This is a ``preproof'' accepted article for *Canadian Journal of Mathematics*
This version may be subject to change during the production process.
DOI: 10.4153/S0008414X25000021

Canad. J. Math. Vol. **00** (0), 2025 pp. 1–44
http://dx.doi.org/10.4153/xxxx

On the $(k+2,k)$-problem of Brown, Erdős and Sós for $k=5,6,7$

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Abstract. **Let $f^{(r)}(n;s,k)$ denote the maximum number of edges in an n-vertex r-uniform hypergraph containing no subgraph with k edges and at most s vertices. Brown, Erdős and Sós [*New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan 1971)*, pp. 53–63, Academic Press 1973] conjectured that the limit $\lim_{n\to\infty} n^{-2}f^{(3)}(n;k+2,k)$ exists for all k. The value of the limit was previously determined for $k=2$ in the original paper of Brown, Erdős and Sós, for $k=3$ by Glock [*Bull. Lond. Math. Soc.*, 51 (2019) 230–236] and for $k=4$ by Glock, Joos, Kim, Kühn, Lichev and Pikhurko [*Proc. Amer. Math. Soc., Series B*, 11 (2024) 173–186] while Delcourt and Postle [*Proc. Amer. Math. Soc.*, 152 (2024), 1881–1891] proved the conjecture (without determining the limiting value).
In this paper, we determine the value of the limit in the Brown–Erdős–Sós Problem for $k\in\{5,6,7\}$. More generally, we obtain the value of $\lim_{n\to\infty} n^{-2}f^{(r)}(n;rk-2k+2,k)$ for all $r\geqslant 3$ and $k\in\{5,6,7\}$. In addition, by combining these new values with recent results of Bennett, Cushman and Dudek [arxiv:2309.00182, 2023] we obtain new asymptotic values for several generalised Ramsey numbers.**

1 Introduction

Given a family $\mathcal{F}$ of r-uniform hypergraphs (in short, *r-graphs*), denote by $\mathrm{ex}(n;\mathcal{F})$ the *Turán number* of $\mathcal{F}$, i.e. the maximum number of edges in an n-vertex r-graph containing no element of $\mathcal{F}$ as a subgraph. Turán problems for hypergraphs are notoriously difficult and we still lack an understanding of even seemingly simple instances such as when $\mathcal{F}$ forbids the complete 3-graph on 4 vertices. We refer the reader to the surveys [19, 28] for more background. In this paper, we focus on the family $\mathcal{F}^{(r)}(s,k)$ of all r-graphs with k edges and at most s vertices.

Brown, Erdős and Sós [4] launched the systematic study of the function

$$f^{(r)}(n;s,k) := \mathrm{ex}(n;\mathcal{F}^{(r)}(s,k)).$$

The case $r=2$ (resp. $r=3$) of the problem was previously studied by Erdős [9] (resp. Brown, Erdős and Sós [5]). Since then, the asymptotics of $f^{(r)}(n;s,k)$ as $n\to\infty$ have been intensively investigated for various natural choices of parameters r,s,k (see e.g. [1, 6, 11, 13, 15, 18, 23, 25, 27, 29]). For instance, it includes the celebrated $(6,3)$-theorem of

AMS subject classification: 05C65, 05C70, 05C35.

*Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 542321564.
†Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) No. RS-2023-00210430.
‡Supported by ERC Advanced Grant 101020255.
§Supported by ERC Advanced Grant 101020255.

https://doi.org/10.4153/S0008414X25000021 Published online by Cambridge University Press

Ruzsa and Szemerédi [25] (namely, when $(r, s, k) = (3, 6, 3)$), as well as the notoriously difficult $(7, 4)$-problem (namely, when $(r, s, k) = (3, 7, 4)$). Beyond its significance of being a fundamental Turán problem, the Brown–Erdős–Sós function is closely related to problems from other areas such as additive combinatorics (see e.g. [25]), coding theory (e.g. the case $k = 2$), hypergraph packing and designs (see below).

Brown, Erdős and Sós [4] proved that

$$\Omega(n^{(rk-s)/(k-1)}) = f^{(r)}(n; s, k) = O(n^{\lceil (rk-s)/(k-1)\rceil}).$$

In this paper, we are interested in the case when the exponent in both the lower and the upper bound is equal to 2, i.e. $s = rk - 2k + 2$. In this setting, the natural question is whether $n^{-2} f^{(r)}(n; rk - 2k + 2, k)$ converges to a limit as $n \to \infty$; in fact, already Brown, Erdős and Sós [4] considered this question and conjectured that the limit exists for $r = 3$. They verified their conjecture for $k = 2$ by showing that the limit is $1/6$. Glock [13] proved that, when $k = 3$, the limit exists and is equal to $1/5$. Recently, Glock, Joos, Kim, Kühn, Lichev and Pikhurko [15] solved the case $k = 4$ by showing that the limit equals to $7/36$.

Already the original work of Brown, Erdős and Sós [4, 5] pointed connections to (approximate) designs: in particular, it was observed by them that Steiner triple systems (when they exist) give extremal examples for $k = 2$. More generally, the celebrated theorem of Rödl [24] which solved the Erdős–Hanani problem from 1965 on asymptotically optimal clique coverings of complete hypergraphs can be phrased as

$$\lim_{n\to\infty} n^{-t} f^{(r)}(n; 2r - t, 2) = \frac{(r-t)!}{r!}. \tag{1.1}$$

Furthermore, the more recent results in [13, 15] linked the Brown–Erdős–Sós Problem to almost optimal graph packings. Namely, in [13], F-packings of complete graphs with some special graph F are used, and in [15], a significant strengthening was needed to find "high-girth" packings. These structures are related to another famous problem of Erdős in design theory, namely the existence of high-girth Steiner triple systems, which was recently resolved by Kwan, Sah, Sawhney and Simkin [20].

In a recent breakthrough, Delcourt and Postle [8] proved the Brown–Erdős–Sós conjecture, namely, that for $r = 3$ and any $k \geqslant 2$ the limit exists, without determining its value. Moreover, as observed by Shangguan [26], their approach generalises to every uniformity $r \geqslant 4$. Thus the limit $\lim_{n\to\infty} n^{-2} f^{(r)}(n; rk - 2k + 2, k)$ exists for all $r \geqslant 3$ and $k \geqslant 2$.

