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The computability of group constructions II

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Finitely presented groups having word problem solvable by functions in the relativized Grzegorczyk hierarchy, $\{E^n(A) \mid n \in N,\ A \subset N$ (N the natural numbers)$\}$ are studied. Basically the class E^3 consists of the elementary functions of Kalmar and E^{n+1} is obtained from E^n by unbounded recursion. The relativization $E^n(A)$ is obtained by adjoining the characteristic function of A to the class E^n .

It is shown that the Higman construction embedding, a finitely generated group with a recursively enumerable set of relations into a finitely presented group, preserves the computational level of the word problem with respect to the relativized Grzegorczyk hierarchy. As a corollary it is shown that for every $n \geq 4$ and $A \subset N$ recursively enumerable there exists a finitely presented group with word problem solvable at level $E^n(A)$ but not $E^{n-1}(A)$. In particular, there exist finitely presented groups with word problem solvable at level E^n but not E^{n-1} for $n \geq 4$, answering a question of Cannonito.

Introduction

In Part I [0] the concept of the relativized Grzegorczyk hierarchy of

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computable functions was defined, each class being denoted by $E^n(A)$ for $n \geq 2$ and $A \subset N$, the natural numbers. Following Rabin, [9], $E^n(A)$ groups were defined as groups having "indices" which are $E^n(A)$ computable. A particular form of $E^n(A)$ index, called a standard index, was developed, and it was shown that for finitely generated (f.g.) groups the standard index reflected the computability level of the word problem and was independent of the particular finitely generated presentation involved. The effect of the constructions of direct product, free product, and free product with amalgamation on the computability levels was investigated leading to the Higman, Neumann, Neumann Theorem for $E^n(A)$ groups: every $E^n(A)$ group for $n \geq 3$ can be embedded in a f.g. $E^{n+1}(A)$ standard group and, for $n \geq 4$, every $E^n(A)$ standard group can be embedded in a f.g. $E^n(A)$ standard group.

In Part II the group constructions developed in Part I for countable groups are applied to the Higman construction embedding f.g. groups in finitely presented (f.p.) groups, [6]. We show that every f.g. $E^n(A)$ standard group, for $n \geq 4$ and A recursively enumerable, can be embedded in a f.g. $E^n(A)$ standard group. This result is a generalization of Clapham, [12] and [13], that the Higman construction preserves recursively enumerable degrees of unsolvability and also of Gatterdam, [4], that the Higman construction preserves primitive recursive levels of computability. Whereas Clapham's proof parallels the original construction [6], our proof uses the technique of Shoenfield [15] as modified in [4]. It was shown in [4] that this technique also produces the Clapham result. The problems of generalizing [4] to groups in the Grzegorczyk hierarchy were discussed in [14]. The stronger results for the usual group constructions and the techniques developed in Part I, together with a restriction to standard indices has permitted the resolution of these difficulties.

We also show, as a corollary, the existence of f.p. E^n standard groups for $n \geq 4$ which are not E^{n-1} standard, answering a question raised by Cannonito, [1].

Our notation and definitions will be taken from Part I and used without further explanation. Frequent references will be made to statements found in Part I. In particular we begin our numbering of sections and statements at 6. Reference to sections or statements preceeding 6 (for example, Theorem 5.5) are to Part I.

6. Higman groups and benign subgroups

We consider the question of embedding particular finitely generated (f.g.) $E^n(A)$ groups in finitely presented (f.p.) groups. We assume that the groups in question are given standard indices or indices related to the standard by identity isomorphisms which are $E^n(A)$ computable relative to either index. In view of Theorem 5.5 the above restriction is not severe since for $n \geq 4$ any $E^{n-1}(A)$ group can be embedded in a f.g. $E^n(A)$ group. Also, by Proposition 4.4, Corollary 4.9 and Corollary 5.4 we may use the indices given for the construction of direct products, free products, free products with amalgamation and strong Britton extensions since these indices are related to the standard by identity isomorphisms computable at the appropriate level. As in Sections 4 and 5 we require all embeddings to be $E^n(A)$ embeddings in the sense of Definition 4.1.

DEFINITION 6.1. Let G be a f.g. $E^n(A)$ standard group. We say G is $E^n(A)$ *Higman* if there is an $E^n(A)$ embedding of G into a f.p. $E^n(A)$ standard group.

The following is immediate from the definition and Proposition 4.2.

PROPOSITION 6.2. *If G_1 and G_2 are f.g. $E^n(A)$ standard and $E^n(A)$ Higman for $n \geq 3$ then $G_1 \times G_2$ is f.g., $E^n(A)$ standard and $E^n(A)$ Higman.* □

Similarly from Theorem 4.6, Lemma 4.20 and the fact that $E^n(A)$ decidable subgroups are $E^n(A)$ compatible with the group embedding we have the following.

PROPOSITION 6.3. *Let G_1 and G_2 be f.g., $E^n(A)$ standard and*

and $E^n(A)$ *Higman groups for* $n \geq 3$. *Also assume* $H_a < G_a$ *are f.g.* $E^n(A)$ *decidable subgroups for* $a = 1, 2$ *and* $\phi : H_1 \to H_2$ *is an isomorphism such that* ϕ *and* ϕ^{-1} *are* $E^n(A)$ *computable. Then* $G_1 *_\phi G_2$ *is f.g.,* $E^{n+1}(A)$ *standard and* $E^{n+1}(A)$ *Higman.* □

In Proposition 6.3 the requirement that the H_a are f.g. is needed so that if $G_a \to L_a$ are the original embeddings, the embedding $G_1 *_\phi G_2 < L_1 *_\phi L_2$ is such that $L_1 *_\phi L_2$ is f.p.

Recall the construction of strong Britton extensions, G_ϕ , discussed in Section 5. For G an $E^n(A)$ group and $H < G$ an $E^n(A)$ decidable subgroup let G_H denote the group G_ϕ for ϕ the identity isomorphism on G restricted to H . By Theorem 5.3, if G is f.g. and $E^n(A)$ standard for $n \geq 3$ and $H < G$ is $E^n(A)$ decidable then G_H is $E^{n+1}(A)$ standard. We make use of the following slightly stronger result.

PROPOSITION 6.4. *Let* G *be f.g. and* $E^n(A)$ *standard for* $n \geq 4$ *and* $H < G$ $E^n(A)$ *decidable. Then* G_H *is* $E^n(A)$ *standard.*

Proof. Consider $L = (G * \langle r; \ \rangle) *_\Psi (G_1 * \langle s; \ \rangle)$ for G_1 a copy of by $g \mapsto g_1$ and $\Psi : G * rHr^{-1} \to G_1 * sH_1s^{-1}$ by $g \mapsto g_1$ and $rhr^{-1} \mapsto sh_1s^{-1}$. Since G and G_1 have standard indices the encoded multiplication and inverse in them is bounded by an E^3 function. Clearly the encoded Ψ is bounded by an E^3 function and so using the coset representative function given by minimalization, Lemma 2.6 can be used to bound the recursion involved in the encoding of multiplication and inverse by an E^4 function. Since $n \geq 4$, L is $E^n(A)$ with respect to this normal form index. An almost identical argument shows that the process of rewriting a word in the generators of L into normal form, when encoded, is bounded by an E^4 function and so L is $E^n(A)$ standard with the identity isomorphism $E^n(A)$ computable relative to these indicies.

Then $G_H < L$ is the set fixed by the homomorphism $\tau : L \to L$ by $g \mapsto g$, $r \mapsto s^{-1}$, $g_1 \mapsto g_1$ and $s \mapsto 1$, which is clearly $E^n(A)$ computable relative to the standard hence normal form index, so $G_H < L$ is $E^n(A)$ decidable. Thus by Corollary 3.7 it is $E^n(A)$ standard. $\square$

The construction G_ϕ and particularly the special case G_H will be used extensively. In some of the proofs which follow it will be convenient to refer to the construction and notation used in the proof of Proposition 6.4 without specific reference. For more details the interested reader is referred to the proof of Theorem 3.1 of [4].

DEFINITION 6.5. Let G be a f.g. $E^n(A)$ standard and $E^n(A)$ Higman group for $n \geq 4$. An $E^n(A)$ decidable subgroup $H < G$ is said to be $E^n(A)$ *benign* if G_H is $E^n(A)$ Higman.

Notice that by Proposition 6.4, for $n \geq 4$, G_H in the definition above is f.g. and $E^n(A)$ standard and so it makes sense to consider the question of it being $E^n(A)$ Higman. The following characterizations of $E^n(A)$ benign subgroups are of technical use.

LEMMA 6.6. *Let G be a f.g. $E^n(A)$ standard and $E^n(A)$ Higman group for $n \geq 4$ and let $H < G$ be an $E^n(A)$ decidable subgroup. The following are equivalent:*

(i) H is $E^n(A)$ benign;

(ii) for G_1 a copy of G , $G *_H G_1 = \langle G, G_1; h = h_1 \ \forall h \in H\rangle$ is $E^n(A)$ Higman;

(iii) there exists an $E^n(A)$ embedding of G into a f.p. $E^n(A)$ standard group K which has a f.g. $E^n(A)$ decidable subgroup $M < K$ such that $M \cap G = H$ and G is M, $E^n(A)$ compatible

in K *via a right coset representative system having an* E^3 *bound.*

Proof. (Reference the proof of Proposition 4.3 of [4].)

(i) $\Rightarrow$ *(ii)* $G_H < L$ is $E^n(A)$ decidable and L is $E^n(A)$ standard from the proof of Proposition 6.4. We consider $\{G, s^{-1}rGr^{-1}s\} < G_H < L$. Since G_H is assumed to be $E^n(A)$ Higman and since $\{G, s^{-1}rGr^{-1}s\} < G_H$ is f.g. we show $G *_H G_1 \simeq \{G, s^{-1}rGrs^{-1}\}$ by an $E^n(A)$ computable isomorphism and $\{G, s^{-1}rGr^{-1}s\} < G_H$ is $E^n(A)$ decidable.

