ON REGULAR SURFACES OF GENUS THREE

PATRICK DU VAL

ENRIQUES, in his posthumous *magnum opus* (1), devotes a chapter to the canonical or (where the genus is small) bi- or tricanonical models of regular surfaces, for various values of the genus $p_g = p_a = p$ and of the linear genus $p^{(1)}$. If, however, the cases he deals with are tabulated as follows (* marking the surfaces described by Enriques):

$p \backslash p^{(1)} =$	2	3	4	5	6	7	8	9	10
$p = 0$	*	*							
1	*	*							
2	*	*							
3		*							
4				*	*	*	*	*	
5								*	*

it is immediately clear that the scheme has remarkable gaps. The triangular space in the lower left-hand part corresponds, by the inequalities

$$p^{(1)} \geqslant 2p - 3, \qquad p^{(1)} \geqslant 3p - 6$$

(the former holding if the canonical system is irreducible, the latter if it is also simple) to a real absence of surfaces with the genera in question, except that the case $p = 5$, $p^{(1)} = 7$ has been omitted; this is easily seen to be a double normal rational ruled cubic, branching along a C^{16} (curve of order 16) which meets each generator in six and the directrix in four points; there is an analogous surface for every value of $p \geqslant 4$ satisfying the first inequality above with equality, consisting of a double normal rational ruled surface of order $p - 2$, branching along a C^{4p-4} which meets each generator in 6 points, intersection of the surface with a sextic hypersurface residual to $2p - 8$ generators. The rectangular gap in the upper right-hand part of the scheme, however, represents surfaces which presumably exist but have not been investigated; and this gap penetrates so far that the body of surfaces described is cut into two isolated parts; no surface with $p^{(1)} = 4$ is mentioned in the book, and only one with $p = 3$. It is as a first effort to fill in some of these lacunae that I offer this investigation of surfaces of genus $p = 3$, which (as might be expected) becomes less complete with increase of $p^{(1)}$.

The canonical regular surface of genus $p = 3$ and $p^{(1)} = n + 1$ is of course an n-ple plane, branching along a C^{2n+4} of some sort. The case $n = 2$ is well

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known, the branch curve being the most general C^8; the case $n = 3$ I studied myself some years ago (2), and showed that the branch curve has 24 cusps at the intersections of a quartic and a sextic, its equation being linearly dependent on the square of the sextic and the cube of the quartic, and is the most general curve of this kind. I also proved the case $n = 3$ of the following theorem, which I shall now prove generally, and of its converse, which I can only prove in its entirety (and which is probably only true) for $n \leqslant 5$. (We use $[n]$ to denote a projective space of n dimensions.)

THEOREM I. *In $[n+3]$ if V_n^4 is the cone projecting a Veronese surface V_2^4 from an $[n-3]$, Ω, and U_5^n a five dimensional manifold whose general $[n]$ section is the del Pezzo surface of order n, then the intersection of V_n^4 with U_5^n is in general a surface F^{4n}, the bicanonical model of a surface with $p = 3$, $p^{(1)} = n+1$.*

For F^{4n} has on it a net $|C|$ of curves C^{2n}, traced on it by the quadric cones Γ_{n-1}^2 of V_n^4 which project the conics of V_2^4 from the vertex Ω. Each of these lies in the $[n]$ containing the cone Γ_{n-1}^2, and is the intersection of Γ_{n-1}^2 with a del Pezzo surface S_2^n, the section of U_5^n by $[n]$. Now a quadric section of S_2^n is a canonical curve of genus $n+1$, since when the surface is mapped on a plane in the ordinary way, it appears as a sextic whose adjoint cubics are precisely the system mapping the hyperplane sections of the surface. Thus $|C'|$, the adjoint system of $|C|$, consists of the hyperplane sections of the surface F^{4n}, which are clearly the system $|2C|$, i.e.

$$|C| = |C' - C|$$

is the canonical system on the surface. Thus F^{4n} is bicanonical; it has $p = 3$ since $|C|$ is a net, and $p^{(1)} = n+1$ since this is the genus of C^{2n}.

