, following [**BKLR21**, Theorem 2.2], by decreasing induction, let $n_k \in \mathbb{Z}_{\geq 1}$ be the maximal positive integer such that there exists $j_1, \ldots, j_{n_k} \in \mathbb{Z}_{\geq 1}$ such that for $\ell = 1, \ldots, n_k$, the set $\{j \in \mathbb{Z}_{\geq \varphi_k(m_k)} : Y_{j_{\ell-1}} \geq R Y_j\}$ is non-empty and for $\ell = 1, \ldots, n_k + 1$, the integer j_ℓ is its largest element. Set $\varphi'_k(i) = j_{n_k+1-i}$ for $i = 1, \ldots, n_k + 1$. Then, the sequence $(\varphi'_k(i))_{1 \leq i \leq n_k+1}$ is contained in $[\varphi_k(m_k), \varphi_{k+1}(1)]$ and satisfies that for $i = 1, \ldots, n_k$,

$$Y_{\varphi'_k(i+1)} \geq R Y_{\varphi'_k(i)} \quad \text{and} \quad M_{\varphi'_k(i)} Y_{\varphi'_k(i+1)} \leq R \tag{6.9}$$

from the proof of [**BKLR21**, Theorem 2.2].

Now, putting alternatively together the sequences $(\varphi_k(i))_{1 \leq i \leq m_k - 1}$ and $(\varphi'_k(i))_{1 \leq i \leq r_k}$ as k ranges over $\mathbb{Z}_{\geq 1}$, we define $N_k = \sum_{\ell=1}^{k-1} (m_\ell - 1 + n_\ell)$ and

$$\varphi(i) = \begin{cases} \varphi_k(i - N_k) & \text{if } 1 + N_k \leq i \leq m_k - 1 + N_k, \\ \varphi'_k(i + 1 - m_k - N_k) & \text{if } m_k + N_k \leq i \leq r_k - 1 + m_k + N_k. \end{cases}$$

Here, we use the standard convention that an empty sum is zero. With equation (6.8) for $i = 1, \ldots, m_k - 2$ and equation (6.9) for $i = 1, \ldots, n_k$, since $\varphi'_k(n_k + 1) = \varphi_{k+1}(1)$, it is enough to show the following lemma to prove that the map φ satisfies equation (6.4).

LEMMA 6.10. *For every $k \in \mathbb{Z}_{\geq 1}$, we have*

$$Y_{\varphi'_k(1)} \geq R Y_{\varphi_k(m_k - 1)} \quad \textit{and} \quad M_{\varphi_k(m_k-1)} Y_{\varphi'_k(1)} \leq R. \tag{6.10}$$

Proof. Since $\varphi'_k(1) \geq \varphi_k(m_k)$ and equation (6.8) with $i = m_k - 1$, we have

$$Y_{\varphi'_k(1)} \geq Y_{\varphi_k(m_k)} \geq S Y_{\varphi_k(m_k - 1)} \geq R Y_{\varphi_k(m_k - 1)},$$

which proves the left-hand side of equation (6.10). If $\varphi'_k(1) = \varphi_k(m_k)$, then equation (6.8) with $i = m_k - 1$ gives the right-hand side of equation (6.10).

Now assume that $\varphi'_k(1) > \varphi_k(m_k)$. By the maximality of n_k, we have $Y_{\varphi'_k(1)} \leq R Y_{\varphi_k(m_k)}$. First, we will prove that $\varphi_k(m_k) = \psi_k(m_k)$. For a contradiction, assume that

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$\varphi_k(m_k) = \psi_k(m_k+1) - 1 > \phi_k(m_k)$. Following the third subcase of the proof of equation (6.7), we have

$$\frac{Y_{\psi_k(m_k+1)}}{Y_{\psi_k(m_k+1)-1}} = \frac{M_{\psi_k(m_k)}Y_{\psi_k(m_k+1)}}{M_{\psi_k(m_k)}Y_{\psi_k(m_k+1)-1}} \geq S.$$

Hence, by the construction of $\varphi'_k(1)$, we have $\varphi'_k(1) = \varphi_k(m_k)$, which is a contradiction to our assumption $\varphi'_k(1) > \varphi_k(m_k)$.