While the existence of the limits is an important step forward, it would be very interesting to actually determine the limiting values, in particular in view of the fact that only few asymptotic results on degenerate hypergraph Turán problems of quadratic growth are currently known.

In this paper, we determine the limit for $k = 5, 6, 7$ and arbitrary uniformity $r \geqslant 3$, as given by the following four theorems. (Recall that the limit for $k = 2$ is given in (1.1) while the cases $k = 3, 4$ were settled in [13, 15, 27].)

The following two results show that, for $k = 5, 7$, the limiting value is the same as for $k = 3$.

***Theorem* 1.1** *For every $r \geqslant 3$, we have $\lim_{n\to\infty} n^{-2} f^{(r)}(n; 5r - 8, 5) = \frac{1}{r^2-r-1}$.*

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***Theorem* 1.2** *For every $r \geqslant 3$, we have $\lim_{n\to\infty} n^{-2} f^{(r)}(n; 7r-12, 7) = \frac{1}{r^2-r-1}$.*

However, the case $k=6$ exhibits different behaviour when $r=3$ and $r \geqslant 4$ (which parallels the situation for $k=4$), as established by the following two theorems.

***Theorem* 1.3** $\lim_{n\to\infty} n^{-2} f^{(3)}(n; 8, 6) = \frac{61}{330}$.

***Theorem* 1.4** *For every $r \geqslant 4$, we have $\lim_{n\to\infty} n^{-2} f^{(r)}(n; 6r-10, 6) = \frac{1}{r^2-r}$.*

Very recently, Letzter and Sgueglia [21] proved various results on the existence and the value of the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n, k(r-t)+t, k)$. In particular, for $t=2$, they independently re-proved our upper bounds in Theorems 1.1 and 1.2 when r is sufficiently large, and showed that $f^{(r)}(n; kr-2k+2, k) = (\frac{1}{r^2-r} + o(1))n^2$ when k is even and $r \geqslant r_0(k)$ is large enough.

An application to generalised Ramsey numbers. The following generalisation of Ramsey numbers was introduced by Erdős and Shelah [10], and its systematic study was initiated by Erdős and Gyárfás [12]. Fix integers p, q such that $p \geqslant 3$ and $2 \leqslant q \leqslant \binom{p}{2}$. A (p,q)*-colouring* of K_n is a colouring of the edges of K_n such that every p-clique has at least q distinct colours among its edges. The *generalised Ramsey number* $\mathrm{GR}(n,p,q)$ is the minimum number of colours such that K_n has a (p,q)-colouring. One relation to the classical Ramsey numbers is that $\mathrm{GR}(n,p,2) > t$ if and only if every t-colouring of the edges of K_n yields a monochromatic clique of order p.

In their work, Erdős and Gyárfás [12] showed that, for every $p \geqslant 3$ and $q_{\mathrm{lin}} := \binom{p}{2} - p + 3$,

$$\mathrm{GR}(n,p,q_{\mathrm{lin}}) = \Omega(n) \quad \text{and} \quad \mathrm{GR}(n,p,q_{\mathrm{lin}}-1) = o(n),$$

while for every $p \geqslant 3$ and $q_{\mathrm{quad}} := \binom{p}{2} - \lfloor p/2 \rfloor + 2$,

$$\mathrm{GR}(n,p,q_{\mathrm{quad}}) = \Omega(n^2) \quad \text{and} \quad \mathrm{GR}(n,p,q_{\mathrm{quad}}-1) = o(n^2).$$

Thus, q_{lin} and q_{quad} are the thresholds (that is, the smallest values of q) for $\mathrm{GR}(n,p,q)$ to be respectively linear and quadratic in n.

Very recently, Bennett, Cushman and Dudek [2] found the following connection between generalised Ramsey numbers and the Brown–Erdős–Sós function.

***Theorem* 1.5** ([2, Theorem 3]) *For all even $p \geqslant 6$, we have*

$$\lim_{n\to\infty} \frac{\mathrm{GR}(n,p,q_{\mathrm{quad}})}{n^2} = \frac{1}{2} - \lim_{n\to\infty} \frac{f^{(4)}(n; p, p/2-1)}{n^2}.$$

In particular, the limit on the left exists by [26].

By combining this with our results, we obtain the following new asymptotic values for the generalised Ramsey numbers at the quadratic threshold.

***Theorem* 1.6** *The following equalities hold:*

$$\lim_{n\to\infty} \frac{\mathrm{GR}(n,12,62)}{n^2} = \frac{9}{22}, \ \lim_{n\to\infty} \frac{\mathrm{GR}(n,14,86)}{n^2} = \frac{5}{12}, \ \lim_{n\to\infty} \frac{\mathrm{GR}(n,16,114)}{n^2} = \frac{9}{22}. \quad \blacksquare$$

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Bennett, Cushman and Dudek [2, Theorem 4] also proved that, for all $p \geqslant 3$, it holds that

$$\liminf_{n\to\infty} \frac{\mathrm{GR}(n,p,q_{\mathrm{lin}})}{n} \geqslant 1 - \lim_{n\to\infty} \frac{f^{(3)}(n;p,p-2)}{n^2}. \tag{1.2}$$

Using our above results in the cases when $r = 3$ and $k = p - 2$ is in $\{5, 6, 7\}$, we get the following lower bounds at the linear threshold:

$$\liminf_{n\to\infty} \frac{\mathrm{GR}(n,7,17)}{n} \geqslant \frac{4}{5}, \quad \liminf_{n\to\infty} \frac{\mathrm{GR}(n,8,23)}{n} \geqslant \frac{269}{330}, \quad \liminf_{n\to\infty} \frac{\mathrm{GR}(n,9,30)}{n} \geqslant \frac{4}{5}. \tag{1.3}$$

Note that (1.2) gives only a one-sided inequality. It happens to be tight for $p = 3$ (trivially) and for $p = 4$ by the result of Bennett, Cushman, Dudek and Prałat [3] that $\mathrm{GR}(n, 4, 5) = (\frac{5}{6}+o(1))n$. However, (1.2) is not tight for $p = 5$: Gomez-Leos, Heath, Parker, Schwieder and Zerbib [16] showed that $\mathrm{GR}(n, 5, 8) \geqslant \frac{6}{7}(n-1)$ while $f^{(3)}(n; 5, 3) = (\frac{1}{5}+o(1))n^2$, as proved in [13]. We do not know if the bounds in (1.3) are sharp.

