Canad. Math. Bull. Vol. **54** (1), 2011 pp. 180–192
doi:10.4153/CMB-2010-088-8

Additive Families of Low Borel Classes and Borel Measurable Selectors

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Abstract. An important conjecture in the theory of Borel sets in non-separable metric spaces is whether any point-countable Borel-additive family in a complete metric space has a σ-discrete refinement. We confirm the conjecture for point-countable $\mathbf{\Pi}^0_3$-additive families, thus generalizing results of R. W. Hansell and the first author. We apply this result to the existence of Borel measurable selectors for multivalued mappings of low Borel complexity, thus answering in the affirmative a particular version of a question of J. Kaniewski and R. Pol.

1 Introduction

The theory of non-separable metric spaces, as developed by A. H. Stone, R. W. Hansell, and others (see, *e.g.*, [18] or [5]), very often relies upon the possibility of decomposing a given family of sets into countably many discrete pieces. If this is the case, standard methods of descriptive set theory of separable spaces can be applied to get results analogous to the separable ones. In particular, the existence of measurable selectors for multivalued mappings on non-separable metric spaces often depends on the existence of a σ-discrete decomposition of a certain kind (see [11, Section 3] or [9, Theorem 4.1]).

A classical result of Stone, stating that any open cover of a metric space has a σ-discrete locally finite open refinement (for the proof see [1, Theorem 4.4.1] and for the definitions of notions not explained here see the next section), is a basic example of a decomposition of a large family of sets into countably many discrete pieces.

The question of decomposability of a family of sets that are not necessarily open is much more difficult and, naturally, some additional assumptions on the given family must be imposed.

Hansell showed in [4, Theorem 2] that any Suslin-additive disjoint cover of an absolute Suslin metric space is σ-discretely decomposable. By an improvement due to J. Kaniewski and R. Pol (see [11, Theorem 1]), every point-finite Suslin-additive cover of an absolute Suslin space is σ-discretely decomposable.

If the assumption of point-finiteness is weakened to point-countability, a more appropriate notion of decomposability is the one of σ-discrete refinement (see [9, p. 366] for the reason). By [6, Theorem 3.1(b)], under suitable set-theoretical assumptions, there exists a point-countable Suslin-additive family in a Polish space

Received by the editors January 24, 2008.
Published electronically August 3, 2010.
The work is a part of the research project MSM 0021620839 financed by MSMT and partly supported by the grants GA ČR 201/06/0198, GA ČR 201/06/0018, and GA ČR 201/07/0388.
AMS subject classification: **54H05**, 54E35.
Keywords: σ-discrete refinement, Borel-additive family, measurable selection.

that is not σ-discretely refinable. On the other hand, Pol showed in [13, Theorem 1.3] that any point-countable Borel-additive family of sets in an arbitrary metrizable space admits a σ-discrete refinement provided each member of the family is of weight at most $\aleph_1$. W. G. Fleissner showed in [2] (see also [3, Theorem 3N]) that under an additional axiom of set theory every point-countable Suslin-additive family is σ-discretely refinable.

Nevertheless, the central problem we are trying to solve is still wide open (see survey papers [3, 8].)

Question 1.1 Is it provable in ZFC that every point-countable Borel-additive cover of a complete metric space has a σ-discrete refinement?

In metrizable spaces, we use $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ for the standard Borel classes of sets (see [12, Chapter II, Section 11.A]). In particular, $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$ stand for the families of all F_σ and all G_δ sets respectively.

Hansell answered Question 1.1 affirmatively if the family is $\mathbf{\Sigma}^0_2$-additive (see [9, Theorem 3.3]). He also showed in [9, Example] that a $\mathbf{\Sigma}^0_2$-additive cover of a complete space need not admit a σ-discrete refinement if the assumption of point-countability is omitted.

Using a method different from the one used in [9], the first author obtained in [16, Theorem 6] that a $\mathbf{\Pi}^0_2$-additive cover of a complete metric space has a σ-discrete refinement. The method of the proof was an application of a Hurewicz-like construction.

The aim of our paper is to show that this construction can be refined to yield the following result further supporting the belief of [9, p. 366] that the answer to Question 1.1 is affirmative.

Theorem 1.2 *Let $\mathcal{A}$ be an $\mathbf{\Pi}^0_3$-additive family in an absolute Suslin space. If*

(i) *$\mathcal{A}$ is point-countable, or*

(ii) *every partial selector set for $\mathcal{A}$ is σ-discrete,*

then $\mathcal{A}$ is σ-discretely refinable.

Assumption (ii) of Theorem 1.2 is rather natural, since this condition is satisfied when one looks for measurable sections (see the proof of Theorem 6.2 or of [15, Proposition 3.3, Theorem 4.3]).

Two important ingredients of our proof are the aforementioned [13, Theorem 1.3] on decomposability of families consisting of sets of low weight and Fremlin's theorem [3, Proposition 8A] on point-countable families of meager sets in complete metric spaces. We imitate Fremlin's proof in Theorem 3.3 to get an analogous result for families with σ-discrete partial selector sets.

Unfortunately, when trying to prove Theorem 1.2 for Borel sets of the second additive class, we met obstacles that we have not been able to surmount.

As mentioned in the abstract, Theorem 1.2 is the key tool for obtaining Borel measurable selectors for multivalued mappings of low Borel complexity. Thus we are able to answer at least a very particular case of the problem formulated in [11, Question 1] or in [3, 13C(f)]. The precise formulations of our results on Borel measurable selectors are contained in Section 6.

2 Preliminaries

By a *space* we always mean a nonempty metrizable topological space.

Let X be a space and $\mathcal{A}$ be a family of sets in X. We define properties mentioned in the introduction and used in the sequel.