To show the right-hand side of equation (6.10), we consider two possible values of $\varphi_k(m_k - 1)$.

Assume that $\varphi_k(m_k-1) = \psi_k(m_k-1)$. If $\psi_k(m_k-1) > \psi_k(m_k) - 1$, then by the definition of $\varphi_k(m_k-1)$, we have $M_{\psi_k(m_k-1)}Y_{\psi_k(m_k)} \leq R/S$. If $\psi_k(m_k-1) = \psi_k(m_k)-1$, then $M_{\psi_k(m_k-1)}Y_{\psi_k(m_k)} \leq R/S^3 \leq R/S$, since $\psi_k(m_k)-1 \in \mathcal{J}$. Since $\varphi_k(m_k) = \psi_k(m_k)$, we have

$$M_{\varphi_k(m_k-1)}Y_{\varphi'_k(1)} = M_{\psi_k(m_k-1)}Y_{\psi_k(m_k)}\bigg(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\bigg) \leq R,$$

which proves the right-hand side of equation (6.10).

Assume that $\varphi_k(m_k-1) = \psi_k(m_k) - 1$. Since $\varphi_k(m_k) = \psi_k(m_k)$ and $\psi_k(m_k) - 1 \in \mathcal{J}$, we have

$$M_{\varphi_k(m_k-1)}Y_{\varphi'_k(1)} = M_{\psi_k(m_k)-1}Y_{\psi_k(m_k)}\bigg(\frac{Y_{\varphi'_k(1)}}{Y_{\varphi_k(m_k)}}\bigg) \leq R,$$

which proves the right-hand side of equation (6.10), and concludes the proof of Lemma 6.10. □

Finally, we will show equation (6.5) for the map φ. Since there exists an element of $\mathcal{J}^c$ in any finite sequence of $3\lceil\log_2 S\rceil V + 1$ consecutive integers in the complement of $\bigcup_{k\geq 1} C_{i(k)}$, there exists $c_0 \geq 0$ such that for every $k \geq 1$, we have

$$\frac{|\{j \leq \varphi(k) : j \notin \bigcup_{k\geq 1} C_{i(k)}\}|}{\log Y_{\varphi(k)}} \leq \frac{c_0 + (3\lceil\log_2 S\rceil V + 1)|\{j \leq \varphi(k) : j \in \mathcal{J}^c\}|}{\log Y_{\varphi(k)}},$$

which converges to 0 as $k \to +\infty$ by equation (6.6). Let us define

$$n(k) = |\{i \leq k : Y_{\varphi(i)} \geq SY_{\varphi(i+1)}\}|.$$

For each integer $\ell \geq 1$, since $Y_{i+\lceil\log_2 S\rceil V} \geq SY_i$ for every $i \geq 1$ by Lemma 6.2, and by the maximality of m_ℓ in the construction of $(\varphi_\ell(i))_{1\leq i\leq m_\ell}$, we have $|\{j \in C_{i(\ell)} : j \geq \varphi_\ell(m_\ell)\}| \leq 2\lceil\log_2 S\rceil V$. If $\varphi(i)$ belongs to $C_{i(\ell)}$ but $\varphi(i+1)$ does not, then $\varphi(i) \geq \varphi_\ell(m_\ell)$. If $\varphi(i)$ and $\varphi(i+1)$ belong to $C_i(\ell)$, then φ and φ_ℓ coincide on i and $i+1$. Thus, by equation (6.8), we have

$$\begin{aligned} k - n(k) &= |\{i \leq k : Y_{\varphi(i)} < SY_{\varphi(i+1)}\}| \\ &\leq (2\lceil\log_2 S\rceil V)\bigg|\bigg\{j \leq \varphi(k) : j \notin \bigcup_{k\geq 1} C_{i(k)}\bigg\}\bigg|. \end{aligned}$$

Therefore, we have

$$\limsup_{k\to\infty} \frac{k}{\log Y_{\varphi(k)}} = \limsup_{k\to\infty} \frac{n(k)+k-n(k)}{\log Y_{\varphi(k)}} = \limsup \frac{n(k)}{\log Y_{\varphi(k)}}$$
$$\le \limsup_{k\to\infty} \frac{n(k)}{\log S^{n(k)-1} Y_{\varphi(1)}} = \frac{1}{\log S}.$$

This proves equation (6.5) and concludes the proof of Proposition 6.9.