Organisation of the paper. The remainder of this paper is organised as follows. Section 2 introduces some notation. An overview of our proofs can be found in Section 3. The lower bounds are proved in Section 4, and the upper bounds are proved in Section 5. The proof of the upper bound of Theorem 1.3, while using the same general proof strategy, is rather different from the other proofs in detail, so it is postponed until the end. The final section is dedicated to some concluding remarks.

2 Notation

Throughout the paper, we use the following notation and definitions. Let $\mathbb{N}$ denote the set of positive integers. For $m, n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$ and by $[m, n]$ the set $[n] \setminus [m-1] = \{m, \ldots, n\}$. For a set X, we let $\binom{X}{s} := \{Y \subseteq X : |Y| = s\}$ be the family of all s-subsets of X. We will often write an unordered pair $\{x, y\}$ (resp. triple $\{x, y, z\}$) as xy (resp. as xyz). Moreover, for three real numbers a, b and $c \geqslant 0$, we write $a = b \pm c$ to say that $a \in [b - c, b + c]$. Also, we write $a \gg b > 0$ to mean that b is a sufficiently small positive real depending on a.

Given an r-graph G, we denote by $V(G)$ the vertex set of G and by $E(G)$ its edge set. Moreover, we define $|G|$ as the number of edges of G and $v(G)$ as the number of vertices of G. When it is notationally convenient, we may identify an r-graph with its set of edges. If we specify only the edge set $E(G)$, then the vertex set is assumed to be the union of these edges, that is, $V(G) := \bigcup_{X\in E(G)} X$. For r-graphs F and H, their *union* $F \cup H$ and *difference* $F \setminus H$ have edge sets respectively $E(F) \cup E(H)$ and $E(F) \setminus E(H)$ (with their vertex sets being the unions of these edges). We reserve the lowercase letter r to denote the uniformity of our hypergraphs.

For positive integers s and k, an (s, k)-*configuration* is an r-graph with k edges and at most s vertices, that is, an element of $\mathcal{F}^{(r)}(s, k)$. An r-graph is called (s, k)-*free* if it contains no (s, k)-configuration. Let us define another r-graph family

$$\mathcal{G}_k^{(r)} := \mathcal{F}^{(r)}(rk - 2k + 2, k) \cup \left(\bigcup_{\ell=2}^{k-1} \mathcal{F}^{(r)}(r\ell - 2\ell + 1, \ell) \right). \tag{2.1}$$

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Thus, $\mathcal{G}_k^{(r)}$ includes the family $\mathcal{F}^{(r)}(rk-2k+2,k)$, whose Turán function is the main object of study of this paper, as well as all analogous r-graphs for smaller sizes that are "denser" (that is, are subject to a stronger restriction on the number of vertices). Note that the family $\mathcal{F}^{(r)}(r\ell-2\ell+1,\ell)$ that appears in the right-hand side of (2.1) for $\ell=2$ happens to be empty when $r=3$ (however, we include it to have a single formula that works for all pairs (r,k)). The family $\mathcal{G}_k^{(r)}$ is of relevance for both the lower and the upper bounds (see Theorem 4.1 and Lemma 3.1).

For an r-graph G, a pair xy of distinct vertices (not necessarily in $\binom{V(G)}{2}$) and $A\subseteq\mathbb{N}\cup\{0\}$, we say that G *A-claims* the pair xy if, for every $i\in A$, there are i distinct edges $X_1,\ldots,X_i\in E(G)$ such that $|\{x,y\}\cup(\bigcup_{j=1}^i X_j)|\leqslant ri-2i+2$. If $xy\in\binom{V(G)}{2}$, this is the same as the existence of an $(ri-2i+2,i)$-configuration $J\subseteq G$ with $\{x,y\}\subseteq V(J)$ for every $i\in A$. When $A=\{i\}$ is a singleton, we just say *i-claims* instead of $\{i\}$-claims. By definition, any r-graph 0-claims any pair (which will be notationally convenient, see e.g. Lemma 5.1). For $i\geqslant 1$, let $P_i(G)$ be the set of all pairs in $\binom{V(G)}{2}$ that are i-claimed by G. For example, if $i=1$, then $P_1(G)$ is the usual 2-*shadow* of G consisting of all pairs of vertices uv such that there exists some edge $X\in E(G)$ with $u,v\in X$. Also, let $C_G(xy)$ be the set of those $i\geqslant 0$ such that the pair xy is i-claimed by G, that is,

$$C_G(xy):=\left\{i\geqslant 0:\exists\text{ distinct }X_1,\ldots,X_i\in E(G)\ \left|\{x,y\}\cup(\textstyle\bigcup_{j=1}^i X_j)\right|\leqslant ri-2i+2\right\}. \tag{2.2}$$

More generally, for disjoint subsets $A,B\subseteq\mathbb{N}$, we say that G *$\overline{A}B$-claims* a pair xy if $A\cap C_G(xy)=\emptyset$ and $B\subseteq C_G(xy)$. In the special case when $A=\{1\}$ and $B=\{i\}$ we just say *$\overline{1}i$-claims*; also, we let $P_{\overline{1}i}(G):=P_i(G)\setminus P_1(G)$ denote the set of pairs in $\binom{V(G)}{2}$ that are $\overline{1}i$-claimed by G.

A *diamond* is an r-graph consisting of two edges that share exactly 2 vertices. Thus, a $(2r-3,2)$-free r-graph G $\overline{1}2$-claims a pair of vertices xy if and only if $xy\notin P_1(G)$ and there is a diamond $\{X_1,X_2\}\subseteq G$ such that $x,y\in X_1\cup X_2$.

3 Overview of the proofs

For the lower bounds, we combine a result from [15] that allows us to build relatively dense $\mathcal{F}^{(r)}(rk-2k+2,k)$-free r-graphs G from a fixed $\mathcal{G}_k^{(r)}$-free r-graph F. Namely, G will be the union of many edge-disjoint copies of F and, of course, the main issue is to avoid forbidden subgraphs coming from different copies of F. In order to attain the desired lower bound on $|G|$, the packed copies of F will be allowed to share pairs (but not triples) of vertices. Pairs inside $V(F)$ that will be allowed to be shared will be limited to those uv for which $C_F(uv)$ does not contain any i with $1\leqslant i\leqslant k/2$. This will automatically exclude forbidden subgraphs in G coming from at most 2 copies of F. Then, a result from [15] will be used to eliminate any forbidden configurations whose edges come from at least 3 different copies of F.