The family $\mathcal{A}$ is said to be *discrete* if every point $x \in X$ has a neighbourhood meeting at most one member of $\mathcal{A}$. If $\mathcal{A} = \bigcup_{n\in\omega} \mathcal{A}_n$, such that each $\mathcal{A}_n$ is a discrete family, $\mathcal{A}$ is called σ*-discrete*. The family $\mathcal{A}$ is called σ*-discretely decomposable* if for each $A \in \mathcal{A}$, there exist sets $A(n)$, $n \in \omega$, such that $A = \bigcup_{n\in\omega} A(n)$ and $\{A(n)\ ;\ A \in \mathcal{A}\}$ is a discrete family for each $n \in \omega$. A family $\mathcal{R}$ is called a *refinement of* $\mathcal{A}$ if $\bigcup \mathcal{R} = \bigcup \mathcal{A}$ and for every $R \in \mathcal{R}$, there exists $A \in \mathcal{A}$ with $R \subset A$. A family $\mathcal{A}$ is said to admit a σ*-discrete refinement* if there exists a refinement of $\mathcal{A}$ that is σ-discrete (we also say that $\mathcal{A}$ is σ*-discretely refinable*). If $\mathcal{B}$ is a family of sets in a space X, the family $\mathcal{A}$ is called $\mathcal{B}$*-additive* if $\bigcup \mathcal{A}' \in \mathcal{B}$ for every $\mathcal{A}' \subset \mathcal{A}$. The family $\mathcal{A}$ is said to be *point-finite* (respectively *point-countable*) if the set $\{A \in \mathcal{A}\ ;\ x \in A\}$ is finite (respectively countable) for every $x \in X$.

Let $\mathcal{A}' \subset \mathcal{A}$. We say that a family $\mathcal{S} = (\{x_A\})_{A\in\mathcal{A}'}$ is a *partial selector* for the family $\mathcal{A}$ if $x_A \in A$ for every $A \in \mathcal{A}'$. The set $\{x_A\ ;\ A \in \mathcal{A}'\}$ is called the *set of the partial selector* $\mathcal{S}$ and such sets are said to be *partial selector sets* for $\mathcal{A}$.

For any $F \subset X$, we denote by $\mathcal{A}\restriction_F$ the family $\{A \cap F\ ;\ A \in \mathcal{A}\}$. The following auxiliary notion will be helpful later on. We say that $\mathcal{A}$ is *nowhere* σ*-discretely refinable* if $\bigcup \mathcal{A} \neq \varnothing$ and $\mathcal{A}\restriction_U$ is not σ-discretely refinable for any open $U \subset X$ intersecting $\bigcup \mathcal{A}$.

A space X is *absolute Suslin* if X is homeomorphic to a Suslin subset of a complete metric space. It follows from [10, Theorem 1.1] and [5, Theorem 4.1] that X is an absolute Suslin space if and only if there exists a complete metric space Y and a continuous mapping f of Y onto X such that f preserves σ-discretely decomposable families.

We recall that $A \subset X$ has the *Baire property* in X if $A = B \cup N$, where B is $\mathbf{\Pi}^0_2$ in X and N is meager in X. As it is well known, any Suslin set has the Baire property (see, *e.g.,* [14, Corollary 2.9.4]).

We denote by $2^{<\omega}$ the space of finite sequences of digits 0 and 1. Let $|s|$ be the length of s. We denote by $\varnothing$ the empty sequence and adopt the convention that the length of the empty sequence is 0. For $s \in 2^{<\omega}$ and $i \in \{0, 1\}$, we write $s^\wedge i$ for the sequence $(s_0, \dots, s_{|s|-1}, i)$. For a sequence σ in the Cantor set 2^ω and $n \in \omega$, we write $\sigma\restriction_n$ for the finite sequence $(\sigma_0, \dots, \sigma_{n-1})$. We adopt the convention that $\sigma\restriction_0 = \varnothing$.

3 Covers Consisting of Meager Sets

The aim of this section is to prove Theorem 3.3. We start with a simple lemma.

Lemma 3.1 *Let $\mathcal{A}$ be a cover of a separable space X such that every partial selector set of $\mathcal{A}$ is σ-discrete. Then there exists a countable subfamily $\mathcal{A}_0 \subset \mathcal{A}$ that covers X.*

Proof Assuming the contrary, we can find sets $A_\alpha \in \mathcal{A}$, $\alpha \in [0, \omega_1)$, such that $A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta \neq \varnothing$. For each $\alpha < \omega_1$, we select a point $x_\alpha \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$. Then $S = \{x_\alpha\ ;\ \alpha \in [0, \omega_1)\}$ is an uncountable partial selector set for $\mathcal{A}$. By the assumption, S is σ-discrete, which is impossible by virtue of separability of X. This concludes the proof. ■

Lemma 3.2 *Let X be a space of weight not exceeding $\aleph_1$. Let $\mathcal{A}$ be a cover of X such that every partial selector set of $\mathcal{A}$ is σ-discrete. Then $\mathcal{A}$ has a σ-discrete refinement.*

Proof Let $\mathcal{A}$ be a cover of X as in the premise. Let $\mathcal{A} = \{A_\alpha ; \alpha \in [0, \kappa)\}$ be an enumeration of the family $\mathcal{A}$. We set

$$B_0 = A_0 \quad \text{and} \quad B_\alpha = A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta, \quad \alpha \in (0, \kappa).$$

Then $\mathcal{B} = \{B_\alpha ; \alpha \in [0, \kappa)\}$ is a disjoint cover of X such that its each partial selector set is σ-discrete. By the weight restriction on X, we may use [13, Theorem 1.1] to deduce that $\mathcal{B}$ has a σ-discrete refinement $\mathcal{C}$. Then $\mathcal{C}$ is a refinement of $\mathcal{A}$, and we are done. ■

The proof of the following theorem imitates a method used by Fremlin in [3, Proposition 8A].

Theorem 3.3 *Let $\mathcal{A}$ be a cover of a completely metrizable space such that every partial selector set of $\mathcal{A}$ is σ-discrete. Then $\mathcal{A}$ contains a non-meager set.*

Proof We assume that $\mathcal{A}$ is as in the hypothesis and does not contain a non-meager set. For every $A \in \mathcal{A}$, we select closed nowhere dense sets $F(A, n)$, $n \in \omega$, such that $A \subset \bigcup_{n\in\omega} F(A, n)$. For every point $x \in X$, $A \in \mathcal{A}$, and $n \in \omega$, we find a sequence $(y(x, A, n, k))_{k\in\omega}$ converging to x such that its elements are not in $F(A, n)$.