First, $\{G, s^{-1}rGr^{-1}s\} \simeq \left\{rGr^{-1}, sG_1s^{-1}\right\} < L$ by an inner automorphism which is therefore $E^n(A)$ computable. By Lemma 4.19 and the "spelling" argument of Proposition 3.2 of [4], rGr^{-1} is $G * rHr^{-1}$, $E^n(A)$ compatible in $G * \langle r;\ \rangle$ and similarly sG_1s^{-1} is $G_1 * sH_1s^{-1}$, $E^n(A)$ compatible in $G_1 * \langle s;\ \rangle$. An inspection of this "spelling" argument reveals that the resulting right coset representative systems in $G * \langle r;\ \rangle$ and $G_1 * \langle s;\ \rangle$ have an E^3 bound. Moreover

$$\Psi\left(rGr^{-1} \cap G * rHr^{-1}\right) = \Psi\left(rHr^{-1}\right) = sH_1s^{-1} = sG_1s^{-1} \cap G_1 * sH_1s^{-1} .$$

Thus by Lemma 4.20, with Lemma 2.6 bounding the recursion, L is given an $E^n(A)$ index which is related to the standard index by the $E^n(A)$ computable identity isomorphism and in which $\left\{rGr^{-1}, sG_1s^{-1}\right\}$ is $E^n(A)$ decidable. Thus $\{G, s^{-1}rGr^{-1}s\} < L$ is $E^n(A)$ decidable.

Now suppose G has k generators and let F_{2k} be the free group on $2k$ generators, $a_1, \ldots, a_k$ and $a_1', \ldots, a_k'$. Then the homomorphism $F_{2k} \to L$ by $a_i \mapsto ra_ir^{-1} \in rGr^{-1}$ and $a_i' \mapsto s(a_i)_1s^{-1} \in sG_1s^{-1}$ is $E^n(A)$ computable (L having standard index). Its image is

$$\left\{rGr^{-1},\ sG_1s^{-1}\right\} \simeq rGr^{-1} *_{\Psi|} sG_1s^{-1} \simeq G *_H G_1$$

where $\Psi| = \Psi|rHr^{-1}$. Thus $G *_H G_1$ is $E^n(A)$ standard and is isomorphic to $\{G,\ s^{-1}rGr^{-1}s\}$ (which is $E^n(A)$ Higman) by an $E^n(A)$ computable isomorphism.

(ii) ⇒ *(iii)* Let $G *_H G_1 \to K$ be the $E^n(A)$ embedding implied by *(ii)* for K f.p. and $E^n(A)$ standard. Then the embeddings $G \to G *_H G_1 \to K$ and $G_1 \to G *_H G_1 \to K$ are also $E^n(A)$ embeddings. In particular $G_1 < K$ is $E^n(A)$ decidable, f.g. and satisfies $G \cap G_1 = H$. That G is G_1, $E^n(A)$ compatible follows from Lemma 4.18 and the "spelling" argument in the proof of Proposition 4.3 of [*4*]. From the spelling argument it is clear that the resulting coset representative system has an E^3 bound.

(iii) ⇒ *(i)* Let K and M be as in *(iii)*. Then K_M is $E^n(A)$ standard and f.p. since K is f.p. and M f.g. Moreover since G is M, $E^n(A)$ compatible in K by a right coset representative system bounded by an E^3 function and $n \geq 4$, $G_{M \cap G} = G_H < K_M$ is an $E^n(A)$ embedding by Lemma 5.9 with Lemma 2.6 used to bound the recursion. □

PROPOSITION 6.7. *Let G be a f.g. $E^n(A)$ standard $E^n(A)$ Higman group for $n \geq 4$ and let $H < G$ be a f.g. $E^n(A)$ decidable subgroup. Then H is $E^n(A)$ benign.*

Proof. Let $G \to M$ be an $E^n(A)$ embedding for M f.p. and $E^n(A)$ standard. Since $H < G < M$, G is H, $E^n(A)$ compatible in M and therefore G is $E^n(A)$ invariant under the identity isomorphism restricted to H. Notice that the encoded identity isomorphism on H has an E^3 bound as does the right coset representative function. Therefore Lemma

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2.6 applies to bound the recursion used in Lemma 5.9 and the embedding $G_H \to M_H$ is an $E^n(A)$ embedding. Moreover M being f.p. and H f.g., M_H is f.p. $\square$

A particular technique used in the proofs of Lemma 6.6 and Proposition 6.7 is worth note as it will be used again. Namely we applied Lemma 5.9 to the situation of ϕ the identity restricted to H and KH, $E^n(A)$ compatible by an $E^n(A)$ right coset representative system having an E^3 bound (this occurs whenever $H < K$ and also when the coset representative system arises from a "spelling" argument involving a normal form index). We were able to conclude that for $n \geq 4$ the embedding $K_{H \cap K} < G_H$ is an $E^n(A)$ embedding.

LEMMA 6.8. *Let G be a f.g., $E^n(A)$ standard, $E^n(A)$ Higman group for $n \geq 4$ and $K < G$ be a f.g. $E^n(A)$ decidable subgroup. Then an $E^n(A)$ decidable subgroup $H < K$ is $E^n(A)$ benign in K iff $H < G$ is $E^n(A)$ benign in G.*

Proof. If H is $E^n(A)$ benign in G, say $G_H \to M$ an $E^n(A)$ embedding for M f.p. and $E^n(A)$ standard then it suffices to show $K_H < G_H$ is an $E^n(A)$ embedding for then $K_H \to G_H \to M$ shows H is $E^n(A)$ benign in K. As before, the bounds being E^3 and $n \geq 4$, Lemma 5.9 applies and the embedding $K_H < G_H$ is an $E^n(A)$ embedding.

Conversely let

$$L' = (K * \langle r;\ \rangle) *_{\Psi|} \left(K_1 * \langle s;\ \rangle\right) < L = (G * \langle r;\ \rangle) *_{\Psi} \left(G_1 * \langle s;\ \rangle\right)$$

for $\Psi : G * rHr^{-1} \to G_1 * sH_1s^{-1}$ by $g \mapsto g_1$ and $rhr^{-1} \mapsto sh_1s^{-1}$. The embeddings $K_H \to L'$, $L' \to L$ and $G_H \to L$ are all $E^n(A)$ embeddings as we have seen (relative to the obvious indices). Now

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$$K_H * \langle s;\ \rangle = \langle K,\ t,\ s;\ tHt^{-1} = H\rangle$$
$$\simeq \left\langle K,\ K_1,\ r,\ s;\ k = k_1,\ \forall k \in K,\ rhr^{-1} = sh_1s^{-1}\ \forall h \in H\right\rangle = L' .$$

Here the indices involved are related by rewriting processes having E^3 bounds when encoded and therefore the isomorphism is $E^n(A)$ using Lemma 2.6. Thus K_H being $E^n(A)$ Higman, say by $K_H \to M$, L' is $E^n(A)$ Higman by $L' = K_H * \langle s;\ \rangle \to M * \langle s;\ \rangle$.

Define $G \simeq G_2$ by $g \mapsto g_2$ and $\omega : K \to G_2$ by $k \mapsto k_2$. Consider

$$L' *_\omega G_2 = \left\langle K,\ K_1,\ r,\ s,\ G_2;\ k = k_1,\ rhr^{-1} = sh_1s^{-1}\ \forall h \in H,\ k = k_2\ \forall k \in K\right\rangle$$
$$= \left\langle G_1,\ r,\ s,\ K,\ G_2;\ g_1 = g_2,\ rhr^{-1} = sh_1s^{-1}\ \forall h \in H,\ k = k_2\ \forall k \in K\right\rangle$$
$$= \left\langle G,\ r,\ s,\ G_1;\ g = g_1,\ rhr^{-1} = sh_1s^{-1}\ \forall h \in H\right\rangle = L .$$

Again the rewriting process when encoded has E^3 bounds and so by Lemma 2.6 the isomorphism is $E^n(A)$ computable relative to the indices implied by the constructions. Notice in this regard that ω when encoded will have an obvious E^3 bound.

As we saw, L' is $E^n(A)$ Higman, by $L' \to M * \langle s;\ \rangle$ and by assumption G is $E^n(A)$ Higman say by $G \to P$. Since $M * \langle s;\ \rangle$ and P are $E^n(A)$ standard and since ω has an E^3 bound, $(M*\langle s;\ \rangle) *_\omega P$ is $E^n(A)$ standard and by our special usage of Lemma 5.9 the embedding $G_H < L \simeq L' *_\omega G_2 \to (M*\langle s;\ \rangle) *_\omega P$ is an $E^n(A)$ embedding. Thus since M and P are f.p. and domain $\omega = K$ is f.g., $(M*\langle s;\ \rangle) *_\omega P$ is f.p. and G_H is $E^n(A)$ Higman. $\square$

In view of Lemma 6.8 we may speak of an $E^n(A)$-benign subgroup without regard to the group it is a subgroup of so long as all groups and subgroups involved are $E^n(A)$ decidable in some suitable f.g. $E^n(A)$

standard, $E^n(A)$ Higman group.

The next three lemmas are useful for the construction of benign subgroups. They are similar to Lemmas 4.2, 4.3 and 4.4 of [4] modified and strengthened for our purposes. The proofs are almost identical to those in [4] so we only sketch them below indicating the slight modifications.