If $r=3$ and $k\in\{5,7\}$, then we take for F the union of many diamonds $\{x_iy_ia,x_iy_ib\}$ sharing only the pair ab of vertices. This is a straightforward generalisation of the construction for $k=3$ by Glock [13]. However, if $r\geqslant 4$ and $k\in\{5,7\}$, then finding a suitable F is a new difficult challenge, not present in [15]. The initial idea that eventually led to its resolution was to take two sufficiently sparse $(r-2)$-graphs

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with edge sets $\{K_1^1, \ldots, K_1^t\}$ and $\{K_2^1, \ldots, K_2^t\}$, and let F be the union of diamonds $\{\{x_i, y_i\} \cup K_1^i, \{x_i, y_i\} \cup K_2^i\}$, $1 \leqslant i \leqslant t$, for new vertices x_i, y_i. Some further ideas are needed to fix the two big issues of this construction: namely, avoiding any subgraph in $\mathcal{G}_k^{(r)}$ and having "overhead" (like the pair $\{a, b\}$ in the construction for $r = 3$) of size neglibile compared to $|F|$. We refer the reader to Section 4.1 for details.

For the case $(r, k) = (3, 6)$, we provide an explicit construction of a 3-graph F_{63} on 63 vertices and 61 edges, while for $r \geqslant 4$, $k = 6$, the lower bound comes from the trivial construction when F is a single edge.

Concerning the upper bounds, we will need the following result (proved for $r = 3$ in [8, Theorem 1.7] and then extended to any r in [26, Lemma 5]) which allows us to get rid of smaller "denser" structures.

***Lemma* 3.1** ([26, Lemma 5]) *For all fixed $r \geqslant 3$ and $k \geqslant 3$,*

$$\limsup_{n\to\infty} \frac{f^{(r)}(n; rk - 2k + 2, k)}{n^2} \leqslant \limsup_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{G}_k^{(r)})}{n^2}. \tag{3.1}$$

Since $\mathcal{F}^{(r)}(rk - 2k + 2, k) \subseteq \mathcal{G}_k^{(r)}$, the opposite inequality in (3.1) trivially holds. Also, the main results of [8, 26] show that both ratios in (3.1) tend to a limit (which is the same for both) as $n \to \infty$. By Lemma 3.1, in order to obtain an upper bound on $f^{(r)}(n; rk - 2k + 2, k)$, it is enough to consider only those r-graphs G on $[n]$ which are $\mathcal{G}_k^{(r)}$-free. For any such r-graph G, we define a partition of the edge set $E(G)$ by starting with the trivial partition into single edges and iteratively merging parts as long as possible using some merging rules (that depend on k and r). Then, we specify a set of weights that each final part (which is a subgraph $F \subseteq G$) attributes to some of the pairs in $\binom{V(F)}{2}$ and use combinatorial arguments to show that every vertex pair receives total weight at most 1. Thus, the total weight assigned by the parts is at most $\binom{n}{2}$ which translates into an upper bound on $|G|$. The main difficulty lies in designing the merging and weighting rules, which have to be fine enough to detect even the extremal cases (which are quite intricate constructions) but coarse enough to be still analysable.

The most challenging case here is $(r, k) = (3, 6)$, where our solution uses a rather complicated weighting rule with values in $\{0, \frac{6}{61}, \frac{11}{61}, \frac{25}{61}, \frac{1}{2}, \frac{36}{61}, \frac{55}{61}, 1\}$. Note that any weighting rule that gives the correct limit value of $\frac{61}{330}$ has to be tight on optimal packings of the 63-vertex configuration F_{63} from the lower bound. Unfortunately, this seems to force any such rule to be rather complicated.

4 Lower Bounds

To prove our lower bounds, we use the following result, which is derived from [15].

***Theorem* 4.1** ([15, Theorem 3.1]) *Let $k \geqslant 2$, $r \geqslant 3$ and let F be a $\mathcal{G}_k^{(r)}$-free r-graph. Then,*

$$\liminf_{n\to\infty} n^{-2} f^{(r)}(n; rk - 2k + 2, k) \geqslant \frac{|F|}{2\,|P_{\leqslant \lfloor k/2 \rfloor}(F)|},$$

where we define $P_{\leqslant t}(F) := \{xy \in \binom{V(F)}{2} : C_F(xy) \cap [t] \neq \emptyset\}$ to consist of all pairs xy of vertices of F such that $C_F(xy)$ contains some i with $1 \leqslant i \leqslant t$.

We remark that the $\mathcal{G}_k^{(r)}$-freeness captures the two conditions needed for F in [15, Theorem 3.1] and that the choice of $J := P_{\leqslant \lfloor k/2 \rfloor}(F)$ there guarantees that J contains the 2-shadow of F and that, using the notation from [15], the pair (F, J) has non-edge girth greater than $k/2$.

To give the reader a little bit of motivation for this theorem, we briefly sketch where the ratio $\frac{|F|}{2|J|}$ comes from, where $J = P_{\leqslant \lfloor k/2 \rfloor}(F)$. The proof goes via packing many edge-disjoint copies of the graph J and then putting a copy of F "on top" of each J. Note that the total number of edge-disjoint copies of J that we can find in K_n is roughly $\binom{n}{2}/|J|$, and each copy of F adds new $|F|$ edges to our r-graph. Hence, in total, we will have roughly $\frac{|F|}{2|J|}n^2$ edges, as desired. To ensure that the resulting r-graph remains $(rk - 2k + 2, k)$-free, the recently developed theory of conflict-free hypergraph matchings [7, 14] is used, and the $\mathcal{G}_k^{(r)}$-freeness condition of Theorem 4.1 is necessary to apply this method. We also remark that Theorem 4.1 was used in [15] to settle the case $k = 4$, and by Delcourt and Postle [8] and by Shangguan [26] to prove the existence of the limits.