We construct inductively, for $\alpha \in [0, \omega_1)$, countable sets $X_\alpha \subset X$ and countable families $\mathcal{A}_\alpha \subset \mathcal{A}$ as follows. We set $X_0 = \{z\}$, where z is a point of X, and $\mathcal{A}_0 = \varnothing$.

Let $\alpha \in [1, \omega_1)$ and assume that the objects have been constructed for all ordinals smaller than α.

If α is a successor ordinal, say $\alpha = \beta + 1$, then we set

$$X_\alpha = X_\beta \cup \{y(x, A, n, k) ; x \in X_\beta, A \in \mathcal{A}_\beta, k, n \in \omega\}.$$

Further, the family $\{\overline{X}_\beta \cap A ; A \in \mathcal{A}\}$ is a cover of the separable space $\overline{X}_\beta$ that satisfies the assumption of Lemma 3.1. Thus, we can select a countable subfamily $\mathcal{A}'_\beta \subset \mathcal{A}$ such that $\overline{X}_\beta \subset \bigcup \mathcal{A}'_\beta$. Let $\mathcal{A}_\alpha = \mathcal{A}_\beta \cup \mathcal{A}'_\beta$. This finishes the inductive step of the construction for a nonlimit ordinal.

If $\alpha < \omega_1$ is a limit ordinal, set

$$X_\alpha = \bigcup_{\eta<\alpha} X_\eta \quad \text{and} \quad \mathcal{A}_\alpha = \bigcup_{\eta<\alpha} \mathcal{A}_\eta.$$

This completes the construction.

We define

$$Y = \overline{\bigcup_{\alpha<\omega_1} X_\alpha} \quad \text{and} \quad \mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha.$$

https://doi.org/10.4153/CMB-2010-088-8 Published online by Cambridge University Press

Then
$$Y = \bigcup_{\alpha<\omega_1} \overline{X}_\alpha,$$
since
$$\bigcup_{\alpha<\omega_1} X_\alpha \subset \bigcup_{\alpha<\omega_1} \overline{X}_\alpha$$
and the latter set is an increasing union of ω_1 many closed sets, hence closed as well.

Further, Y is a complete metric space of weight not exceeding $\aleph_1$, and $\mathcal{B}\restriction_Y$ is its cover such that each partial selector set is σ-discrete.

Moreover, $\mathcal{B}\restriction_Y$ consists of sets meager in Y. Indeed, it is enough to show that $F(A,n) \cap Y$ is nowhere dense in Y for every $A \in \mathcal{B}$ and $n \in \omega$. Let $A \in \mathcal{B}$, $n \in \omega$, and $x \in F(A,n) \cap Y$ be given. We find $\alpha < \omega_1$ such that $x \in \overline{X}_\alpha$ and $A \in \mathcal{A}_\alpha$. We can choose a sequence (x_j) of elements of X_α so that $x_j \to x$.

Then
$$\{y(x_j, A, n, k) \,;\, j, k \in \omega\} \subset Y \setminus F(A,n) \quad \text{and} \quad x \in \overline{\{y(x_j, A, n, k) \,;\, j, k \in \omega\}}.$$
Hence x cannot be in the interior (relative to Y) of $F(A,n) \cap Y$.

According to Lemma 3.2, $\mathcal{B}\restriction_Y$ admits a σ-discrete refinement $\mathcal{R}$. Then $\bigcup \mathcal{R}$, as a union of a σ-discrete family of meager sets in Y, is meager in Y (see [3, 7B(e)]). On the other hand, $Y = \bigcup \mathcal{R}$ is not meager in Y. This contradiction finishes the proof. ■

4 Auxiliary Lemmas

Lemma 4.1 *Let $\mathcal{A}$ be a cover of a completely metrizable space X such that*

(i) *$\mathcal{A}$ is a Borel-additive family and point-countable, or*

(ii) *$\mathcal{A}$ consists of sets with the Baire property and every partial selector set for $\mathcal{A}$ is σ-discrete.*

Then there exists $A \in \mathcal{A}$ that is comeager in some nonempty open set $V \subset X$.

Proof Assume first that $\mathcal{A}$ satisfies the assumptions of (i). By virtue of [3, Proposition 8A], there exists a set $A \in \mathcal{A}$ that is not meager. Since A, as a Borel subset of X, has the Baire property, A is comeager in some nonempty open set V.

We proceed analogously in (ii) except that we use Theorem 3.3 and finish the proof as above. ■

The following easy assertion can be found, *e.g.*, in [16, Lemma 2].

Lemma 4.2 *Let $\mathcal{A}$ be a family of subsets of a metric space X and $C \subset X$. If $\mathcal{A}\restriction_C$ is not σ-discretely refinable, then there exists a set $\tilde{C} \subset C$ closed in C such that $\mathcal{A}\restriction_{\tilde{C}}$ is nowhere σ-discretely refinable.*

Proof Setting
$$G := \bigcup\{U \subset C \,;\, U \text{ is open in } C, \mathcal{A}\restriction_U \text{ has a } \sigma\text{-discrete refinement}\}$$
and $\tilde{C} := C \setminus G$, one can easily check that $\tilde{C}$ is the desired set. ■

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Lemma 4.3 *Let $\mathcal{A}$ be a $\mathbf{\Pi}^0_3$-additive cover of a completely metrizable space X such that*

(i) *$\mathcal{A}$ is point-countable, or*
(ii) *every partial selector set for $\mathcal{A}$ is σ-discrete.*

Let $H \subset X$ be a $\mathbf{\Pi}^0_2$ set such that $\mathcal{A}\restriction_H$ is not σ-discretely refinable and let $\mathcal{A}_0 \subset \mathcal{A}$ be finite. Then there exist a nonempty $\mathbf{\Pi}^0_2$ set $G \subset H \setminus \bigcup \mathcal{A}_0$ and $A \in \mathcal{A}$ such that $\mathcal{A}\restriction_G$ is nowhere σ-discretely refinable and $A \cap G$ is comeager in G.