6.4. *Dimension estimates.* Following the notation in [**BHKV10**], given a sequence $\{\mathbf{y}_i\}$ in $\mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $\alpha \in (0, 1/2)$, let

$$\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_i\}} \overset{\text{def}}{=} \{\theta \in \mathbb{R}^m : |\theta \cdot \mathbf{y}_i|_{\mathbb{Z}} \ge \alpha \text{ for all } i \ge 1\}.$$

PROPOSITION 6.11. [**CGGMS20**] *Let $A \in M_{m,n}$ be a matrix and let $(\mathbf{y}_k)_{k\ge1}$ be a sequence of weighted best approximations to ${}^tA$, and let $R > 1$ and $\alpha \in (0, 1/2)$ be given. Suppose that there exists an increasing function $\varphi : \mathbb{Z}_{\ge1} \to \mathbb{Z}_{\ge1}$ such that for any integer $i \ge 1$,*

$$M_{\varphi(i)} Y_{\varphi(i+1)} \le R.$$

Then, $\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_{\varphi(i)}\}}$ *is a subset of* $\mathrm{Bad}_A(\epsilon)$*, where* $\epsilon = (1/R)(\alpha^2/4mn)^{1/\delta}$ *and* $\delta = \min\{r_i, s_j : 1 \le i \le m, 1 \le j \le n\}$.

Proof. In the proof of [**CGGMS20**, Theorem 1.11], the condition $Y_{\varphi(i)+1} \ge R^{-1} Y_{\varphi(i+1)}$ is used. However, the assumption $M_{\varphi(i)} Y_{\varphi(i+1)} \le R$ also implies the same conclusion. □

PROPOSITION 6.12. [**CGGMS20**] *For any $\alpha \in (0, 1/2)$, there exists $R(\alpha) > 1$ with the following property. Let $(\mathbf{y}_k)_{k\ge1}$ be a sequence in $\mathbb{Z}^m \setminus \{\mathbf{0}\}$ such that $\|\mathbf{y}_{k+1}\|_{\mathbf{r}}/\|\mathbf{y}_k\|_{\mathbf{r}} \ge R(\alpha)$ for all $k \ge 1$. Then,*

$$\dim_H(\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_i\}}) \ge m - C \limsup_{k\to\infty} \frac{k}{\log \|\mathbf{y}_k\|_{\mathbf{r}}}$$

for some positive constant $C = C(\alpha)$.

Proof. The proof of [**CGGMS20**, Theorem 6.1] concludes this proposition. □

The two propositions are used in [**BKLR21**, Theorem 5.1] in the unweighted setting.

Proof of Theorem 1.3 (2) $\Longrightarrow$ (1). Suppose A is singular on average. By Corollary 6.7, ${}^tA$ is also singular on average. Let $(\mathbf{y}_k)_{k\ge1}$ be a sequence of weighted best approximations to ${}^tA$. Then, by Propositions 6.9, 6.11, and 6.12, for each $S > R(\alpha) > 1$, we have

$$\dim_H(\mathrm{Bad}_A(\epsilon)) \ge \dim_H(\mathrm{Bad}^{\alpha}_{\{\mathbf{y}_{\varphi(i)}\}})$$
$$\ge m - C \limsup_{k\to\infty} \frac{k}{\log Y_{\varphi(k)}}$$
$$\ge m - \frac{C}{\log S},$$

where $\epsilon = (1/R)(\alpha)(\alpha^2/4mn)^{1/\delta}$. Taking $S \to \infty$, we have $\dim_H(\mathrm{Bad}_A(\epsilon)) = m$ for $\epsilon = (1/R(\alpha))(\alpha^2/4mn)^{1/\delta}$. □

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