The next problem of course is, what is the nature of the canonical n-ple plane, the projective model of $|C|$? In this connexion we prove

THEOREM II. *The canonical model of the surface F^{4n} is an n-ple plane, branching along a curve of order $4n$ with $24(n-2)$ cusps and $8(n-2)(n-3)$ nodes.*

For F^{4n} is projected from the vertex Ω_{n-3} of V_n^4 into an n-ple Veronese surface V_2^4, which when mapped on the plane in the ordinary way gives the required n-ple plane; and this n-ple V_2^4 is just the section by V_2^4 of the n-ple [5], projection of U_5^n from Ω_{n-3}; the nature of the branching of this can in turn be found by considering that of its general n-ple plane, the general projection of a del Pezzo surface of order n from $[n-3]$, whose branch curve is a C^{2n} of genus $n+1$, the projection of a general quadric section of the surface, and of class 12, since the del Pezzo surface is of class 12. From these data we see that this C^{2n} has $6\ (n-2)$ cusps and $2\ (n-2)(n-3)$ nodes; and the n-ple [5] accordingly branches along a hypersurface of order $2n$ with three dimensional cuspidal and nodal loci of orders $6(n-2)$, $2(n-2)(n-3)$ respectively; and from the intersections of these with a general V_2^4 and the mapping of the latter on the plane, Theorem II follows.

It is not likely, of course, that the branch curve is completely characterized by the mere number of its singularities. For $n = 3$ I have already given the stronger result, and for $n = 4$ it will appear in the sequel (Theorem V). It may be noted that Theorem II accords with the general formula

$$p_n = n + \tfrac{1}{2}(\beta - 1)(\beta - 2) - \delta - \kappa - 2$$

for an n-ple plane with 2β-ic branch curve having 4δ nodes and 3κ cusps.

Meanwhile, let us consider to what extent we can establish the converse of Theorem I. For a general value of n probably the best we can look for is the following:

THEOREM III. *Every regular surface with* $p = 3$, $p^{(1)} = n + 1$, *has as its bicanonical model a surface of order* $4n$ *lying on* V_n^4, *the canonical system being traced on it by the quadric cones* Γ_{n-1}^2.

For the characteristic series of the canonical system $|C|$ is a semicanonical g_n^1. On the canonical model of a general C^{2n}, each set of g_n^1 is joined by an $[n-2]$, and any two of these sets are together a hyperplane section of C^{2n}, i.e. any two of the ∞^1 $[n-2]$'s are joined by a hyperplane in the ambient $[n]$ of C^{2n}; and as clearly not all of these $[n-2]$'s lie in any one hyperplane, they all pass through one $[n-3]$ and generate a cone with this vertex, which is quadric since any hyperplane through one of them contains just one other. (We have made use of the dual of the familiar theorem that any set of lines, of which every two meet, either all pass through one point or all lie in one plane.)

Now on the bicanonical model of the surface there are ∞^2 sets of n points, any two of which belong to the semicanonical involution on a C^{2n}, so that the $[n-2]$'s joining them intersect in an $[n-3]$. The theorem just quoted about sets of lines every two of which meet, can easily be generalized to read: "Every set of $[k]$'s, every two of which meet in a $[k-1]$ and are joined by a $[k+1]$, either all lie in one $[k+1]$ or all pass through one $[k-1]$." Thus the ∞^2 $[n-2]$'s in the ambient F^{4n} (since they manifestly do not all lie in one $[n]$) all pass through one $[n-3]$ Ω, the common vertex of the quadric cones containing the individual curves C^{2n}; the ∞^2 $[n-2]$'s thus generate the cone V_n^4 with vertex Ω, and Theorem III is proved.