Proof Let H be as in the premise. As $\mathcal{A}$ is $\mathbf{\Pi}^0_3$-additive, we can write

$$H \setminus \bigcup \mathcal{A}_0 = \bigcup_{n \in \omega} G_n,$$

where $G_n \subset H$, $n \in \omega$, are $\mathbf{\Pi}^0_2$ subsets of X. Since $\mathcal{A}\restriction_H$ is not σ-discretely refinable, we can find $j \in \omega$ such that $\mathcal{A}\restriction_{G_j}$ is not σ-discretely refinable.

We use Lemma 4.2 to find a relatively closed set $D \subset G_j$ (hence D is $\mathbf{\Pi}^0_2$ in X) such that $\mathcal{A}\restriction_D$ is nowhere σ-discretely refinable. We use Lemma 4.1 on a completely metrizable space D and on the family $\mathcal{A}\restriction_D$ to find an open set $V \subset X$ intersecting D and $A \in \mathcal{A}$ such that $A \cap V \cap D$ is comeager in $V \cap D$. By setting $G := V \cap D$, we finish the proof. ■

Notation 4.4 Let $n, k \in \omega$, $n \geq k$. Then $\langle n, k\rangle$ stands for the natural number $\frac{1}{2}n(n+1)+k+1$. Note that for given $p \in \omega$, the numbers $n, k \in \omega$ with $\langle n, k\rangle - 1 = p$ are uniquely determined.

Lemma 4.5 *The set*

$$\mathbf{P} := \{\nu \in 2^\omega \ ; \exists k \in \omega : \ (\nu_{\langle n,k\rangle})_{n \geq k} \textit{ is not eventually zero}\}$$

is a true $\mathbf{\Sigma}^0_3$ set.

Proof Denote $\mathbf{Q} = \{\alpha \in 2^\omega \ ; (\alpha_n) \text{ is eventually } 0\}$. It is well known that $\mathbf{Q}^\omega$ is a true $\mathbf{\Pi}^0_3$ subset of $(2^\omega)^\omega$ (see [12, Exercise 23.1]). Define the mapping $\varphi\colon 2^\omega \to (2^\omega)^\omega$ by

$$\varphi(\nu)(k) = (\nu_{\langle n+k,k\rangle -1})_{n \geq 0}.$$

Using the fact that each $l \in \omega$ can be uniquely expressed as $\langle n+k, k\rangle - 1$, $n, k \in \omega$, it is routine to verify that φ is a homeomorphism of 2^ω and $(2^\omega)^\omega$. Now it is easy to check that $\varphi(2^\omega \setminus \mathbf{P}) = \mathbf{Q}^\omega$, and we are done. ■

5 Proof of Theorem 1.2

Lemma 5.1 *Let $\mathcal{A}$ be a $\mathbf{\Pi}^0_3$-additive cover of a complete metric space X such that*

(i) *$\mathcal{A}$ is point-countable, or*
(ii) *every partial selector set for $\mathcal{A}$ is σ-discrete.*

Then the family $\mathcal{A}$ is σ-discretely refinable.

Proof Let ρ be the metric on X. On the contrary, assume that $\mathcal{A}$ is not σ-discretely refinable. For $s \in 2^{<\omega}$, we will construct

- an open ball $B(s) \subset X$,
- $A(s) \in \mathcal{A}$,
- $\mathbf{\Sigma}_2^0$ sets $F^j(s) \subset X$, $j \in \omega$, such that $(F^j(s))_{j\in\omega}$ is non-increasing and $A(s) = \bigcap_{j\in\omega} F^j(s)$,
- $\mathbf{\Pi}_2^0$ set $G(s) \subset X$,
- a complete metric $\rho(s)$ on $G(s)$ that is equivalent on $G(s)$ to the original metric ρ.

For $s \in 2^{<\omega}$, we require:

(i) $\overline{B(s^\wedge 0)} \cup \overline{B(s^\wedge 1)} \subset B(s)$, $B(s^\wedge 0) \cap B(s^\wedge 1) = \varnothing$,
(ii) $B(s) \cap G(s) \neq \varnothing$,
(iii) $\operatorname{diam}_{\rho(s)}\big(\overline{B(s)} \cap G(s)\big) < 2^{-|s|}$, $\operatorname{diam}_{\rho}\big(\overline{B(s)}\big) < 2^{-|s|}$,
(iv) $A(s) \cap G(s)$ is comeager in $G(s)$,
(v) $G(s) \cap \bigcup\{A(t) \; ; \; t \in 2^{<\langle k,k\rangle}\} = \varnothing$ whenever $|s| = \langle n,k\rangle$ for some $n,k \in \omega$, $k \leq n$,
(vi) $G(s^\wedge\iota) \subset G(s)$ whenever $|s| = \langle n,k\rangle$ with $k < n$ and $\iota \in \{0,1\}$,
(vii) if $|s| = \langle n,k\rangle$, $k < n$, and $s_{\langle n,l\rangle} = 0$ for all $l < k$, then

$$A(s) = A(s\restriction_{\langle n-1,k\rangle}), \quad F^j(s) = F^j(s\restriction_{\langle n-1,k\rangle}),\ j \in \omega,$$
$$G(s) = G(s\restriction_{\langle n-1,k\rangle}), \quad \rho(s) = \rho(s\restriction_{\langle n-1,k\rangle}),$$

(viii) $G(s^\wedge 1) \cap B(s^\wedge 1) \subset F^{|s|+1}(s^\wedge 1)$ whenever $|s| = \langle n,k\rangle$ with $k < n$,
(ix) $\mathcal{A}\restriction_{G(s)}$ is nowhere σ-discretely refinable,
(x) if $|s| = \langle n,k\rangle$, $k < n$, and $s_{\langle n,l\rangle} = 0$ for all $l \leq k$, then

$$B(s) \cap G(s\restriction_{\langle n-1,n-1\rangle}) \neq \varnothing.$$

We point out that condition (vii) also applies for $s \in 2^{<\omega}$ with $|s| = \langle n,0\rangle$, $n \geq 1$.