4.1 Lower bounds in Theorems 1.1 and 1.2

We apply Theorem 4.1 to derive first the lower bounds in Theorems 1.1 and 1.2 (that is, when $k \in \{5, 7\}$). Note that if F is a diamond then

$$\frac{|F|}{2\,|P_1(F)|} = \frac{2}{2(2\binom{r}{2} - 1)} = \frac{1}{r^2 - r - 1}$$

is exactly the bound we are aiming for. The problem is that if $k \geqslant 4$ and we apply Theorem 4.1 for this F then $P_{\leqslant \lfloor k/2 \rfloor}(F)$ includes all pairs inside $V(F)$ and the theorem gives a weaker bound. Thus, we essentially want F to consist of many edge-disjoint diamonds in such a way that the $\overline{1}2$-claimed pairs are "reused" by many different diamonds. To illustrate our approach, we start with the simpler 3-uniform case.

Proof of the lower bounds in Theorems 1.1 and 1.2 with $r = 3$ Recall that we forbid $\mathcal{F}^{(3)}(k+2, k)$ for $k = 5, 7$ here. Fix a positive integer t and consider the 3-graph F consisting of t diamonds $\{x_i y_i a, x_i y_i b\}$ where the $2t$ vertices $x_1, \ldots, x_t, y_1, \ldots, y_t$ are all distinct.

Let us show that F is $\mathcal{G}_5^{(3)}$-free and $\mathcal{G}_7^{(3)}$-free. Take any set $X \subseteq V(F)$ of size ℓ. If $\{a, b\} \subseteq X$ then X can contain at most $\lfloor (\ell - 2)/2 \rfloor$ of the pairs $x_i y_i$ and thus spans at most twice as many edges in F. If X is disjoint from $\{a, b\}$ then X spans no edges. In the remaining case $|X \cap \{a, b\}| = 1$, the set X contains at most $\lfloor (\ell - 1)/2 \rfloor$ of the pairs $x_i y_i$ and thus spans at most this many edges in F. Thus, for $\ell = 4, 5, 6, 7, 9$, we see that X spans at most $2, 2, 4, 4, 6$ edges, respectively. Thus, F is $\mathcal{G}_5^{(3)}$-free and $\mathcal{G}_7^{(3)}$-free, as claimed.

The above argument gives that F is $(5, 3)$-free and that every $(4, 2)$-configuration in F is $\{x_i y_i a, x_i y_i b\}$ for some $i \in [t]$. Thus, $P_{\leqslant 3}(F) \setminus P_1(F)$ consists only of the pair ab.

As a result, Theorem 4.1 implies that, for $k = 5, 7$,

$$\liminf_{n\to\infty} n^{-2} f^{(3)}(n; k+2, k) \geqslant \frac{|F|}{2\,|P_{\leqslant 3}(F)|} = \frac{2t}{2(5t+1)}.$$

By taking $t \longrightarrow \infty$, we conclude that the lim-inf is at least $1/5$, as desired. ■

Let us now informally describe how one can generalise the above construction to higher uniformity $r \geqslant 4$. We will often use the following definition. Given an r-graph G, the *girth* of G is the smallest integer $\ell \geqslant 2$ such that there exist edges $X_1, \dots, X_\ell$ spanning at most $(r-2)\ell + 2$ vertices. For example, the girth is strictly larger than 2 if and only if G is *linear* (that is, every two edges intersect in at most one vertex). In informal discussions, we use the phrase "high girth" to assume that the girth is at least an appropriate constant.

Similar to the above $r = 3$ case, we would like to find a suitable r-graph F as the union of diamonds such that the set of 2- or 3-claimed pairs not in $P_1(F)$ is much smaller than $|F|$. The difficulty here is that, for example, if a pair is $\overline{1}2$-claimed by two diamonds, then, by the $(4r-7, 4)$-freeness requirement of Theorem 4.1, these two diamonds cannot share any other vertices. In particular, we cannot simply replace a and b by two $(r-2)$-sets in the above construction for $r = 3$. One approach would be to consider two linear $(r-2)$-graphs $\mathcal{K}_1$ and $\mathcal{K}_2$ on disjoint vertex sets A_1 and A_2 with $|\mathcal{K}_1| = |\mathcal{K}_2|$ and $m := |A_1| = |A_2|$. Let us pick a matching M between some edges of $\mathcal{K}_1$ and $\mathcal{K}_2$, say consisting of the pairs $\{K_1^j, K_2^j\}$ for $j = 1, \dots, |M|$. We add new vertices x_j and y_j for each $j = 1, \dots, |M|$ and define $F = F(M)$ as the r-graph with edge set

$$E(F) := \left\{K_1^j \cup \{x_j, y_j\} : j = 1, \dots, |M|\right\} \bigcup \left\{K_2^j \cup \{x_j, y_j\} : j = 1, \dots, |M|\right\}. \tag{4.1}$$

Thus, F is a union of $|M|$ diamonds. It is easy to show that F is necessarily $(4r-7, 4)$-free (see Claim 4.7) and that $P_{\overline{1}2}(F)$ is the union of $\{ab : a \in K_1^j, b \in K_2^j\}$ for $1 \leqslant j \leqslant |M|$. Moreover, we can additionally ensure that $\mathcal{K}_1$ and $\mathcal{K}_2$ have large girth, which follows from the recent results on conflict-free hypergraph matchings [7, 14]. As a direct consequence of this high-girth assumption, we can see that we do not get any forbidden configurations in F when we use edges only from one "side" of the construction, say $\mathcal{K}_i$. Indeed, for any $\ell \geqslant 2$ such edges, by the high-girth assumption, the union of the corresponding $(r-2)$-sets in $\mathcal{K}_i$ has more than $(r-2-2)\ell + 2$ vertices, and when adding the 2ℓ new vertices x_j and y_j as above, we get more than $r\ell - 2\ell + 2$ vertices, that is, there is no $(r\ell - 2\ell + 2, \ell)$-configuration.

There are still two serious issues even for $k = 5$. First, we have not guaranteed that the number of $\overline{1}2$-claimed pairs is much smaller than the number of edges in F. Indeed, even if $|M| = \Theta(m^2)$ (which is the largest possible order of magnitude by $|M| \leqslant \binom{m}{2}/\binom{r-2}{2}$), the set $P_{\overline{1}2}(F)$ may have size comparable to $|F| = 2\,|M|$ since potentially a positive fraction of pairs between A_1 and A_2 could be $\overline{1}2$-claimed. To ensure that $|P_{\overline{1}2}(F)|$ is much smaller than $|F|$, we form a random bipartite graph G_3 with parts A_1 and A_2 where every edge is included with small probability α. Then, we allow $K_1^j \in \mathcal{K}_1$ to be matched to $K_2^j \in \mathcal{K}_2$ only if all pairs in $K_j^1 \times K_j^2$ are edges of G_3. This ensures that $P_{\overline{1}2}(F)$ is a subgraph of G_3 and hence $|P_{\overline{1}2}(F)| \leqslant |G_3| \approx \alpha m^2$.