LEMMA 6.9. *Let G be a f.g. $E^n(A)$ standard, $E^n(A)$ Higman group for $n \geq 4$ and $H < G$, $K < G$ $E^n(A)$ decidable and $E^n(A)$ benign subgroups. Then $H \cap K$ is $E^n(A)$ benign.*

Proof. Clearly $H \cap K$ is $E^n(A)$ decidable. Using the characterization of Lemma 6.6 let $G *_H G_1 \to M$ and $G *_K G_2 \to N$ be $E^n(A)$ embeddings and M, N $E^n(A)$ standard and f.p. Then $M *_G N$ is f.p. (since G f.g.) and $E^n(A)$ standard (since M and N are $E^n(A)$ standard and the amalgamating isomorphism has an E^3 bound). In $M *_G N$, $G_1 \cap G_2 = (G \cap G_1) \cap (G \cap G_2) = H \cap K$ and so by Lemma 6.6 *(iii)* it suffices to show G_1 is G_2, $E^n(A)$ compatible in $M *_G N$. This compatibility is shown by a "spelling" argument having the property that for any $w \in M *_G N$ the right coset representative function is bounded by the index w. □

Notice that in Lemma 6.6 *(iii)*, M and G being f.g. and $E^n(A)$ decidable in K are then $E^n(A)$ benign in K by Proposition 6.7 and so $H = M \cap G$ is $E^n(A)$ benign in K by Lemma 6.9 and $E^n(A)$ benign in G by Lemma 6.8. Thus the condition "G is M, $E^n(A)$ compatible in K ..." in Lemma 6.6 *(iii)* is superfluous.

LEMMA 6.10. *Let G be a f.g. $E^n(A)$ standard, $E^n(A)$ Higman group for $n \geq 4$ and $H < G$, $K < G$ $E^n(A)$ decidable and $E^n(A)$ benign subgroups. Then if $\{H, K\} < G$ is $E^n(A)$ decidable it is $E^n(A)$ benign.*

Proof. Under the notation used in the proof of Lemma 6.9 it is shown that $\{H, K\} = \{G_1, G_2\} \cap G < M *_G N$. Then since $\{G_1, G_2\}$ and G are f.g. they are $E^n(A)$ benign if $E^n(A)$ decidable and so $\{H, K\}$ is $E^n(A)$ benign by Lemma 6.9. Thus it is required that

$$\{G_1, G_2\} < (G *_H G_1) *_G (G *_K G_2)$$

be $E^n(A)$ decidable. This follows from Lemma 4.19,

$$\{G_1, K\} \cap G = \{H, K\} = \{G_2, H\} \cap G$$

and the fact that $\{G_1, K\}$ is G, $E^n(A)$ compatible in $G *_H G_1$ by an $E^n(A)$ right coset representative function having an E^3 bound (similarly for $\{G_2, H\}$ in $G *_K G_2$). □

LEMMA 6.11. *Let G and G' be f.g., $E^n(A)$ standard and $E^n(A)$ Higman groups for $n \geq 4$ and $\phi : G \to G_1$ be an $E^n(A)$ computable homomorphism. If $H < G$ is an $E^n(A)$ decidable, $E^n(A)$ benign subgroup and $\phi(H) < G'$ is $E^n(A)$ decidable then $\phi(H)$ is $E^n(A)$ benign. If $K < G'$ is an $E^n(A)$ decidable, $E^n(A)$ benign subgroup then $\phi^{-1}(K)$ is $E^n(A)$ decidable and $E^n(A)$ benign.*

Proof. We consider $G \times G'$ which is f.g., $E^n(A)$ standard and $E^n(A)$ Higman by Proposition 6.2. Then $Q = \{(g, \phi(g)) \mid g \in G\} < G \times G'$ is $E^n(A)$ decidable and isomorphic to G hence f.g. and $E^n(A)$ benign. It is shown in [4] that $\{H, G'\} < G \times G'$ and $\{\{H, G'\} \cap Q, G\} < G \times G'$ are $E^n(A)$ decidable and $\phi(H) = \{\{H, G'\} \cap Q, G\} \cap G'$ which is then $E^n(A)$ benign by Lemmas 6.9 and 6.10.

Clearly $\phi^{-1}(K)$ is $E^n(A)$ decidable. It is shown that $\{G, K\} < G \times G'$ and $\{\{G, K\} \cap Q, G'\} < G \times G'$ are $E^n(A)$ decidable and so $\phi^{-1}(K) = \{\{G, K\} \cap Q, G'\} \cap G$ is $E^n(A)$ benign by Lemmas 6.9 and 6.10. □

The following proposition provides an example of an E^4 benign subgroups which is not f.g. This particular subgroup is also of use later.

PROPOSITION 6.12. *Let* $F_2 = \langle a, b; \rangle$ *be the free group on two generators and* $U < F_2$ *the subgroup generated by all elements* $a^{-i}ba^{i}$ *for* $i > 0$. *Then* U *is* E^4 *decidable and* E^4 *benign in* F_2.

Proof. By Lemma 3.1, F_2 is E^3 standard and by Proposition 3.1 of [4], $U < F_2$ is E^3 decidable. We consider $\alpha : F_2 \to F_2$ by $a \mapsto a^2$, $b \mapsto b$ and $\beta : F_2 \to F_2$ by $a \mapsto a^2$ and $b \mapsto aba^{-1}$. It is easy to see α and β are E^3 isomorphisms in F_2 (see the proof of Proposition 4.5 of [4]) and so $(F_2)_{\alpha,\beta}$ is E^4 standard by Lemma 5.10. Moreover by a spelling argument U is E^3 invariant under α and β and so by Lemma 5.11, for

$$(F_2)_{\alpha,\beta} = \left\langle a, b, t_1, t_2; t_1at_1^{-1} = a^2, t_1bt_1^{-1} = b, t_2at_2^{-1} = a^2, t_2bt_2^{-1} = aba^{-1}\right\rangle ,$$

$\{U, t_1, t_2\} < (F_2)_{\alpha,\beta}$ is E^4 decidable and $\{U, t_1, t_2\} \cap F_2 = U$. By the construction $\{U, t_1, t_2\}$ is generated by $a^{-1}ba$, t_1 and t_2 and so $\{U, t_1, t_2\}$ is E^4 benign. By Lemma 6.9, U is E^4 benign. □

The crucial relationship between $E^n(A)$ benign subgroups and $E^n(A)$ Higman groups now can be established.

LEMMA 6.13. *Let* G *be a f.g.* $E^n(A)$ *standard group for* $n \geq 4$ *and* $1 \to K \to F \xrightarrow{\sigma} G \to 1$ *a presentation with* F *f.g. and free,* $K < F$ $E^n(A)$ *decidable. Then* G *is* $E^n(A)$ *Higman iff* K *is* $E^n(A)$ *benign in* F.

Proof. If G is $E^n(A)$ Higman then $\{1\} < G$ is $E^n(A)$ benign and so $K = \sigma^{-1}(\{1\})$ is $E^n(A)$ benign by Lemma 6.11.

Conversely if K is $E^n(A)$ benign then by Lemma 6.6 for F' a copy of F there is an $\mathcal{E}^n(A)$ embedding $F *_K F' \to M$ for M f.p. and $\mathcal{E}^n(A)$ standard. Now $F *_K F' < M \times G$ is $\mathcal{E}^n(A)$ decidable and $\phi : F *_K F' \to M \times G$ by $f \mapsto (f, \sigma(f))$ $\forall f \in F$ and $f' \mapsto (f', 1)$ $\forall f' \in F'$ is $\mathcal{E}^n(A)$ computable (notice for $k \in K = K'$, $k \mapsto (k, 1)$). By considering the first factor we see ϕ is monic. Moreover for $q = kp_1q_1 \ldots p_rq_r \in F *_K F'$, $k \in K$, $p_i \in F$, $q_i \in F'$ in normal form $\phi(q) = (q, \sigma(p_1 \ldots p_r))$ so range ϕ is $\mathcal{E}^n(A)$ decidable and ϕ^{-1} is $\mathcal{E}^n(A)$ computable, that is, ϕ is an $\mathcal{E}^n(A)$ isomorphism in $M \times G$. Also observe that since G is $\mathcal{E}^n(A)$ standard the encoding of σ is by minimalization and so ϕ and ϕ^{-1} have $\mathcal{E}^3$ bounds. Thus since $M \times G$ is $\mathcal{E}^n(A)$ standard and $n \geq 4$, Lemma 2.6 can be used to show $(M{\times}G)_\phi$ is $\mathcal{E}^n(A)$ standard (the proof would be almost a copy of the proof of Proposition 6.4 with ϕ replacing the restricted identity).

Now $G \to (M{\times}G)_\phi$ is clearly an $\mathcal{E}^n(A)$ embedding and $(M{\times}G)_\phi$ is f.g. The relations of $(M{\times}G)_\phi$ are those of M (finitely many), those of G, the commutators of generators of M with generators of G (finitely many), and those of the form $t^{-1}ft = (f, \sigma(f))$ for f a generator of F and $t^{-1}f't = f'$ for f' a generator of F' (finitely many). Let $w \in F$, w' correspond in F' and assume $w \in K$. Then

$$w\sigma(w) = (w, \sigma(w)) = t^{-1}wt = t^{-1}w't = w' = w$$

and so $\sigma(w) = 1$. That is, the relations $w = 1$ for $w \in K$ are redundant in $(M{\times}G)_\phi$ which is then f.p. $\square$

7. Benign sets and benign predicates

Lemma 6.13 reduces the study of Higman groups to a consideration of benign subgroups of f.g. free groups. As a next step we reduce further to a study of subsets of f.g. free groups.

DEFINITION 7.1. Let $F = \langle a_1, \ldots, a_m; \rangle$ be a f.g. free group and $P \subset F$ be an $E^n(A)$ decidable subset for $n \geq 4$ (that is, P is any $E^n(A)$ decidable set of freely reduced words on the a_i). Define $E_P = \{XzX^{-1} \ \forall X \in P\} < F * \langle z; \rangle$. Then P is $E^n(A)$ *benign* iff E_P is $E^n(A)$ benign as a subgroup of $F * \langle z; \rangle$.