Using Lemma 4.3 for $H = X$ and $\mathcal{A}_0 = \varnothing$, we get a nonempty $\mathbf{\Pi}_2^0$ set $G(\varnothing) \subset X$ and $A(\varnothing) \in \mathcal{A}$ such that $\mathcal{A}\restriction_{G(\varnothing)}$ is nowhere σ-discretely refinable and $A(\varnothing) \cap G(\varnothing)$ is comeager in $G(\varnothing)$. Since $A(\varnothing) \in \mathcal{A}$, there exists a nonincreasing sequence $(F^j(\varnothing))_{j\in\omega}$ of $\mathbf{\Sigma}_2^0$ sets with $A(\varnothing) = \bigcap_{j\in\omega} F^j(\varnothing)$. We choose a complete metric $\rho(\varnothing)$ on $G(\varnothing)$ that is equivalent on $G(\varnothing)$ to ρ. Let $B(\varnothing)$ be an arbitrary open ball of X centered at some point of $G(\varnothing)$ such that

$$\operatorname{diam}_{\rho(\varnothing)}\big(\overline{B(\varnothing)} \cap G(\varnothing)\big) < 1 \quad \text{and} \quad \operatorname{diam}_{\rho}\big(\overline{B(\varnothing)}\big) < 1.$$

This finishes the first step of our construction, since properties (i)–(x) are vacuous or clearly satisfied.

Let $s \in 2^{<\omega}$. Suppose that our construction has been done for all $t \in 2^{<\omega}$ with $|t| \leq |s|$. Let n,k be uniquely determined natural numbers with $|s| + 1 = \langle n,k\rangle$. To define our auxiliary objects for $s^\wedge 0$ and $s^\wedge 1$, we distinguish the following cases.

Case 1. Suppose that $k = n$ or there is $l < k$ such that $s_{\langle n,l\rangle} = 1$.

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By (ii) and (ix), we have that $\mathcal{A}\restriction_{B(s)\cap G(s)}$ is not σ-discretely refinable. Applying Lemma 4.3 for

$$H := G(s) \cap B(s), \qquad \mathcal{A}_0 := \{A(t)\,;\, t \in 2^{\leq|s|}\},$$

we find a nonempty $\mathbf{\Pi}^0_2$ set $G \subset H \setminus \bigcup \mathcal{A}_0$ and $A \in \mathcal{A}$ such that $\mathcal{A}\restriction_G$ is nowhere σ-discretely refinable and $A \cap G$ is comeager in G. Since $A \in \mathcal{A}$, one can find a nonincreasing sequence $(F^j)_{j\in\omega}$ of $\mathbf{\Sigma}^0_2$ sets such that $A = \bigcap_{j\in\omega} F^j$. By taking $F^0 = \cdots = F^{|s|+1} = X$, we may achieve that $G \subset F^{|s|+1}$. Further, we find a complete metric τ on G that is equivalent on G to ρ. For $\iota \in \{0,1\}$, we set

$$A(s^\wedge\iota) := A, \quad F^j(s^\wedge\iota) := F^j,\ j \in \omega, \quad G(s^\wedge\iota) := G, \quad \rho(s^\wedge\iota) := \tau.$$

Finally, it is easy to find $B(s^\wedge 0)$ and $B(s^\wedge 1)$ such that conditions (i)–(iii) are satisfied. By the choice of $F^{|s|+1}$, (viii) is satisfied. Conditions (iv)–(vi) and (ix) are easy to check, and conditions (vii) and (x) are vacuous.

Case 2. Suppose that $k < n$ and $s_{\langle n,l\rangle} = 0$ for all $l < k$.

Denote $v = s\restriction_{\langle n-1,k\rangle}$. We define the desired objects for $s^\wedge\iota$, $\iota \in \{0,1\}$, by

$$A(s^\wedge\iota) := A(v), \quad F^j(s^\wedge\iota) := F^j(v),\ j \in \omega, \quad G(s^\wedge\iota) := G(v), \quad \rho(s^\wedge\iota) := \rho(v).$$

We immediately get that conditions (iv), (v), and (ix), where we replace s by $s^\wedge 0$ and $s^\wedge 1$, are satisfied.

Claim 5.2 *We have $B(s) \cap G(s\restriction_{\langle n-1,n-1\rangle}) \neq \varnothing$ and $G(s\restriction_{\langle n-1,n-1\rangle}) \subset G(s)$.*

Proof If $k = 0$, then $|s| = \langle n-1, n-1\rangle$, and thus $s\restriction_{\langle n-1,n-1\rangle} = s$. Hence

$$B(s) \cap G(s\restriction_{\langle n-1,n-1\rangle}) = B(s) \cap G(s) \neq \varnothing$$

by (ii). The inclusion $G(s\restriction_{\langle n-1,n-1\rangle}) \subset G(s)$ is obvious.

If $k > 0$, then $|s| = \langle n, k-1\rangle$, and by (x), we get $B(s) \cap G(s\restriction_{\langle n-1,n-1\rangle}) \neq \varnothing$ again. Using (vi) and (vii) we get

$$G(s\restriction_{\langle n-1,n-1\rangle}) \subset G(s\restriction_{\langle n-1,k-1\rangle}) = G(s\restriction_{\langle n,k-1\rangle}) = G(s),$$

concluding the proof. ∎

The set $A(s^\wedge 1) \cap G(s^\wedge 1)$ is comeager in $G(s^\wedge 1)$ because we have already verified (iv) for $s^\wedge 1$. The set $F^{|s|+1}(s^\wedge 1)$ is $\mathbf{\Sigma}^0_2$ and contains $A(s^\wedge 1)$ by definition. Since $G(s^\wedge 1) = G(v) \supset G(s\restriction_{\langle n-1,n-1\rangle})$ (by (vi)), Claim 5.2 implies $B(s) \cap G(s^\wedge 1) \neq \varnothing$. By the Baire category argument, there is an open set $V \subset B(s)$ such that $\varnothing \neq V \cap G(s^\wedge 1) \subset F^{|s|+1}(s^\wedge 1)$.