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The second (much more complicated) problem is that we have to avoid dense configurations when using edges from both sides of the construction, which could overlap significantly in the "middle layer" formed by the vertices x_j, y_j. Hence, roughly speaking, when we have two collections of $(r-2)$-sets in $\mathcal{K}_1$ and $\mathcal{K}_2$ that contain few vertices, we want to avoid matching many of these $(r-2)$-sets with each other to form diamonds. Formally, we construct an auxiliary bipartite graph H where $K_1 \in \mathcal{K}_1$ and $K_2 \in \mathcal{K}_2$ are adjacent if $\{\{a,b\} : a \in K_1, b \in K_2\}$ is a subset of $E(G_3)$ (so a diamond could be attached to K_1, K_2), and we define a family C of sets of disjoint edges of H such that if the matching M avoids C, then $F(M)$ avoids all forbidden configurations. Then, the goal is to find a large matching in H which avoids each of the problematic configurations in C. However, this would still not be possible with the current construction. Roughly speaking, the problem is that, for every $u_1 \in K_1^i \in \mathcal{K}_1$ and $u_2 \in K_2^i \in \mathcal{K}_2$, being able to attach a diamond to K_1^i, K_2^i implies that $u_1u_2 \in E(G_3)$. However, the presence of u_1u_2 in G_3 increases significantly the probability that, for any fixed pair $K_1^j \in \mathcal{K}_1$ and $K_2^j \in \mathcal{K}_2$ such that $K_1^i \cap K_1^j = \{u_1\}$ and $K_2^i \cap K_2^j = \{u_2\}$, a diamond can be attached to the pair K_1^i, K_2^i. As it turns out, the number of such pairs (K_1^j, K_2^j) happens to be too large. To fix this, we first randomly sparsify the complete graphs on A_1 and A_2 with a well-chosen probability, and then restrict our attention to $(r-2)$-sets in A_1, A_2 which form cliques in the underlying random graphs. This allows better control on the number of edges $K_1^j \in \mathcal{K}_1$ and $K_2^j \in \mathcal{K}_2$ containing a pair $u_1u_2 \in E(G_3)$.

We will use the following concentration inequality for functions of independent coordinates satisfying a Lipschitz condition, which is known as *the Bounded Difference Inequality* or *McDiarmid's inequality*; it can be also derived from the Azuma-Hoeffding Inequality.

***Lemma* 4.2** ([22, Lemma 1.2]) *Let $X_1, X_2, \dots, X_n$ be independent random variables with X_i taking values in Λ_i, and let $f : \Lambda_1 \times \cdots \times \Lambda_n \to \mathbb{R}$ be a function that satisfies the following Lipschitz condition for some numbers $(c_i)_{i=1}^n$: for every $i \in [n]$ and every two vectors $x, \tilde{x} \in \Lambda_1 \times \cdots \times \Lambda_n$ that differ only in the i-th coordinate, it holds that $|f(x) - f(\tilde{x})| \leqslant c_i$.*

Then, the random variable $Z := f(X_1, \dots, X_n)$ satisfies

$$\mathbb{P}\left[\, |Z - \mathbb{E}[\,Z\,]| > s \,\right] \leqslant 2\exp\left(-\frac{2s^2}{\sum_{i=1}^n c_i^2}\right).$$

Let us also recall the classical *Chernoff bound* stating that, for every binomial random variable X and every $t \geqslant 0$,

$$\mathbb{P}\left[\, |X - \mathbb{E}[X]| \geqslant t \,\right] \leqslant 2\exp\left(-\frac{t^2}{2(\mathbb{E}[X]+t/3)}\right),$$

see e.g. [17, Theorem 2.1].

Furthermore, we will need a simplified version of a result from [14] on the existence of approximate clique packings of high girth.

***Theorem* 4.3** ([14, Theorem 1.4]) *For all $c_0 > 0$, $\ell \geqslant 2$ and $r \geqslant 3$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there exists m_0 such that the following holds for all $m \geqslant m_0$ and $c \geqslant c_0$. Let G be a graph on m vertices such that every edge of G is contained in $(1 \pm m^{-\varepsilon})cm^{r-2}$*

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cliques of order r. Then, there exists a K_r-packing $\mathcal{K}$ in G of size $|\mathcal{K}| \geqslant (1 - m^{-\varepsilon^3})|G|/\binom{r}{2}$ such that, for every $j \in [2, \ell]$, any set of j elements in $\mathcal{K}$ spans more than $(r-2)j+2$ vertices.

Note that if we consider the packing $\mathcal{K}$ returned by Theorem 4.3 as an r-graph, then the last requirement is precisely that the girth of $\mathcal{K}$ is larger than ℓ.

Now we are ready to provide the construction which establishes the lower bounds in Theorems 1.1 and 1.2 for $r \geqslant 4$.

***Lemma* 4.4** *Fix any integer $r \geqslant 4$. Then, for a sufficiently small real $\alpha > 0$ and a sufficiently large integer m, that is, for $1/r \gg \alpha \gg 1/m$, there exists an r-graph F satisfying each of the following properties:*

(a) *F is $(5r - 8, 5)$-free and $(7r - 12, 7)$-free,*
(b) *F is $(2r - 3, 2)$-free, $(3r - 5, 3)$-free, $(4r - 7, 4)$-free and $(6r - 11, 6)$-free,*
(c) *$|F| = \Omega(\alpha^{3/4}m^2)$,*
(d) *$|P_{\leqslant 3}(F)| \leqslant \frac{r^2-r-1}{2} |F| + 2\alpha m^2$.*

Proof Let A_1 be a set of size m. Sample every edge of the complete graph on A_1 independently with probability $\beta := \alpha^{3/4}$ to get a random graph G_1 on A_1. In the sequel, implicit constants in the O, Ω, Θ-notation may depend on r but not on α and m unless the dependence is explicitly indicated in a lower index such as O_α. We say that an event holds *with high probability* if its probability tends to 1 as $m \to \infty$.