Notice $F * \langle z; \rangle = \langle a_1, \ldots, a_m, z; \rangle$ is the free group on $m + 1$ generators and so the words XzX^{-1} freely generate E_P which is then an $E^n(A)$ decidable subgroup of $F * \langle z; \rangle$. Thus it makes sense to ask if E_P is $E^n(A)$ benign. If P is in fact a subgroup of F, then it may be said to be $E^n(A)$ benign as a subgroup (that is, if F_P is $E^n(A)$ Higman) or as a subset. The next two lemmas state that for P a subgroup the two notions of benign coincide.

LEMMA 7.2. *Let $P < F = \langle a_1, \ldots, a_m; \rangle$ be an $E^n(A)$ decidable subgroup for $n \geq 4$. If P is $E^n(A)$ benign as a set then it is $E^n(A)$ as a subgroup.*

Proof. (This proof is very similar to the proof of Lemma 5.1 of [4] and is based on Shoenfield [15]; however this lemma being the crucial step in the argument, details are reproduced here.)

The tactic of the proof is to embed F in a f.p. E^4 standard group K by an E^4 embedding and then use the properties of $E^n(A)$ benign subgroups to show $P < K$ is $E^n(A)$ benign in K and hence in F by Lemma 6.8.

Let $G = F * \langle c, d; \rangle = \langle a_1, \ldots, a_m, c, d; \rangle$ and $X_1, X_2, \ldots$ be an E^3 enumeration of the elements of F. For $\{c\} < G$ and $\{dX_q\} < G$, for every $X_q \in F$, define $\phi_q : \{c\} \to \{dX_q\}$ by $c \mapsto dX_q$. Then G is E^3

standard and each ϕ_q is an E^3 isomorphism in G so

$$G_\infty = G_{q_1,q_2,\ldots} = \left\langle G,\ t_X\ \forall X \in F;\ t_X c t_X^{-1} = dX\ \forall X \in F \right\rangle$$

is a countably generated, E^4 standard group (it is E^4 by Lemma 5.9 and the quotient homomorphism $\langle a_1, \ldots, a_m, c, d, t_X\ \forall X \in F;\ \rangle \to G_\infty$ is E^4 computable since G was E^3 standard and each of the ϕ_q, ϕ_q^{-1}, when encoded, has an E^3 bound using Lemma 2.6). Now let $\Psi_j : G_\infty \to G_\infty$ by $a_i \mapsto a_i$, $c \mapsto c$, $d \mapsto da_j$ and $t_X \mapsto t_{a_j X}$ for $j = 1, \ldots, m$.

Observe that the conditions defining Ψ_j yield an E^3 automorphism of $\langle a_1, \ldots, a_m, c, d, t_X\ \forall X \in F;\ \rangle$ which permutes the relations of G_∞ and so the Ψ_j are E^4 automorphisms of G_∞. Let $K = (G_\infty)_{\Psi_1,\ldots,\Psi_m}$. Then G_∞ being E^4 standard and each Ψ_j having an E^3 bound when encoded, $(G_\infty)_{\Psi_1,\ldots,\Psi_m}$ is E^4 standard using Lemma 2.6 as before with respect to the generators $a_1, \ldots, a_m, c, d, t_X\ \forall X \in F$, $a'_1, \ldots, a'_m$ for a'_j corresponding to Ψ_j. We show K is f.p. and so E^4 standard relative to any finite set of generators by Theorem 3.4. The relations of K are

(1) $\quad t_X c t_X = dX, \qquad \forall X \in F,$

(2) $\quad a'_i a_j a'^{-1}_i = a_j, \qquad \forall i = 1, \ldots, m$ and $j = 1, \ldots, m,$

(3) $\quad a'_i c a'^{-1}_i = c, \qquad \forall i = 1, \ldots, m,$

(4) $\quad a'_i d a'^{-1}_i = da_i, \qquad \forall i = 1, \ldots, m,$

(5) $\quad a'_i t_X a'^{-1}_i = t_{a_i X}, \qquad \forall i = 1, \ldots, m$ and $\forall X \in F.$

Let $t = t_\Lambda$ for Λ the identity (empty word) in F. For $X \in P$ an

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arbitrary (freely reduced) word on the a_i, let X' be the corresponding word formed by replacing each a_i by a_i'. By (5), $X'tX'^{-1} = t_X$ and so the generators t_X for $X \neq \Lambda$ are redundant and K is f.g. by $a_1, \ldots, a_m, c, d, t, a_1', \ldots, a_m'$. Also the relations (5) can be deleted and (1) replaced by

(1') $\left(X'tX'^{-1}\right)c\left(X'tX'^{-1}\right)^{-1} = dX$, $\forall X \in F$.

From (3), $X'cX'^{-1} = c$ and $X'^{-1}cX' = x$ and from (1'), $tct^{-1} = d$. Thus

$$\left(X'tX'^{-1}\right)c\left(X'tX'^{-1}\right)^{-1} = X'tX'^{-1}cX't^{-1}X'^{-1} = X'tct^{-1}X'^{-1} = X'dX'^{-1} = dX ,$$

the latter by (2) and (4). Thus (1') can be replaced by

(1") $tct^{-1} = d$,

and K if f.p. and E^4 standard. The embeddings $F \to G_\infty \to K$ are E^4 embeddings.

To show $P < K$ is $E^n(A)$ benign we show $P = K \cap \{c, d, t_X \ \forall X \in P\}$ and $\{c, d, t_X \ \forall X \in P\}$ is $E^n(A)$ benign. Then F is E^4 benign being f.g. and P is $E^n(A)$ benign by Lemma 6.9. As a first step we show $\{c, d, t_X \ \forall X \in P\}$ is $E^n(A)$ decidable in G_∞ and hence in K. Let $G_q = G_{\phi_1, \phi_2, \ldots, \phi_q} < G_\infty$ for $q = 1, 2, \ldots$ and

$$G_{(P)} = \left\langle G, t_X \ \forall X \in P; \ t_X c t_X^{-1} = dx \ \forall X \in P \right\rangle .$$

Clearly from the construction of G_∞ (see Lemma 5.10), $G_q < G_\infty$ are E^4 decidable for all q and for any $g \in G_\infty$ the minimal q so that $g \in G_q$ can be obtained by an E^3 process. Thus to show $G_{(P)} < G_\infty$ is $E^n(A)$ decidable it suffices to show $G_{(P)} \cap G_q < G_q$ is $E^n(A)$ decidable for all q. We proceed by induction on q noting that for $q = 0$, $G_q = G$ and $G_{(P)} \cap G = G$. Assume $G_{(P)} \cap G_q < G_q$ is $E^n(A)$ decidable. If $X_{q+1} \notin P$

then $G_{(P)} \cap G_{q+1} = G_{(P)} \cap G_q < G_q < G_{q+1}$ and is $E^n(A)$ decidable since P is. If $X_{q+1} \in P$, $\{c\} < G_{(P)} \cap G_q$ and $\{dX_{q+1}\} < G_{(P)} \cap G_q$ so $G_{(P)} \cap G_q$ is $E^n(A)$ invariant under ϕ_{q+1}. Moreover ϕ_{q+1} is E^3 computable and the right coset representative functions for $\{c\} < G < G_{(P)} \cap G_q$ and $\{dX_{q+1}\} < G < G_{(P)} \cap G_q$ can be given by minimalization and so have obvious E^3 bounds. By Lemma 2.6 and Lemma 5.9, $G_{(P)} \cap G_{q+1} = \left(G_{(P)} \cap G_q\right)\phi_{q+1} < \left(G_q\right)\phi_{q+1} = G_{q+1}$ is $E^n(A)$ decidable. By relation (1), $\{c, d, t_X \ \forall X \in P\} = \{c, d, t_X \ \forall X \in P, P\}$. Since $P < F$ is $E^n(A)$ decidable, $\{c, d, P\} = P * \langle c, d; \ \rangle < G$ is $E^n(A)$ decidable. Moreover for all $X \in P$, $\{c\} < \{c, d, P\}$ and $\{dX\} < \{c, d, P\}$ so $\{c, d, P\}$ is $E^n(A)$ invariant under ϕ_q for all $X_q \in P$. Again the required right coset representative functions can be given by minimalization and so have E^3 bounds as do the ϕ_q so by Lemma 2.6 and Lemma 5.11, $\{c, d, t_X \ \forall X \in P\} = \{c, d, t_X \ \forall X \in P, P\} < G_\infty < K$ is $E^n(A)$ decidable in K. Also by Lemma 5.11, $\{c, d, t_X \ \forall X \in P\} \cap G = \{c, d, P\}$ so $\{c, d, t_X \ \forall X \in P\} \cap F = \{c, d, P\} \cap F = P$.