Claim 5.2 gives $B(s) \cap G(s\restriction_{\langle n-1,n-1\rangle}) \neq \varnothing$. According to condition (ix), the restriction of $\mathcal{A}$ to both sets $B(s) \cap G(s\restriction_{\langle n-1,n-1\rangle})$ and $V \cap G(s^\wedge 1)$ is nowhere σ-discretely refinable. Thus we can choose different centers of $B(s^\wedge 0)$ and $B(s^\wedge 1)$ in $B(s) \cap$

$G(s\restriction_{\langle n-1,n-1\rangle})$ and $V \cap G(s^\wedge 1)$ respectively. Radii of the balls are chosen sufficiently small that $B(s^\wedge 1) \subset V$ and conditions (i) and (iii) are satisfied.

Validity of conditions (vii), (viii), and (x) for $s^\wedge 0$ and $s^\wedge 1$ follows immediately from the construction. It remains to verify (ii) and (vi). Since $G(s^\wedge 0) = G(v) \supset G(s\restriction_{\langle n-1,n-1\rangle})$, we have $B(s^\wedge 0) \cap G(s^\wedge 0) \neq \varnothing$. Clearly, we have $G(s^\wedge 1) \cap B(s^\wedge 1) \neq \varnothing$. Thus we have verified (ii).

As for (vi), if $k = 0$, then there is nothing to prove. If $k > 0$, we use inductive assumption (vii) for $\langle n-1, k-1\rangle$ to get

$$G(s^\wedge \iota) = G(v) \subset G(s\restriction_{\langle n-1,k-1\rangle}) = G(s).$$

This finishes the construction of the auxiliary objects.

Now we define a continuous mapping $\varphi\colon 2^\omega \to X$ by

$$\varphi(\nu) = \bigcap_{p\in\omega} B(\nu\restriction_p).$$

This is a standard construction leading to a continuous mapping according to (i) and (iii). Following notation from Lemma 4.5, we set

$$\mathbf{P} = \{\nu \in 2^\omega \,;\, \exists k \in \omega : \ (\nu_{\langle n,k\rangle})_{n\geq k} \text{ is not eventually zero}\}.$$

Claim 5.3 $\mathbf{P} = \varphi^{-1}(\bigcup\{A(s) \,;\, s \in 2^{<\omega}\})$.

Granting this claim we get a contradiction. Indeed, the set $\bigcup\{A(s) \,;\, s \in 2^{<\omega}\}$ is $\mathbf{\Pi}^0_3$ by $\mathbf{\Pi}^0_3$-additivity of $\mathcal{A}$, hence $\varphi^{-1}(\bigcup\{A(s) \,;\, s \in 2^{<\omega}\})$ is $\mathbf{\Pi}^0_3$, but on the other hand, $\mathbf{P}$ is a true $\mathbf{\Sigma}^0_3$ set by Lemma 4.5, a contradiction.

To prove Claim 5.3, choose $\nu \in \mathbf{P}$. Find the smallest $k \in \omega$ such that $(\nu_{\langle n,k\rangle})_{n\geq k}$ is not eventually zero. Then there exists $q \in \omega$ such that $\nu_{\langle n,j\rangle} = 0$ for all $n \geq q$ and $j < k$. By (vii), we have that there are $A \in \mathcal{A}$, a nonincreasing sequence $(F^j)_{j\in\omega}$, a $\mathbf{\Pi}^0_2$ set G, and a metric d on G such that

$$A(\nu\restriction_{\langle n,k\rangle}) = A, \quad F^j(\nu\restriction_{\langle n,k\rangle}) = F^j, \quad G(\nu\restriction_{\langle n,k\rangle}) = G, \quad \rho(\nu\restriction_{\langle n,k\rangle}) = d$$

for all $n \geq q$. The sequence $\left(\overline{B(\nu\restriction_{\langle n,k\rangle})} \cap G\right)_{n\geq q}$ is a nonincreasing sequence of nonempty closed sets in G such that

$$\operatorname{diam}_d\left(\overline{B(\nu\restriction_{\langle n,k\rangle})} \cap G\right) \to 0$$

for $n \to \infty$. This and the completeness of d give $\varphi(\nu) \in G$. Since the sequence $(\nu_{\langle n,k\rangle})_{n\geq q}$ contains 1 infinitely many times, we have

$$\varphi(\nu) \in G \cap B(\nu\restriction_{\langle n,k\rangle}) \subset F^{\langle n,k\rangle}$$

for infinitely many n by (viii). This implies $\varphi(\nu) \in A \subset \bigcup\{A(s) \,;\, s \in 2^{<\omega}\}$.

Now assume that $\nu \in 2^\omega \setminus \mathbf{P}$. Take an arbitrary $t \in 2^{<\omega} \setminus \{\varnothing\}$. Find $k, n \in \omega$, $k \leq n$, with $|t| = \langle n, k\rangle$. Again there is $q \in \omega$, $q > n$, such that $\nu\restriction_{\langle m,j\rangle} = 0$ for all

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$m \geq q$ and $j \leq n+1$. As in the previous case we get $\varphi(\nu) \in G(\nu\restriction_{\langle q,n+1\rangle})$. By (v) we have

$$(\star) \qquad G(\nu\restriction_{\langle q,n+1\rangle}) \cap \{A(u)\ ;\ u \in 2^{<\langle n+1,n+1\rangle}\} = \varnothing.$$

Consequently, $\varphi(\nu) \notin A(t)$ since $\langle n,k\rangle < \langle n+1,n+1\rangle$.

From $(\star)$, it also follows that $\varphi(\nu) \notin A(\varnothing)$. Hence we have proved that

$$\varphi(\nu) \notin \bigcup\{A(s)\ ;\ s \in 2^{<\omega}\}.$$

Thus we get the assertion of Claim 5.3. ■

Proof of Theorem 1.2. Let $\mathcal{A}$ be a $\mathbf{\Pi}^0_3$-additive family in an absolute Suslin metrizable space X. By replacing X with $\bigcup\mathcal{A}$ if necessary, we may assume that $\mathcal{A}$ is a cover of X. According to results mentioned in Section 2, there exists a continuous mapping $f\colon Y \to X$ of a complete metric space Y onto X such that f preserves σ-discretely decomposable families. Then $\mathcal{B} = \{f^{-1}(A)\ ;\ A \in \mathcal{A}\}$ is a $\mathbf{\Pi}^0_3$-additive cover of Y that inherits property (i) or (ii) of Theorem 1.2 from $\mathcal{A}$.