In the cases $n = 2, 3$ we can of course go further than this, and assert that the surface given by Theorem I is the most general with the given genera (for $n = 2$ this follows at once from the fact that the most general canonical surface is the double plane with octavic branch curve; for $n = 3$ it was proved in my former paper). We shall now show that the same thing is true for $n = 4, 5$. For $n = 4$ we have in fact

THEOREM IV. *The bicanonical model of the most general regular surface with* $p = 3$, $p^{(1)} = 5$ *is the intersection of* V_4^4 *with* U_5^4, *i.e. the complete intersection of* V_4^4 *with two general quadric hypersurfaces in* [7].

By Theorem III we already know that F^{16} lies on V_4^4, and its canonical system $|C|$ is traced by the quadric cones Γ_3^2. Now each C_8 of $|C|$, being a canonical curve of genus 5, is the complete intersection of a net of quadrics in its ambient [4], one of which is of course Γ_3^2. The rest trace on each generating plane of Γ_3^2 a pencil of conics (whose base points are the four points in which the plane meets F^{16}), on Γ_3^2 itself a pencil of Segre (quartic del Pezzo) surfaces S_2^4 (whose base curve is C^8), and on the vertex line Ω of V_4^4 an involution. Thus (since any two generating planes of V_4^4 belong to a Γ_3^2) the pencils of conics in all these planes with base points at the intersections with F^{16} trace the same involution on Ω; and isolating in each pencil the conic tracing a particular pair of the involution, we see that the locus of these is a three-dimensional variety on V_4^4, which traces on each generating plane a conic, and on each Γ_3^2 an S_2^4; this variety does not contain Ω, but meets it in a pair of points, and must accordingly be a quadric section, since every $(n - 1)$ dimensional variety on V_n^4 is either a complete intersection or residual to a Γ_{n-1}^2, and in the latter case must contain Ω_{n-3}; and it contains F^{16}. Thus the ∞^1 pairs of the involution on Ω give a pencil of quadric sections of V_4^4, all containing F^{16}, which is thus the complete intersection of V_4^4 with a pencil of quadric hypersurfaces in its ambient [7], i.e. with their base U_5^4. Theorem IV is thus proved.

Since V_4^4 is itself the base of an ∞^5 linear system of quadrics, the pencil whose intersection is U_5^4 is not unique, but is an arbitrary pencil skew to the ∞^5 system within an ∞^7 linear system containing the latter. F^{16} thus lies on ∞^{12} U_5^4's.

Theorem IV enables us to specify more precisely the branch curve of the canonical quadruple plane as follows:

THEOREM V. *The branch curve of the canonical surface with $p = 3$, $p^{(1)} = 5$, is the envelope of a family of quartic curves of index* 3 (*i.e. depending cubically on a parameter, so that three curves of the family pass through a general point of the plane*) *of which two members reduce to double conics, the eight branch points on each of these being the* 16 *nodes of the branch curve. Conversely the envelope of the most general family of this kind is the branch curve of a quadruple plane, which is a canonical surface with $p = 3$, $p^{(1)} = 5$.*

For the discriminant of the pencil of conics traced by the pencil of quadrics Q_6^2 on any plane through Ω can be written in the form

$$\alpha\lambda^3 + 3\beta\lambda^2 + 3\gamma\lambda + \delta$$

where λ is the parameter in the pencil, and α, β, γ, δ are quadratic functions of the coordinates of the plane, i.e. of those of the point in which it meets the [5] onto which we project. Thus the branch hypersurface of the quadruple [5] is the envelope of the family of quadrics

$$(\dagger) \qquad \alpha\lambda^3 + 3\beta\lambda^2 + 3\gamma\lambda + \delta = 0$$

and its cuspidal locus is the complete intersection of the three quartics

$$\alpha\gamma = \beta^2,\ \alpha\delta = \beta\gamma,\ \beta\delta = \gamma^2.$$