The proof is completed by showing $\{c, d, t_X \ \forall X \in P\}$ is $E^n(A)$ benign in K. Let $E = \langle a_1, \ldots, a_m, c, d, z; \ \rangle = F * \langle c, d, z; \ \rangle$. As previously observed $\{XzX^{-1} \ \forall X \in P\} < F * \langle z; \ \rangle$ is $E^n(A)$ decidable and by assumption $E^n(A)$ benign. Since $F * \langle z; \ \rangle < E$ is obviously an E^3 embedding, $\{XzX^{-1} \ \forall X \in P\}$ is $E^n(A)$ decidable and $E^n(A)$ benign in E by Lemma 6.8. Now $\{c, d\} < E$ is E^3 decidable and E^3 benign so $\{c, d, XzX^{-1} \ \forall X \in P\} = \{XzX^{-1} \ \forall X \in P\} * \langle c, d; \ \rangle < E$ is $E^n(A)$ decidable and $E^n(A)$ benign by Lemma 6.10. Let $\eta : E \to K$ by $a_i \mapsto a_i'$, $c \mapsto c$, $d \mapsto d$, $z \mapsto t$. Clearly η is E^4 computable and

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$\eta(\{c, d, XzX^{-1}\ \forall X \in P\}) = \{c, d, t_X\ \forall X \in P\}$ since $X'tX'^{-1} = t_X$ in K. Therefore since $\{c, d, t_X\ \forall X \in P\} < K$ is $E^n(A)$ decidable, it is $E^n(A)$ benign by Lemma 6.11. □

LEMMA 7.3. *If $P < F = \langle a_1, \ldots, a_m; \ \rangle$ is an $E^n(A)$ decidable, $E^n(A)$ benign subgroup for $n \geq 4$ then P is an $E^n(A)$ benign subset.*

Proof. Consider $\phi : F * \langle z; \ \rangle \to F$ by $a_i \mapsto a_i$ and $z \mapsto 1$. Then ϕ is E^3 computable and $\ker\phi = \{XzX^{-1}\ \forall X \in F\}$ is E^3 decidable and E^4 benign by Lemma 6.11 since $\{\Lambda\} < F$ is E^4 benign. Moreover, $P < F$ being $E^n(A)$ benign, $\{P, z\} < F * \langle z; \ \rangle$ is $E^n(A)$ decidable and $E^n(A)$ benign by Lemma 6.10. Therefore

$$E_P = \{XzX^{-1}\ \forall X \in P\} = \{XzX^{-1}\ \forall X \in F\} \cap \{P, z\}$$

is $E^n(A)$ benign by Lemma 6.9. □

It should be observed that for $P \subset F = \langle a_1, \ldots, a_m; \ \rangle$ the decidability of $E_P < F * \langle z; \ \rangle$ is the same as the decidability of P. This observation together with Proposition 6.7, Lemma 6.8, Lemma 6.9 and Lemma 6.10 yield the following statements.

PROPOSITION 7.4. *Every finite subset of $F = \langle a_1, \ldots, a_m; \ \rangle$ is E^4 benign.* □

PROPOSITION 7.5. *If $P \subset F = \langle a_1, \ldots, a_m; \ \rangle$ is $E^n(A)$ decidable and $E^n(A)$ benign in F for $n \geq 4$ then*

$$P \subset G = \langle a_1, \ldots, a_m, b_1, \ldots, b_q; \ \rangle$$

is $E^n(A)$ decidable and $E^n(A)$ benign in G and conversely. □

PROPOSITION 7.6. *If $P \subset F = \langle a_1, \ldots, a_m; \ \rangle$ and $Q \subset F$ are $E^n(A)$ decidable and $E^n(A)$ benign for $n \geq 4$ then $P \cap Q$ and $P \cup Q$ are*

$E^n(A)$ decidable and $E^n(A)$ benign. $\square$

Let $F = \langle a_1, \ldots, a_m; \ \rangle$ and $G = \langle b_1, \ldots, b_q; \ \rangle$. We consider set maps $\phi : F \to G$. The reader is warned that in the following the above notation is used for set maps which may or may not be homomorphisms. We restrict our attention to a certain class of computable set maps from F to G.

DEFINITION 7.7. Let $F = \langle a_1, \ldots, a_m; \ \rangle$, $G = \langle b_1, \ldots, b_q; \ \rangle$ and $\phi : F \to G$ be an $E^n(A)$ computable set map for $n \geq 3$. An $E^n(A)$ *associate* of ϕ is an $E^n(A)$ computable homomorphism $\Psi : F * \langle z; \ \rangle \to G * \langle z; \ \rangle$ such that for $X \in F$, $\Psi(XzX^{-1}) = \phi(X)z\phi(X)^{-1}$. If $\phi : F \to F$ is an $E^n(A)$ computable bijection such that ϕ^{-1} is also $E^n(A)$ computable then ϕ is $E^n(A)$ *nice* if it has an $E^n(A)$ associate Ψ which is an automorphism of $F * \langle z; \ \rangle$ with Ψ^{-1} $E^n(A)$ computable.

Of course if ϕ is an $E^n(A)$ computable homomorphism then $\Psi : F * \langle z; \ \rangle \to G * \langle z; \ \rangle$ by $a_i \mapsto \phi(a_i)$, $z \mapsto z$ is an $E^n(A)$ associate. In particular if ϕ is an $E^n(A)$ computable automorphism of F such that ϕ^{-1} is $E^n(A)$ computable then ϕ is $E^n(A)$ nice. It is immediate from the definition that the composition of $E^n(A)$ computable maps having $E^n(A)$ associates has an $E^n(A)$ associate and the composition of $E^n(A)$ nice bijections is an $E^n(A)$ nice bijection.

PROPOSITION 7.8. *Let $F = \langle a_1, \ldots, a_m; \ \rangle$ and $Y \in F$. Define $L_Y : F \to F$ by $X \mapsto YX$ $\forall X \in F$ and $R_Y : F \to F$ by $X \mapsto XY$ $\forall X \in F$. Then L_Y and R_Y are E^3 computable and E^3 nice.*

Proof. Clearly L_Y and R_Y are E^3 computable bijections with E^3 computable inverses. Inner automorphism by $Y \in F * \langle z; \ \rangle$ is an E^3 computable automorphism with E^3 computable inverse and is an E^3

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associate of L_Y since $XzX^{-1} \mapsto YXzX^{-1}Y^{-1} = YXz(YX)^{-1}$. To construct an associate for R_Y let $\Psi : F * \langle z; \rangle \to F * \langle z; \rangle$ by $a_i \mapsto a_i$ and $z \mapsto YzY^{-1}$. Then Ψ is an E^3 computable automorphism with E^3 computable inverse given by $a_i \mapsto a_i$, $z \mapsto Y^{-1}zY$ and an E^3 associate of R_Y since $XzX^{-1} \mapsto XYzY^{-1}X^{-1} = XYz(XY)^{-1}$. □

The following is analagous to Lemma 6.11.

LEMMA 7.9. *Let* $F = \langle a_1, \ldots, a_m; \rangle$, $G = \langle b_1, \ldots, b_q; \rangle$ *and* $\phi : F \mapsto G$ *an* $E^n(A)$ *computable set map having an* $E^n(A)$ *associate for* $n \geq 4$. *Then if* $P \subset F$ *is* $E^n(A)$ *decidable and* $E^n(A)$ *benign and* $\phi(P) \subset G$ *is* $E^n(A)$ *decidable,* $\phi(P)$ *is* $E^n(A)$ *benign.*

Proof. $E_P < F * \langle z; \rangle$ is $E^n(A)$ decidable and $E^n(A)$ benign, $E_{\phi(P)} < G * \langle z; \rangle$ is $E^n(A)$ decidable and there is an $E^n(A)$ homomorphism $\Psi : F * \langle z; \rangle \to G * \langle z; \rangle$ satisfying

$$\Psi\left(XzX^{-1}\right) = \phi(X)z\phi(X)^{-1} \ \forall X \in F$$

by assumption. Then $E_{\phi(P)} = \Psi\left(E_P\right)$ which is $E^n(A)$ benign by Lemma 6.11. □

The following definition and Lemma describe a technique used to construct $E^n(A)$ benign sets.

DEFINITION 7.10. Let $F = \langle a_1, \ldots, a_m; \rangle$, $P \subset F$ and $Q \subset F$ and $\phi : F \to F$ be a set map having an associate. We say P is (ϕ, Q) *invariant* if $\forall X \in Q \ \left(X \in P \leftrightarrow \phi(X) \in P\right)$. We say P is *invariant under* ϕ if it is (ϕ, F) invariant.

LEMMA 7.11. *Let* $F = \langle a_1, \ldots, a_m; \rangle$ *and* $P, Q_1, \ldots, Q_q$ *be* $E^n(A)$ *decidable* $E^n(A)$ *benign subsets of* F *for* $n \geq 4$. *Let* $\phi_1, \ldots, \phi_q$ *be* $E^n(A)$ *computable bijections of* F *onto* F *which have* $E^n(A)$ *computable* *inverses and are* $E^n(A)$ *nice via associates* $\Psi_1, \ldots, \Psi_q$. *Further assume the* Ψ_i *and their inverses have* $E^{n-1}(A)$ *bounds. Let* R *be the smallest subset of* F *containing* P *and* (ϕ_i, Q_i) *invariant for* $i = 1, \ldots, q$ *and assume* R *is* $E^n(A)$ *decidable. Then* R *is* $E^n(A)$ *benign. In particular, if the smallest subset of* F *containing* P *and invariant under the* ϕ_i *is* $E^n(A)$ *decidable, it is* $E^n(A)$ *benign.*

Proof. (This proof is similar to the proof of Lemma 5.4 of [4]; some details are reproduced here for completeness.)

Let $G = F * z$; and $\overline{\Psi}_i = \Psi_i | E_{Q_i}$. Since the Ψ_i and Ψ_i^{-1} are $E^n(A)$ computable and $E^{n-1}(A)$ bounded, the $\overline{\Psi}_i$ are $E^n(A)$ isomorphisms in G also having $E^{n-1}(A)$ bounds. Thus by Lemma 2.6 and Theorem 5.3,

$$G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q} = $$
$$= \left\langle F, z, t_1, \ldots, t_q;\ t_i X z X^{-1} t_i^{-1} = \phi_i(X) z \phi_i(X)^{-1}\ \forall X \in Q_i\ \forall i = 1, \ldots, q \right\rangle$$

and

$$G_{\Psi_1,\ldots,\Psi_q} = \Big\langle F, z, s_1, \ldots, s_q;\ s_i a_j s_i^{-1} = \Psi_i(a_j),$$
$$s_i z s_i^{-1} = \Psi_i(z)\ \forall i = 1, \ldots, q\ \forall j = 1, \ldots, m \Big\rangle$$

are $E^n(A)$ standard. Also $\left(G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q}\right)_{\Psi_1,\ldots,\Psi_q}$ is then $E^n(A)$ standard and $\left(G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q}\right)_{\Psi_1,\ldots,\Psi_q} = \left(G_{\Psi_1,\ldots,\Psi_q}\right)_{\iota_1,\ldots,\iota_q}$ for ι_i the identity on E_{Q_i} by Tietze transformations. Moreover since these groups are $E^n(A)$ standard the identity isomorphisms between the various standard and normal form indices are $E^n(A)$ computable. Thus

$G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q} < G_{\Psi_1,\ldots,\Psi_q,\iota_1,\ldots,\iota_q}$ is $E^n(A)$ decidable and, since

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$G_{\Psi_1,\ldots,\Psi_q,\iota_1,\ldots,\iota_q}$ is $E^n(A)$ Higman, it suffices to show E_R is $E^n(A)$ benign in $G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q}$. By assumption R is $E^n(A)$ decidable in F and thus $E_R < G < G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q}$ is $E^n(A)$ decidable.