***Claim* 4.5** *With high probability, G_1 satisfies the following properties.*

(i) *For every vertex $v \in V(G_1)$, we have $d(v) = \Theta(\beta m)$.*
(ii) *For any pair of vertices $u, v \in V(G_1)$, we have $|N(u) \cap N(v)| = \Theta(\beta^2 m)$.*
(iii) *If $r \geqslant 5$, then every edge in G_1 is contained in $(1 \pm m^{-1/3})cm^{r-4}$ cliques of size $r - 2$, where $c := \beta^{\binom{r-2}{2}-1}/(r-4)!$.*
(iv) *There is an $(r - 2)$-graph $\mathcal{K}_1$ of girth at least 8, with vertex set A_1 and edge set being a collection of edge-disjoint $(r - 2)$-cliques in G_1 such that all but $o(m^2)$ edges of G_1 belong to a clique in $\mathcal{K}_1$.*

Proof of Claim 4.5 The first two properties follow easily by noting that for a vertex v (resp. a pair u, v) the probability of failure is $\mathrm{e}^{-\Omega_\alpha(m)}$ by Chernoff's bound and then taking the union bound over all (polynomially many in m) choices.

Let us turn to the third claim. Fix an edge $uv \in G_1$ (that is, we condition on uv being sampled). Let X be the number of $(r-2)$-cliques in G_1 containing uv. Since each potential clique containing uv has $\binom{r-2}{2} - 1$ edges other than uv, each of which appears independently with probability β, we have

$$\mathbb{E}[X] = \binom{m-2}{r-4}\beta^{\binom{r-2}{2}-1} = cm^{r-4} + O_\alpha(m^{r-5}).$$

Let us show that X is concentrated. We use the Bounded Difference Inequality (Lemma 4.2). Altering the state of a pair with one endvertex among u, v may change the value of X by at most m^{r-5}, and every other edge may change the value of X by at most

m^{r-6}. Thus, by Lemma 4.2, we have

$$\mathbb{P}\left[\, X \neq (1 \pm m^{-1/3})cm^{r-4} \,\right] \leqslant \mathbb{P}\left[\, |X - \mathbb{E}\,[\, X\,]\,| \geqslant cm^{r-4-1/3}/2 \,\right]$$
$$\leqslant\ 2\exp\left(-\frac{c^2 m^{2(r-4-1/3)}/2}{2m \cdot m^{2(r-5)} + m^2 \cdot m^{2(m-6)}}\right) \ =\ o(m^{-2}).$$

(If $r=5$, then X is the number of triangles, which is binomially distributed, and Chernoff's bound can be applied instead of Lemma 4.2.) The union bound over all $\binom{m}{2}$ choices of uv finishes the proof.

Let us turn to the existence of $\mathcal{K}_1$. Note first that the case $r=4$ is trivial since we can take each edge of G_1 as a clique of order 2 (and any 2-graph has infinite girth according to our definition of girth), so we assume that $r \geqslant 5$. Theorem 4.3 for $c_0 := c$, $\ell := 8$ and $r-2$ returns some ε_0. Let m_0 be the value returned by the theorem for $\varepsilon := \min(1/3, \varepsilon_0/2)$. Since m_0 depends only on r and α, we can assume that $m > m_0$. Thus, Theorem 4.3 applies to any graph G_1 satisfying Property (iii) and produces $\mathcal{K}_1$ with all the stated properties. ■

We now fix a graph G_1 on the set A_1 and an $(r-2)$-graph $\mathcal{K}_1$ satisfying Claim 4.5. Let $A_2, G_2, \mathcal{K}_2$ be disjoint copies of $A_1, G_1, \mathcal{K}_1$. We identify $\mathcal{K}_1$ and $\mathcal{K}_2$ with their edge sets. Let G_3 be a random bipartite graph with parts A_1 and A_2 where every edge between A_1 and A_2 is sampled independently with probability α.

We set $t := |\mathcal{K}_1| = |\mathcal{K}_2|$ and note that $t = \Theta(\beta m^2)$. We also define an auxiliary bipartite graph H with parts $\mathcal{K}_1$ and $\mathcal{K}_2$ where $K_1 \in \mathcal{K}_1$ and $K_2 \in \mathcal{K}_2$ are adjacent in H if each of the $(r-2)^2$ pairs between $K_1 \subseteq A_1$ and $K_2 \subseteq A_2$ is an edge of G_3. In particular, a pair in $\mathcal{K}_1 \times \mathcal{K}_2$ is an edge of H with probability $\alpha^{(r-2)^2}$. Define

$$d := \alpha^{(r-2)^2} t = \Theta(\beta \alpha^{(r-2)^2} m^2).$$

***Claim* 4.6** *With high probability, all vertices in the graph H have degree $(1 \pm m^{-1/3})d$.*

Proof of Claim 4.6 Take any vertex K of H, that is, an edge of $\mathcal{K}_i$ for $i=1$ or 2. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are fixed, the degree of K in H is a function of the $(r-2)m$ independent Bernoulli variables that encode the edges of G_3 between $K \subseteq A_i$ and the opposite side A_{3-i}. Furthermore, one edge in G_3 can influence the appearance of at most $\frac{m-1}{r-3} \leqslant m$ edges in H containing K, since every pair of vertices in A_{3-i} is contained in at most one clique.

Thus, denoting the degree of K in H by $\deg_H(K)$ and using that its expectation is d, the Bounded Difference Inequality (Lemma 4.2) implies that

$$\mathbb{P}\left[|\deg_H(K) - d| \geqslant \frac{d}{m^{1/3}}\right] \leqslant 2\exp\left(-\frac{2\alpha^{2(r-2)^2} t^2/m^{2/3}}{(r-2)m \cdot m^2}\right) = \exp(-\Omega_\alpha(m^{1/3})).$$

A union bound over all $O(m^2)$ edges in $\mathcal{K}_1 \cup \mathcal{K}_2$ proves the claim. ■

Our goal will be to find a matching M in H with size $\Omega(t) = \Omega(\beta m^2)$ and certain additional properties. Given M, we define the r-graph $F = F(M)$ as in (4.1). Namely, we

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start with the edgeless hypergraph on $A_1 \cup A_2$ and, for every edge $K_1^j K_2^j$ in the matching M, add to F new vertices x_j, y_j and new edges $K_1^j \cup \{x_j, y_j\}$ and $K_2^j \cup \{x_j, y_j\}$ (forming a diamond).