For the two values of λ, for which Q_6^2 touches Ω, the Q_4^2 given by (†) reduces to a double [4], branching along a Q_3^2, and these two Q_3^2's together constitute the nodal locus of the branch hypersurface, since the plane joining Ω to a point of either of them meets the corresponding Q_6^2 in a double line, and hence F^{16} in four points which coincide by pairs. That this branch hyperplane is the envelope of the most general family of quadrics answering to this description follows from the fact that its general hyperplane section, branch curve of the projected Segre surface, is the envelope of the most general family of conics, of index 3, of which two members reduce to double lines. Such a family can in fact be represented by an equation of the form

$$ax^2\lambda^3 + 3\beta\lambda^2 + 3\gamma\lambda + dy^2 = 0$$

which contains 14 homogeneously entering coefficients; and bearing in mind the ∞^4 projective transformations which leave the x and y axes invariant, and the possibility of multiplying the parameter λ by a constant, we see that the number of such envelopes projectively distinct is ∞^8; but the number of projectively distinct figures in [4] consisting of a Segre surface and a line to project it from is also ∞^8, and both systems are irreducible. Since the branch locus of the quadruple V_2^4 is the section of that of the quadruple [5], Theorem V follows.

Turning now to the case $n = 5$ we have

THEOREM VI. *The bicanonical model of the most general regular surface with* $p = 3$, $p^{(1)} = 6$ *is the intersection of* V_5^4 *with* U_5^5.

Up to a point the proof of this is very parallel to that of Theorem IV. The most general canonical curve of genus 6 lies on precisely one del Pezzo surface of order 5, S_2^5, of which it is a quadric section. Each C^{10} of the canonical system $|C|$ on F^{20} is thus the complete intersection of the cone Γ_4^2 with a determinate S_2^5, which of course is itself the complete intersection of a linear system of ∞^4 quadrics Q_4^2; these trace on each generating [3] of Γ_4^2 the ∞^4 quadric surfaces through the five intersections of the [3] with F^{20}, and on the vertex plane Ω a linear system of ∞^4 conics. Thus just as in Theorem IV, in all the ∞^2 generating [3]'s of V_5^4, the quadric surfaces through the intersections of [3] with F^{20} trace the same system of conics in Ω; and the locus of the quadric surface in each of these systems which traces a particular conic in Ω is a quadric section of V_5^4. It is not, of course, the section by a determinate quadric, since V_5^4 is itself the intersection of ∞^5 quadrics; but in the ambient [5], Λ_5 say, of C^{10}, the quadric Q_4^2 containing the S_2^5 on which C^{10} lies and tracing the chosen conic on Ω is determinate, and the locus of these ∞^2 Q_4^2's is clearly a quadric section of the cubic hypersurface K_7^3 generated by the ∞^2 Λ_5's, the cone projecting from Ω the cubic symmetroid generated by the conic planes of V_2^4; it is the section of K_7^3 moreover by a determinate quadric, since of course no quadrics

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contain K_7^3. We have thus in the ambient [8] of the whole figure a linear system of ∞^4 quadric hypersurfaces Q_7^2, whose complete intersection with K_7^3 is the locus Σ_4 of the surfaces S_2^5 in the ∞^2 Λ_5's, and with V_5^4 is the surface F^{20}.

The hypersurface K_7^3 is also generated of course by the ∞^2 tangent [5]'s M_5 of V_5^4, each of which touches it all over a generating [3], and two of which pass through a general generating [3] of K_7^3. Each M_5 is met by ∞^1 Λ_5's in the pencil of [4]'s through its [3] of contact, and accordingly meets Σ_4 in the surface generated by the elliptic quintic curves traced by these [4]'s on the S_2^5's in the corresponding Λ_5's, which is clearly itself an S_2^5, since the curves all meet the [3] of contact in the same five points, the intersections of this [3] with F^{20} Since K_7^3 has [6] sections consisting of a repeated Λ_5 and an M_5, Σ_4 is of order 15.