Let $H < G$ be the smallest subgroup of G containing E_P and satisfying $\overline{\Psi}_i\left(H \cap E_{Q_i}\right) = \overline{\Psi}_i\left(E_{Q_i}\right) \cap H$ for $i = 1, \ldots, q$. Then $\{E_P, t_1, \ldots, t_q\} \cap G < \{H, t_1, \ldots, t_q\} \cap G = H$ by Lemma 5.11. On the other hand one can verify

$$\overline{\Psi}_i\left(\left(\{E_P, t_1, \ldots, t_q\} \cap G\right) \cap E_{Q_i}\right) = \overline{\Psi}_i\left(E_{Q_i}\right) \cap \left(\{E_P, t_1, \ldots, t_q\} \cap G\right)$$

and so H being the smallest subgroup satisfying these conditions, $H < \{E_P, t_1, \ldots, t_q\} \cap G$ so in fact $H = \{E_P, t_1, \ldots, t_q\} \cap G$.

If $g \in E_{Q_i}$ is the product of words of the form $Xz^{\pm 1}X^{-1}$ for $X \in Q_i$ then $\overline{\Psi}(g)$ is a product of words $\phi_i(X)z^{\pm 1}\phi_i(X)^{-1}$ which are free in G . Thus $\overline{\Psi}_i(g) \in E_R$ iff all of the $\phi_i(X)$ involved are in R , iff all of the X involved are in R , iff $g \in E_R$. Thus

$$\overline{\Psi}_i\left(E_R \cap E_{Q_i}\right) = \overline{\Psi}_i\left(E_{Q_i}\right) \cap E_R$$

so $E_R > H$. Conversely if $X \in R$ then $X = \phi_{i_1}^{\varepsilon_1} \ldots \phi_{i_\alpha}^{\varepsilon_\alpha}(Y)$ for $Y \in P$, $\varepsilon_j = \pm 1$ where ϕ_j is applied only to a word in Q_j and ϕ_j^{-1} only to a word in $\phi_j(Q_j)$ since otherwise X could be deleted from R contrary to the minimality of R . Then

$$XzX^{-1} = \phi_{i_1}^{\varepsilon_1} \ldots \phi_{i_\alpha}^{\varepsilon_\alpha}(Y)z\left(\phi_{i_1}^{\varepsilon_1} \ldots \phi_{i_\alpha}^{\varepsilon_\alpha}(Y)\right)^{-1} = \overline{\Psi}_{i_1}^{\varepsilon_1} \ldots \overline{\Psi}_{i_\alpha}^{\varepsilon_\alpha}\left(YzY^{-1}\right) \in H ,$$

so $E_R < H$ and hence $E_R = H$.

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The E_R, E_{Q_i} and $\overline{\Psi}_i\left(E_{Q_i}\right)$ are all generated by words of the form XzX^{-1} for $X \in F$ and are free in G. Thus for $w \in G$, it is an $E^n(A)$ procedure to determine if $w = uv$ for $u \in E_R$ and $v \in E_{Q_i}$ $\left(\text{respectively } \overline{\Psi}\left(E_{Q_i}\right)\right)$ and if so the corresponding u is $E^n(A)$ computable and has index less than that of w. Thus E_R is $E^n(A)$ invariant under the $\overline{\Psi}_i$ with the right coset representative functions having an E^3 bound so by Lemmas 2.6 and 5.11, $\{E_R, t_1, \ldots, t_q\} < G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q}$ is $E^n(A)$ decidable.

Now $\{\Lambda\} < G$ is $E^n(A)$ invariant under the $\overline{\Psi}_i$ with trivial compatibility so $\{\Lambda, t_1, \ldots, t_q\} = \{t_1, \ldots, t_q\} < G_{\overline{\Psi}_1,\ldots,\overline{\Psi}_q}$ is $E^n(A)$ decidable and $E^n(A)$ benign being f.g. Thus since E_P is $E^n(A)$ benign by assumption and $\{E_P, t_1, \ldots, t_q\} = \{E_R, t_1, \ldots, t_q\}$ is $E^n(A)$ decidable it is $E^n(A)$ benign by Lemma 6.10. Therefore $E_R = G \cap \{E_R, t_1, \ldots, t_q\}$ is $E^n(A)$ benign by Lemma 6.9.

Since F is f.g. it is E^4 benign and so may be substituted for the Q_i yielding the final statement. □

Lemma 7.11 is used particularly with the E^3 nice maps L_Y, R_Y and E^3 computable automorphisms of F so the hypotheses are automatically satisfied. We next obtain some useful examples of E^4 benign subsets by the above technique.

PROPOSITION 7.12. *Let* $F = \langle a, b ; \rangle$ *and* $b_i = b^i a b^{-i}$ *for* $i = 0, \pm 1, \pm 2, \ldots$. *Consider* P *the set of all words in* F *of the form* $b_{i_1} b_{i_2} \ldots b_{i_q}$ *for* $0 \leq i_1 < i_2 \ldots < i_q$ *and any (finite)* q . *Then* P

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is $\mathcal{E}^4$ *benign in* F .

Proof. Let $H = \{b_i, i > 0\} < F$ and $H' = \{b_i, i \geq 0\} < F$, $\mathcal{E}^3$ decidable in F since the b_i are free. Then H is $\mathcal{E}^4$ benign by Proposition 6.12 and $H' = \{\{a\}, H\}$ is $\mathcal{E}^4$ benign by Lemma 6.10. Let $\Psi : F \to F$ be the $\mathcal{E}^3$ automorphism defined by $a \mapsto bab^{-1}$ and $b \mapsto b$. Then Ψ is $\mathcal{E}^3$ nice as is L_a of Proposition 7.8. By a simple spelling argument P is the smallest subset of F containing $\{\Lambda\}$ ($\mathcal{E}^4$ benign) and (L_a, H), (Ψ, H') invariant. □

DEFINITION 7.13. Let $F = \langle a_1, \ldots, a_q; \rangle$. A word in F is *positive* if it does not contain any occurances of a_i^{-1} for $i = 1, \ldots, q$.

PROPOSITION 7.14. *Let* $W^+ \subset \langle a_0, \ldots, a_{q-1}; \rangle$ *be the set of positive words. Then* W^+ *is* $\mathcal{E}^4$ *benign.*

Proof. Let

$$D = \left\langle a_0, \ldots, a_{q-1}, z, t; ta_it^{-1} = a_{(i+1)\mathrm{mod}q}, tzt^{-1} = z \right\rangle ,$$

so D is f.p. and $\mathcal{E}^4$ standard being of the form $(\langle a_0, \ldots, a_{q-1}, z; \rangle)_\phi$ for ϕ defined by $a_i \mapsto a_{(i+1)\mathrm{mod}q}$, $z \mapsto z$ an $\mathcal{E}^3$ computable automorphism. Let $F = \langle a, b; \rangle$ and $P \subset F$ be the $\mathcal{E}^4$ benign subset as in Proposition 7.12. Consider $\Psi : F * \langle z; \rangle \to D$ by $a \mapsto a_0$, $b \mapsto t$, $z \mapsto z$ an $\mathcal{E}^4$ computable homomorphism. Then if $w = a_{i_1} \ldots a_{i_k} \in W^+$. $w = \Psi\left(b_{i_1} b_{i_2+n} \ldots b_{i_k+(k-1)n}\right)$ so $\Psi(P) = W^+$. Therefore $\Psi(E_P) = E_{W^+}$ which is $\mathcal{E}^3$ decidable hence $\mathcal{E}^4$ benign by Lemma 6.11 since P and hence E_P are $\mathcal{E}^4$ benign. □

We now restrict our attention to subgroups of $F = \langle a, b; \ \rangle$. The notation $W \subset F$, the set of all positive words, and $W_b \subset W \subset F$, the set of all positive words beginning in b, will be used in the following. Observe that W and W_b are E^3 decidable and also W and $W_b = L_b(W)$ are E^4 benign by Proposition 7.14 and Lemma 7.9. The convention p and q are natural numbers and i is an integer wherever they occur will be in effect.

DEFINITION 7.15. Let $x^{(k)} = (x_1, \ldots, x_k)$ be a k-tuple of natural numbers and associate with $x^{(k)}$ the word $a^{x_1}ba^{x_2}b \ldots ba^{x_k} \in W$. In this way a subset of N^k is associated with a subset of W. A k-ary predicate is $E^n(A)$ *benign* if the corresponding subset of W is $E^n(A)$ decidable and $E^n(A)$ benign.

Observe that by definition an $E^n(A)$ benign predicate must be $E^n(A)$ decidable. Moreover by Proposition 7.6 the conjunction and disjunction of $E^n(A)$ benign predicates are again $E^n(A)$ benign. For $f : N^k \to N$ let P_f denote the k+1-ary predicate $P_f(x^{(k)}, y) \leftrightarrow f(x^{(k)}) = y$.

PROPOSITION 7.16. *The predicates* $=$, P_{f_0} *corresponding to the successor function,* P_{f_1} *corresponding to addition, and* P_{f_2} *corresponding to multiplication are* E^4 *benign.*

Proof. $Q \subset F$ the smallest set containing b and invariant under L_aR_a is E^4 benign so since $=$ corresponds to $W \cap Q$ it is E^4 benign.