Indeed, point-countability is obviously transferred from $\mathcal{A}$ to $\mathcal{B}$. Concerning the second property, let B be a partial selector set for $\mathcal{B}$. Then $f(B)$ is a partial selector set for $\mathcal{A}$, and thus is σ-discrete. Since $B \subset f^{-1}(f(B))$, it easily follows that B is σ-discrete as well.

According to Lemma 5.1, $\mathcal{B}$ has a σ-discrete refinement. Since f preserves σ-discretely decomposable families, $\mathcal{A}$ is σ-discretely refinable as well. This concludes the proof. ■

6 Applications

As already mentioned, existence results for a Borel measurable selector from a set-valued mapping $\varphi\colon X \to Y$ are standard to obtain, once we know that the family $\{\varphi^{-1}(U)\ ;\ U \in \mathcal{U}\}$ is σ-discretely refinable for each discrete family $\mathcal{U}$ of open or closed sets in Y (we recall that $\varphi^{-1}(U) = \{x \in X\ ;\ \varphi(x) \cap U \neq \varnothing\}$). Using classical methods contained, *e.g.,* in [7, Lemma 5.3, Lemma 5.4, and Theorem 5.5], [11, Section 3] or in [15, Section 4] and [16, Theorem 7], we get the following results on Borel measurable selectors.

Theorem 6.1 *Let X be an absolute Suslin space and Y be complete. Let $\varphi\colon X \to Y$ be a set-valued mapping with nonempty closed values such that*

(a1) *$\varphi^{-1}(U)$ is a $\mathbf{\Pi}^0_3$-set in X for each $U \subset Y$ open, or*
(a2) *$\varphi^{-1}(F)$ is a $\mathbf{\Pi}^0_3$-set in X for each $F \subset Y$ closed,*

and

(b1) *$\varphi(x)$ is separable for each $x \in X$.*

Then φ admits a $\mathbf{\Sigma}^0_4$-measurable selector.

Outline of the proof We just mention that, given $\varphi\colon X \to Y$ satisfying (a1) and $\mathcal{U}$ a discrete family of open sets in Y, the family

$$\mathcal{A} = \{\varphi^{-1}(U)\ ;\ U \in \mathcal{U}\}$$

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is $\mathbf{\Pi}_3^0$-additive. Since φ satisfies (b1), $\mathcal{A}$ is point-countable and thus admits a σ-discrete refinement according to Theorem 1.2. If we assume (a2), we use σ-discrete covers of Y that consist of closed sets instead of open sets. In both cases, we may use the standard techniques mentioned above to conclude the proof. ∎

Theorem 6.2 *Let X be an absolute Suslin space and Y be complete. Let $\varphi\colon X \to Y$ be a set-valued mapping with nonempty closed values such that*

(a2) *$\varphi^{-1}(F)$ is a $\mathbf{\Pi}_3^0$-set in X for each $F \subset Y$ closed, and*
(b2) *$\varphi(x) \cap \varphi(x') = \varnothing$ for every couple $x, x' \in X$ of distinct points.*

Then φ admits a $\mathbf{\Sigma}_4^0$-measurable selector.

Outline of the proof Let φ satisfy (a2) along with (b2). We claim that we may use Theorem 1.2 again.

Indeed, let $\mathcal{F}$ be a discrete family of closed sets in Y and $\mathcal{F}' \subset \mathcal{F}$ be given. Let $\mathcal{S} = (x_F)_{F\in\mathcal{F}'}$ be a partial selector for $\varphi^{-1}(\mathcal{F})$ and $S = \{x_F\,;\, F \in \mathcal{F}'\}$.

Without loss of generality, we may assume that the points of $\mathcal{S}$ are pairwise distinct. For every $F \in \mathcal{F}'$, we choose a point $y_F \in F \cap \varphi(x_F)$. Then $\{\{y_F\}\,;\, F \in \mathcal{F}'\}$ is a discrete family of singletons and, by (b2),

$$\{\{x_F\}\,;\, F \in \mathcal{F}'\} = \{\varphi^{-1}(\{y_F\})\,;\, F \in \mathcal{F}'\}.$$

Hence $\{\{x_F\}\,;\, F \in \mathcal{F}'\}$ is a Borel-additive disjoint family of singletons. According to [4, Theorem 2], it is a σ-discretely decomposable family. Hence S is a σ-discrete set.

Using Theorem 1.2, we now finish the proof as above. ∎

An immediate application of the previous theorem is the following result.

Theorem 6.3 *Let $f\colon Y \to X$ be a mapping from a complete space Y to an absolute Suslin space X such that $f^{-1}(x)$ is closed in Y for each $x \in X$, and*

(a3) *f maps closed sets in Y to $\mathbf{\Pi}_3^0$ sets in X.*

Then f admits a $\mathbf{\Sigma}_4^0$-measurable section (i.e., *the mapping $x \mapsto f^{-1}(x)$, $x \in X$, admits a $\mathbf{\Sigma}_4^0$-measurable selector).*

It might be interesting to remark that we cannot replace closed sets in Theorem 6.3 (a3) by open sets.

Example 6.4 There exists a mapping $f\colon X \to Y$ of a complete metric space X onto a complete metric space Y mapping open sets in X to $\mathbf{\Sigma}_2^0$ sets in Y and a discrete family $\mathcal{U}$ of open sets in X such that $f(\mathcal{U})$ has no σ-discrete refinement, and, consequently, f has no Borel measurable section.