We shall now show that the complete intersection of the ∞^4 quadrics Q_7^2 is five dimensional. If it were of more dimensions, its intersection with K_7^3 would be more than four dimensional, which it is not. If it were less, it must consist of Σ_4^{15}, together possibly with some residual variety R which may be two, three, or four dimensional; and any [5] which does not lie on any of the quadrics Q_7^2 must in this case meet them in a linear system of quadrics Q_4^2 whose complete intersection is precisely the section by [5] of $\Sigma_4^{15}+R$, i.e., a curve. Now consider the [5] X_5, joining a generating [3] of K_7^3 to the [3]'s of contact of the two M_5's which intersect in this [3]. (This join is in fact a [5], since the three [3]'s all pass through the plane Ω and do not belong to a pencil.) X_5 meets K_7^3 in three [4]'s, joining the three [3]'s by pairs, and of these, two lie in M_5's and one in a Λ_5, so that each of them meets Σ_4^{15} in an elliptic quintic curve; and these three quintic curves, of which each pair has in common the five points traced by the [3] in which their ambient [4]'s intersect, are the complete intersection of Σ_4^{15} with X_5. Now the number of projectively distinct figures in [5] consisting of three normal elliptic quintic curves each pair of which have a [3] section in common is ∞^{15}, which is also the number of projectively distinct figures consisting of an S_2^5 and three hyperplanes, and both systems are clearly irreducible; thus the three curves just obtained are the sections by their ambient [4]'s of an S_2^5 lying in X_5, the ∞^4 quadrics through which are precisely the sections by X_5 of the ∞^4 quadrics Q_7^2, since a quadric in any space is completely determined when three hyperplane sections are given. Thus the complete intersection of the quadrics Q_7^2 meets X_5 not in a curve but in a surface, and is accordingly five dimensional. Since moreover it is met by the ∞^2 [5]'s Λ_5, the ∞^2 [5]'s M_5 and the ∞^4 [5]'s X_5 in del Pezzo surfaces S_2^5, it must be U_5^5, and F^{20} is its complete intersection with V_5^4; Theorem VI is thus proved.

It is to be noted that there are four descriptively different types of U_5^5; first what we shall regard as the general case, the Grassmannian of a linear complex of lines in [4]; and the cones projecting from a point, line, and plane respectively a [7], [6], and [5] section of this. The last three, however, are all special or limiting forms of the first, although they do not occur among the hyperplane sections of the general U_6^5, Grassmannian of all the lines of [4]; since the equations of the five linearly independent quadrics Q_7^2 can be taken to be

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$$x_{ij}x_{kl} + x_{ik}x_{lj} + x_{il}x_{jk} = 0,$$

where i, j, k, l are any four of 1, 2, 3, 4, 5, and $x_{ij} = -x_{ji}$ are any ten linear functions of the coordinates, the four cases occurring according as these ten functions satisfy one, two, three, or four independent linear identities.

I have not succeeded in this case ($n = 5$) in finding any more precise specification of the branch curve of the canonical quintuple plane (such as was provided for $n = 4$ by Theorem V, and for $n = 3$ is to be found in my former paper) than is given by Theorem II.

It does not seem very probable that for $n = p^{(1)} - 1 \geqslant 6$ the surface F^{4n} of Theorem I is the most general of its genera, since for $n \geqslant 6$ the quadric section of the del Pezzo surface is not the most general canonical curve of its genus, and it is therefore not likely that the section of this surface by a quadric cone with $[n - 3]$ vertex is the most general such curve with a semicanonical g_n^1, a point which was essential in the proofs of Theorems IV, VI, and the analogous result for $n = 3$. In any case, for $n \geqslant 10$ the surface given by Theorem I does not exist, since there is no del Pezzo surface; and if in Theorem I we interpret U_5^n to include as a special case the cone projecting a normal elliptic cone from [3], F^{4n} acquires four elliptic conical nodes, intersections of this [3] with V_n^4, which reduce its arithmetic genus to -1, and is in fact equivalent to an elliptic ruled surface, being generated by an elliptic pencil of rational quartics. The case $n = 6$ ($p^{(1)} = 7$) is perhaps crucial, in the sense that if the analogue of Theorems IV, VI could be established in this case, by some other method than that used hitherto, it might be a plausible speculation whether it held for all values of n; there is of course no reason *a priori* why the canonical curves on a surface of given genera should be the most general compatible with the existence of the semicanonical series required—though I believe this is so in all the cases that have been studied.

References

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University of Georgia