$R \subset F$ the smallest set containing $L_b(W \cap Q)$ and invariant under L_aR_a is E^4 benign and consists of words of the form $a^iba^qba^{i+q}$ so P_{f_1} corresponds to $R \cap W$ and is E^4 benign. Also P_{f_0} corresponds to $R_a(W \cap Q)$ and is E^4 benign.

Let $F' = \langle a, b, c; \rangle$ and $W' \subset F'$ be the set of positive words. Set T equal to the smallest set of words in F' containing b and invariant under $L_a R_c$ and set $T' = L_{cb}(T \cap W')$ so T' consists of words of the form $cba^q bc^q$ and is E^4 benign. Let $\phi = F' \to F'$ by $a \mapsto a$, $b \mapsto b$, $c \mapsto ca$ an E^3 automorphism so T'' the smallest set containing T' and invariant under ϕ is E^4 benign and consists of words of the form $ca^i ba^q b(ca^i)^q$. For $\Psi : F' \to F$ by $a \mapsto a$, $b \mapsto b$ and $c \mapsto \Lambda$, an E^3 homomorphism, P_{f_2} corresponds to $\Psi(T'' \cap W')$ and so is E^4 benign. $\square$

PROPOSITION 7.17. *If* $P(y, x^{(k)})$ *is* $E^n(A)$ *benign for* $n \geq 4$ *and* $Q(x^{(k)}, y) \leftrightarrow P(y, x^{(k)})$ *then* Q *is* $E^n(A)$ *benign.*

Proof. Let $P \subset F$ correspond to the predicate P and P' be the smallest set containing P and invariant under $L_{a^{-1}} R_a$. Then Q corresponds to $L_{b^{-1}}(P' \cap W_b)$ and so is E^4 benign. $\square$

PROPOSITION 7.18. *If* $P(x^{(k)})$ *is* $E^n(A)$ *benign for* $n \geq 4$ *and* $Q(y, x^{(k)}) \leftrightarrow P(x^{(k)})$ *then* Q *is* $E^n(A)$ *benign.*

Proof. Let $P \subset F$ correspond to the predicate P and P' be the smallest set containing $L_b(P)$ and invariant under L_a. Then Q corresponds to $P' \cap W$ so is $E^n(A)$ benign. $\square$

PROPOSITION 7.19. *Assume* $P(y, x^{(k)})$ *is* $E^n(A)$ *benign for* $n \geq 4$ *and* $Q(x^{(k)}) \leftrightarrow \exists y (P(y, x^{(k)}))$. *Then if* Q *is* $E^n(A)$ *decidable it is* $E^n(A)$ *benign.*

Proof. Let $P \subset F$ correspond to the predicate P and P' be the smallest set containing P and invariant under L_a. Then P' consists of words $a^i bw$ such that there exists q so that $a^q bw \in P$ and P' is $E^n(A)$ decidable by assumption and so $E^n(A)$ benign. Q corresponds to

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$L_{b^{-1}}(P' \cap W_b)$ so is $\mathcal{E}^n(A)$ benign. □

PROPOSITION 7.20. *If* $P(y, x^{(k)})$ *is* $\mathcal{E}^n(A)$ *benign for* $n \geq 4$ *and* $Q(z, x^{(k)}) \leftrightarrow (\forall y)_{y<z} P(y, x^{(k)})$ *then* Q *is* $\mathcal{E}^n(A)$ *benign.*

Proof. Let $U_1 \subset F$ be the smallest set containing $\{\Lambda\}$ and invariant under R_a and inductively define U_q to be the smallest set containing $L_b(U_{q-1} \cap W)$ and invariant under R_a. Then $U = U_k \cap W$ consists of all words of the form $a^{n_1}ba^{n_2} \ldots ba^{n_k}$ and is $\mathcal{E}^3$ decidable and $\mathcal{E}^4$ benign.

Let $P \subset F$ correspond to the predicate P and V be the smallest set containing $L_b(U)$ and (L_a, P) invariant. Then Q corresponds to V and is $\mathcal{E}^n(A)$ benign. □

PROPOSITION 7.21. *Let* $g : N \to N$ *be an* $\mathcal{E}^n(A)$ *computable function for* $n \geq 4$ *such that* P_g *is* $\mathcal{E}^n(A)$ *benign. Then if* f *is defined from* g *by pure iteration,* $f(0) = p$, $f(x+1) = g^{x+1}(p) = gf(x)$ *then* P_f *is* $\mathcal{E}^{n+1}(A)$ *benign. If* $f(x)$ *is* $\mathcal{E}^n(A)$ *computable (that is, has an* $\mathcal{E}^n(A)$ *bound) then* P_f *is* $\mathcal{E}^n(A)$ *benign.*

Proof. Let $S \subset F$ correspond to P_g and $F' = \langle a, b, c; \ \rangle$. Then S is $\mathcal{E}^n(A)$ benign in F by Proposition 7.5. Let $S' \subset F'$ be the smallest set containing $L_c(S)$ and invariant under L_a and L_b so S' consists of words $wca^xba^{g(x)}$ for $w \in F$ and $x \in N$. Let $W' \subset F'$ be the set of positive words and $\phi : F' \to F$ by $a \mapsto a$, $b \mapsto b$, $c \mapsto b$, an $\mathcal{E}^3$ homomorphism. Then $S'' = \phi(W' \cap S')$ is $\mathcal{E}^n(A)$ benign and consists of words $wba^xba^{g(x)}$ for $w \in W \subset F$.

Let $T \subset F$ be the smallest set containing ba^pb and (S'', R_b) invariant and invariant under R_a. T consists of words

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$$ba^pba^i,\ ba^pba^{g(p)}ba^i,\ \ldots,\ ba^pba^{g(p)}ba^{g^2(p)}b\ \ldots\ ba^{g^k(p)}ba^i,\ \ldots$$

so is $E^{n+1}(A)$ decidable and $E^{n+1}(A)$ benign. Also $T' = L_{b^{-1}}(T \cap S'')$ consists of words $a^pba^{g(p)}, \ldots, a^pba^{g(p)} \ldots ba^{g^k(p)}, \ldots$ so is $E^n(A)$ decidable and $E^{n+1}(A)$ benign.

Let $F'' = \langle a, b, c, d, e; \ \rangle$ and $W' \subset F''$ be the set of positive words. Then $U \subset F''$ the smallest set containing d and L_aR_c invariant is E^4 benign as is U' the smallest set containing $L_e(U)$ and invariant under L_a and L_b. U' consists of words wea^idc^i for $w \in F$. Define the E^3 homomorphism $\Psi : F'' \to F''' = \langle a, b, c, d; \ \rangle$ by $a \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto d$ and $e \mapsto b$ so $U'' = (U' \cap W'')$ is E^4 benign consisting of words wba^qda^q for $w \in W \subset F$.

In F''', let T'' be the smallest set containing $R_d(T')$ and invariant under R_c. Thus T'' consists of words

$$a^pba^{g(p)}b\ \ldots\ ba^{g^k(p)}dc^i\ ,$$

is $E^{n+1}(A)$ decidable, and $E^{n+1}(A)$ benign. Define the E^3 homomorphism $\eta : F''' \to F$ by $a \mapsto \Lambda$, $b \mapsto a$, $d \mapsto b$ and $c \mapsto a$. Then $T''' = (T'' \cap U'')$ consists of words $a^kba^{g^k(p)} = a^kba^{f(k)}$ for $k > 0$ so is $E^{n+1}(A)$ decidable and $E^{n+1}(A)$ benign. Moreover P_f corresponds to $ba^p \cup T'''$ and is therefore $E^{n+1}(A)$ benign.

If f is $E^n(A)$ computable then T, T', T'' and T''' are $E^n(A)$ decidable hence $E^n(A)$ benign as is $ba^p \cup T'''$. □

LEMMA 7.22. *If* $f : N^k \to N$ *is an* $E^n(A)$ *computable function for* $n \geq 4$ *and* A *recursively enumerable then* P_f *is* $E^n(A)$ *benign.*

Proof. Referring to Definition 2.3, we must show that the predicates

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corresponding to the initial functions

$$Z(x) = 0 \ ,$$

$$U_q^p(x_1, \ldots, x_p) = x_q \ , \text{ for } 1 \le q \le p \ ,$$

$$f_0(x, y) = x + 1 \ ,$$

$$f_1(x, y) = x + y \ ,$$

$$f_n(x, y) \ ,$$

$$E(x) = x - \left[x^{\frac{1}{2}}\right]^2 \ ,$$

$$c_A(x) = \begin{cases} 0 & \text{if } x \in A \ , \\ 1 & \text{if } x \notin A \ , \end{cases}$$

are $E^n(A)$ benign and that the class of $E^n(A)$ computable functions having $E^n(A)$ benign predicates is closed under substitution and limited iteration.

The set corresponding to P_Z consists of words $a^q b$ and is $R_b(Q \cap W)$ for Q the smallest set containing $\{\Lambda\}$ and invariant under L_a so P_Z is E^4 benign. The set corresponding to $P_{U_1^1}$ is the same as that corresponding to "=" and thus is E^4 benign by Proposition 7.16. Then $P_{U_q^p}$ are E^4 benign by repeated applications of Propositions 7.17 and 7.18. P_{f_0} and P_{f_1} are E^4 benign by Proposition 7.16 as is P_{f_2}. Since f_n is obtained from f_{n-1} by iteration, P_{f_n} is E^n benign by induction and Proposition 7.21.