Proof Let $\Omega = \{\alpha\,;\, \alpha < \omega_1\}$ be the discrete space of all countable ordinals. Let Y be the Baire space Ω^ω with the product topology and

$$Y_\alpha = \{(y_n) \in Y\,;\, y_n \le \alpha \text{ for all } n \in \omega\}\,, \quad \alpha < \omega_1.$$

Then $\{Y_\alpha\,;\, \alpha < \omega_1\}$ is an increasing $\mathbf{\Sigma}_2^0$-additive cover of Y by closed separable subsets.

Fix a countable ordinal $\alpha < \omega_1$. Let $\varphi_1 \colon \mathbb{R} \to \mathbb{R}$ be such that $\varphi_1(I) = \mathbb{R}$ for every nonempty open interval $I \subset \mathbb{R}$ and $\varphi_2 \colon \mathbb{R} \to [0, \alpha]$ be a surjective mapping. Then $\varphi = \varphi_2 \circ \varphi_1$ maps any nonempty open set in $\mathbb{R}$ onto $[0, \alpha]$. Set $X_\alpha = \mathbb{R}^\omega$ and define $f_\alpha \colon X_\alpha \to Y$ as $f_\alpha\big((x_n)\big) = \big(\varphi(x_n)\big)$, $(x_n) \in X_\alpha$. Then f_α maps any nonempty open set in X_α onto Y_α.

Let X be the discrete union of spaces X_α, $\alpha < \omega_1$, and $f \colon X \to Y$ be defined as $f(x) = f_\alpha(x)$ for $x \in X_\alpha$. Let $U \subset X$ be a nonempty open set. Let β be the least countable ordinal satisfying $U \cap X_\beta = \varnothing$. If there is no such β, set $\beta = \omega_1$. Then $f(U) = \bigcup\{Y_\alpha \,;\, \alpha < \beta\}$, which is a $\mathbf{\Sigma}^0_2$ set in Y. Thus f maps open sets in X to $\mathbf{\Sigma}^0_2$ sets in Y.

If we put $\mathcal{U} = \{X_\alpha \,;\, \alpha < \omega_1\}$, then $f(\mathcal{U}) = \{Y_\alpha \,;\, \alpha < \omega_1\}$ does not have a σ-discrete refinement. Indeed, if $\mathcal{R}$ were a refinement of $\{Y_\alpha \,;\, \alpha < \omega_1\}$, the family $\mathcal{R}$ would consist of separable sets. Hence Y would be a union of a σ-discrete family of separable sets and thus also σ-locally of weight less than $\aleph_1$. But this contradicts a theorem of A. H. Stone [17, 2.1(7)].

To conclude the proof, we realize that f cannot have a Borel measurable section. Indeed, assume that $g \colon Y \to X$ is a Borel measurable selector from the mapping $y \mapsto f^{-1}(y)$, $y \in Y$. Then $g^{-1}(\mathcal{U})$ is a disjoint Borel-additive cover of Y, and thus σ-discretely decomposable by [4, Theorem 2]. Since $g^{-1}(\mathcal{U})$ is a refinement of $f(\mathcal{U})$, $f(\mathcal{U})$ is σ-discretely refinable.

But this contradicts the first part of the reasoning. ■

References

[1] R. Engelking, *General topology.* Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[2] W. G. Fleissner, *An axiom for nonseparable Borel theory.* Trans. Amer. Math. Soc. **251**(1979), 309–328. doi:10.2307/1998696

[3] D. H. Fremlin, *Measure-additive coverings and measurable selectors.* Dissertationes Math. **260**(1987), 116 pp.

[4] R. W. Hansell, *Borel measurable mappings for nonseparable metric spaces.* Trans. Amer. Math. Soc. **161**(1971), 145–169. doi:10.2307/1995934

[5] ———, *On characterizing non-separable analytic and extended Borel sets as types of continuous images.* Proc. London. Math. Soc. **28**(1974), 683–699. doi:10.1112/plms/s3-28.4.683

[6] ———, *Point-countable Souslin-additive families and σ–discrete reductions.* In: General topology and its relation to modern analysis and algebra. V. (Prague, 1981), Sigma Series in Pure Mathematics, 3, Heldermann, Berlin, 1983, pp. 254–260.

[7] ———, *Hereditarily additive families in descriptive set theory and Borel measurable multimaps.* Trans. Amer. Math. Soc. **278**(1983), no. 2, 725–749. doi:10.2307/1999181

[8] ———, *A measurable selection and representation theorem in nonseparable spaces.* In: Measure theory, Oberwolfach 1983, Lecture Notes in Math., 1089, Springer, Berlin, 1984, pp. 86–94.

[9] ———, *F_σ-set covers of analytic spaces and first class selectors.* Proc. Amer. Math. Soc. **96**(1986), no. 2, 365–371. doi:10.2307/2046182

[10] ———, *Nonseparable analytic metric spaces and quotient maps.* Topology Appl. **85**(1998), no. 1–3, 143–152. doi:10.1016/S0166-8641(97)00146-6

[11] J. Kaniewski and R. Pol, *Borel-measurable selectors for compact-valued mappings in the non-separable case.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **23**(1975), no. 10, 1043–1050.

[12] A. S. Kechris, *Classical descriptive set theory.* Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[13] R. Pol, *Note on decompositions of metrizable spaces. II.* Fund. Math. **100**(1978), no. 2, 129–143.

[14] C. E. Rogers and J. E. Jayne, *K-analytic sets.* In: Analytic sets, Academic Press, London, 1980, pp. 1–181.

[15] J. Spurný, *F_σ-additive families and the invariance of Borel classes.* Proc. Amer. Math. Soc. **133**(2005), no. 3, 905–915. doi:10.1090/S0002-9939-04-07587-2

[16] ———, *G_δ-additive families in absolute Souslin spaces and Borel measurable selectors.* Topology Appl. **154**(2007), no. 15, 2779–2785. doi:10.1016/j.topol.2007.05.012

[17] A. H. Stone, *Nonseparable Borel sets. II.* General Topology and Appl. **2**(1972), 249–270. doi:10.1016/0016-660X(72)90010-4

[18] ———, *Analytic sets in non-separable metric spaces.* In: Analytic sets, Academic Press, London, 1980, pp. 471–480.

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https://doi.org/10.4153/CMB-2010-088-8 Published online by Cambridge University Press