Consider the predicates "≤" and "<" defined by $x \le y \leftrightarrow \exists z P_{f_1}(x, z, y)$ and $x < y \leftrightarrow \exists z\left(z \ne 0 \wedge P_{f_1}(x, z, y)\right)$ which are

E^4 benign by Proposition 7.19. Then for $E'(x) = [x^{\frac{1}{2}}]$,

$$P_{E'}(x, y)$$
$$\leftrightarrow \exists w \exists v \exists u \left(P_{f_2}(y, y, u) \wedge (u \leq x) \wedge P_{f_0}(y, v) \wedge P_{f_2}(v, v, w) \wedge (x < w)\right)$$

which is E^4 benign by Propositions 7.6, 7.16, 7.18 and 7.19. Thus

$$P_E(x, y) \leftrightarrow \exists u \exists z \left(P_{E'}(x, z) \wedge P_{f_2}(z, z, u) \wedge P_{f_1}(y, u, x)\right)$$

is E^4 benign by the same argument.

If $f\left(x^{(k)}\right)$ is defined by $f\left(x^{(k)}\right) = g\left(h_1\left(x^{(k)}\right), \ldots, h_q\left(x^{(k)}\right)\right)$ where g and h_i for $i = 1, \ldots, q$ are $E^n(A)$ computable and P_g and P_{h_i} are $E^n(A)$ benign then

$$P_f\left(x^{(k)}, z\right)$$
$$\leftrightarrow \exists y_1 \exists y_2 \ldots \exists y_q \left(P_{h_1}\left(x^{(k)}, y_1\right) \wedge \ldots \wedge P_{h_q}\left(x^{(k)}, y_q\right) \wedge P_g\left(y_1, \ldots, y_q, z\right)\right)$$

which is $E^n(A)$ decidable so $E^n(A)$ benign by Proposition 7.19. Therefore the class of $E^n(A)$ computable functions having corresponding $E^n(A)$ benign predicates is closed under substitution and, by Proposition 7.21, it is also closed under limited iteration.

It remains to show that if A is recursively enumerable then P_{c_A} is is $E^4(A)$ benign. Let $F : N \to N$ be a recursive function and $A = \text{range } f$. Then by [7], p. 288, $f(y) = U\left(\mu z T(e, y, z)\right)$ for U, E^3 computable and T, E^3 decidable. Using the pairing functions J, K and L of §2, $x \in A \leftrightarrow \exists y \left(T\left(e, K(y), L(y)\right) \wedge UL(y) = x\right)$ and so, since K and L are E^3 computable and the predicate $x \in A$ is $E^3(A)$ decidable the predicate $x \in A$, that is, P_{c_A} is $E^4(A)$ benign.

Recall $P \subset F = \langle a, b; \ \rangle$ of Proposition 7.12. We apply Lemma 7.22 to subsets $R \subset P$ to obtain the following $E^n(A)$ benign sets.

PROPOSITION 7.23. *Let P be as in Proposition 7.12 and $R \subset P$ be an $E^n(A)$ decidable subset for $n \geq 4$ and A recursively enumerable. Then R is $E^n(A)$ benign.*

Proof. Let $F' = \langle a, z; \ \rangle$ and $\phi : F' \to F'$ be the E^3 isomorphism in F' defined by $a \mapsto a^2$, $z \mapsto z$. Then

$$F'_\phi = \langle a, z, b; bab^{-1} = a^2, bzb^{-1} = z \rangle$$

is E^4 standard. Let $\eta : F * \langle z; \ \rangle \to F'_\phi$ be the E^4 computable quotient homomorphism $a \mapsto a$, $b \mapsto b$, $z \mapsto z$. Recall the notation $b_i = b^i a b^{-i}$ and observe $\eta(b_i) = a^{2^i}$. Thus for $w = b_{i_1} \ldots b_{i_k} \in P$, $0 \leq i_1 < \ldots < i_k$, $\eta(w) = a^x$ for $x = 2^{i_1} + \ldots + 2^{i_k}$ and $\eta(wzw^{-1}) = a^x b a^{-x}$. Every $x > 0$, $x \in N$ has a unique representation $x = 2^{i_1} + \ldots + 2^{i_k}$ for $0 \leq i_1 < \ldots < i_k$ and such a representation is E^3 computable. In particular, η restricted to E_P is then monic.

A word in F' is in $\eta(R)$ iff it is of the form a^x for $x = 2^{i_1} + \ldots + 2^{i_k}$ and $b_{i_1} b_{i_2} \ldots b_{i_k} \in R$. Thus $\eta(R) < \langle a; \ \rangle < F$ is $E^n(A)$ decidable and so there exists an $E^n(A)$ computable characteristic function f such that $f(x) = 0$ if $a^x \in \eta(R)$ and $f(x) = 1$ if $a^x \notin \eta(R)$. By Lemma 7.22, P_f is $E^n(A)$ benign corresponding to $T \subset F$ consisting of words of the form $a^x ba$ for $a^x \notin \eta(R)$ and $a^x b$ for $a^x \in \eta(R)$. Thus $\eta(R) = W \cap R_{b^{-1}}(T)$ is $E^n(A)$ benign in F . Moreover

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$\eta(E_R) = \{a^x z a^{-x}$ for $a^x \in \eta(R)\} < F'$ is then $\mathcal{E}^n(A)$ benign in F' by definition and in F'_ϕ by Lemma 6.8.

Since η restricted to E_P is monic, $E_R = E_P \cap \eta^{-1}(\eta(E_R))$. By Lemma 6.11, $\eta^{-1}(\eta(E_R))$ is $\mathcal{E}^n(A)$ benign as then is E_R by Lemma 6.9 and the fact E_P is $\mathcal{E}^4$ benign. $\square$

LEMMA 7.24. *Let $E = \langle a_1, \ldots, a_m; \ \rangle$ and $Q \subset E$ be an $\mathcal{E}^n(A)$ decidable set for $n \geq 4$ and A recursively enumerable. Then Q is $\mathcal{E}^n(A)$ benign.*

Proof. We first assume Q consists only of positive words. We will freely use the notation and observations of the proofs of Propositions 7.12, 7.14 and 7.23. Since Q is assumed positive, $Q \subset \Psi(P)$ so $\Psi^{-1}(Q)$ is $\mathcal{E}^n(A)$ decidable in F. Moreover $\Psi(\Psi^{-1}(Q) \cap P) = Q$ so $\Psi\left(E_{\Psi^{-1}(Q)\cap P}\right) = E_Q$. Since $\Psi^{-1}(Q) \cap \bar{P}$ is an $\mathcal{E}^n(A)$ decidable subset of positive words in F, it is $\mathcal{E}^n(A)$ benign by Proposition 7.23 so $E_{\Psi^{-1}(Q)\cap P}$ is an $\mathcal{E}^n(A)$ benign subgroup of $F * \langle z; \ \rangle$. Then E_Q being $\mathcal{E}^n(A)$ decidable in $E * \langle z; \ \rangle$, it is $\mathcal{E}^n(A)$ decidable in D and hence $\mathcal{E}^n(A)$ benign by Lemma 7.9. Thus every positive, $\mathcal{E}^n(A)$ decidable subset of E is $\mathcal{E}^n(A)$ benign.

Now let Q be an arbitrary $\mathcal{E}^n(A)$ decidable subset of E and $E' = \langle a_1, \ldots, a_m, a'_1, \ldots, a'_m; \ \rangle$ with $W' \subset E'$ the set of positive words. Define an $\mathcal{E}^3$ homomorphism $\zeta : E' \to E$ by $a_i \mapsto a_i$ and $a'_i \mapsto a_i^{-1}$ for $i = 1, \ldots, m$. Then $Q' = \zeta^{-1}(Q) \cap W'$ is an $\mathcal{E}^n(A)$ decidable set of positive words so $\mathcal{E}^n(A)$ benign by the preceeding. Therefore since $Q = \zeta(Q')$ is $\mathcal{E}^n(A)$ decidable and ζ has an $\mathcal{E}^3$ associate, Q is $\mathcal{E}^n(A)$ benign by Lemma 7.9. $\square$

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Our main result is now completed.

THEOREM 7.25. *Every finitely generated* $E^n(A)$ *standard group for* $n \geq 4$ *and* A *recursively enumerable can be embedded in a finitely presented* $E^n(A)$ *standard group by an* $E^n(A)$ *embedding.*

Proof. We must show every f.g. $E^n(A)$ standard group is $E^n(A)$ Higman. By Lemma 6.13, it suffices to show every $E^n(A)$ decidable subgroup of a f.g. free group is $E^n(A)$ benign. By Lemmas 7.2 and 7.3 an $E^n(A)$ decidable subgroup of a f.g. free group is $E^n(A)$ benign iff it is $E^n(A)$ benign as a subset. Finally, by Lemma 7.24, every $E^n(A)$ decidable subset of a f.g. free group is $E^n(A)$ benign. $\square$

Together with the Higman, Neumann, Neumann Theorem, Theorem 5.5 we have the following result.

COROLLARY 7.26. *Every* $E^n(A)$ *group with* $n \geq 3$ *and* A *recursively enumerable can be embedded in a f.p.* $E^{n+1}(A)$ *standard group by an* $E^{n+1}(A)$ *embedding. Every* $E^n(A)$ *standard group with* $n \geq 4$ *and* A *recursively enumerable can be embedded in an* $E^n(A)$ *standard group by an* $E^n(A)$ *embedding.* $\square$

Together with Corollary 4.14 and Proposition 4.16, we have the following results.

COROLLARY 7.27. *For any recursively enumerable* $A \subset N$ *and any* $n \geq 4$ *there exists a f.p.* $E^n(A)$ *standard group* G *which is not* $E^{n-1}(A)$ *standard. If* G *is* $E^m(B)$ *standard for* $m \geq 3$ *, then* A *is* $E^m(B)$ *decidable.* $\square$

COROLLARY 7.28. *For any* $n \geq 4$ *there exists a f.p.* E^n *standard group which is not* E^{n-1} *standard. For* $n \geq 5$ *there exists a f.p.* E^n *standard group which is not* E^{n-2} *computable.* $\square$

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