We remark that, at this point of the proof, we do not fix the choice of G_3 yet but view G_3 (and hence H) as a random graph since we still want to bound the number of certain problematic subconfigurations.

We can rule out some small configurations in F without any additional assumptions on M.

Claim 4.7 *The r-graph $F = F(M)$ is $(4r-7, 4)$-free, $(3r-4, 3)$-free and $(2r-3, 2)$-free.*

Proof of Claim 4.7 Every triple of edges $X_1, X_2, X_3 \in F$ satisfies that each of $|X_1 \cap X_2|$, $|X_2 \cap X_3|$ and $|X_3 \cap X_1|$ is at most 1, or contains a diamond (X_i, X_j) for some $i \neq j$ with the third edge intersecting $X_i \cup X_j$ in at most one vertex; in either case, $|X_1 \cup X_2 \cup X_2| \geqslant 3r - 3$. In particular, this gives that F is $(2r-3, 2)$-free and $(3r-4, 3)$-free.

Now, suppose on the contrary that F has a $(4r-7, 4)$-configuration, say coming from some $(r-2)$-graphs $F_1 \subseteq \mathcal{K}_1$ and $F_2 \subseteq \mathcal{K}_2$. For $i = 1, 2$, let $e_i := |F_i|$ and let

$$d_i := (r-2)|F_i| - |\cup_{X \in F_i} X|,$$

calling it the *defect* of F_i. Thus, $e_1 + e_2 = 4$. By symmetry, assume that $e_1 \geqslant e_2$. By $e_1 \in [2, 4]$ and the high-girth assumption, we obtain that

$$d_1 \leqslant e_1(r-2) - (e_1(r-2-2) + 3) = 2e_1 - 3. \tag{4.2}$$

Let $d' \leqslant e_2$ be the number of pairs in M between F_1 and F_2 (which is exactly the number of diamonds in the hypothetical $(4r-7, 4)$-configuration in F that we started with). Thus we have

$$4r - 7 \geqslant \big(e_1(r-2) - d_1\big) + \big(e_2(r-2) - d_2\big) + 2(4 - d') = 4r - d_1 - d_2 - 2d', \tag{4.3}$$

that is, $d_1 + d_2 \geqslant 7 - 2d'$.

Now, it is routine to derive a contradiction. Although some of the following cases can be combined together, we prefer (here and later) to treat each possible value of e_1 as a separate case for clarity. If $e_1 = 4$, then $e_2 = 0$, $d_2 = 0$, $d' = 0$, and thus $d_1 \geqslant 7$. If $e_1 = 3$, then $e_2 = 1$, $d_2 = 0$, $d' \leqslant 1$ and thus $d_1 \geqslant 5$. If $e_1 = 2$, then $e_2 = 2$, $d_2 \leqslant 1$ and $d' \leqslant 2$ giving $d_1 \geqslant 2$. However, the obtained lower bound on d_1 contradicts (4.2) in each of the three possible cases. Thus, F contains no $(4r-7, 4)$-configuration, as desired. ■

To ensure that F is $(5r-8, 5)$-free, $(6r-11, 6)$-free and $(7r-12, 7)$-free, we have to construct the matching M a bit more carefully. In the following, we will define some problematic configurations and show that there are only few of them with high probability. We will then be able to use a probabilistic argument to construct a matching M that avoids all problematic configurations.

Let us introduce some further terminology. For an $(r-2)$-graph $\mathcal{K}$ and integers e', d', let $\mathcal{S}_{e',d'}(\mathcal{K})$ be the family of all subgraphs of $\mathcal{K}$ with e' edges and defect d'. (Recall that this means that the union of these e' edges has exactly $e'(r-2) - d'$ vertices.) We refer the reader to Figure 1 for some special cases of this definition that will

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play important role in our proof. For integers e_1, e_2, d_1, d_2 with $e_1 \geqslant e_2$, let the set $\mathcal{C}_{e_1,d_1;e_2,d_2}$ consist of all matchings

$$\left\{\{K_i^j, K_{3-i}^j\} : j \in [e_2]\right\} \tag{4.4}$$

in H for some $i = 1$ or 2 such that the $(r-2)$-graph $\{K_{3-i}^1, \ldots, K_{3-i}^{e_2}\}$ is in $\mathcal{S}_{e_2,d_2}(\mathcal{K}_{3-i})$ while $\{K_i^1, \ldots, K_i^{e_2}\}$ extends to an element of $\mathcal{S}_{e_1,d_1}(\mathcal{K}_i)$. (Recall that the set in (4.4) is a matching in H if, for $i = 1, 2$, the $(r-2)$-sets $K_i^1, \ldots, K_i^{e_2}$ are pairwise distinct and, for every $j \in [e_2]$, all pairs in $K_1^j \times K_2^j$ are edges of G_3.) In other words, the set $\mathcal{C}_{e_1,d_1;e_2,d_2}$ can be constructed as follows. Pick a subgraph of size e_1 and defect d_1 in one of $\mathcal{K}_1$ and $\mathcal{K}_2$, then a subgraph of size e_2 and defect d_2 in the other one; viewing these as two sets of vertices in the bipartite graph H, add to $\mathcal{C}_{e_1,d_1;e_2,d_2}$ all matchings in H of size e_2 (that is, fully pairing the smaller subgraph). Note that the definition of $\mathcal{C}_{e_1,d_1;e_2,d_2}$ is symmetric in $\mathcal{K}_1$ and $\mathcal{K}_2$.

The final family of conflicts that our matching M will have to avoid will consists of four families of the type $\mathcal{C}_{e_1,d_1;e_2,d_2}$ plus a related family. In order to illustrate the general proof, we will show first that forbidding $\mathcal{C}_{3,3;2,1}$ alone takes care of all undesired configurations in $F = F(M)$ with at most 6 edges. Note that, by the high-girth assumption, every pair in $\mathcal{C}_{3,3;2,1}$ is a matching of a "loose 2-path" in one of $\mathcal{K}_1$ and $\mathcal{K}_2$ into a "loose 3-cycle" in the other.

***Claim* 4.8** *If M is a matching in H that avoids $\mathcal{C}_{3,3;2,1}$, then the r-graph $F = F(M)$, as defined in (4.1), is $(5r-8,5)$-free and $(6r-11,6)$-free.*

Proof of Claim 4.8 Suppose on the contrary that we have a $(5r-8,5)$-configuration in F. For $i = 1, 2$, let F_i be the $(r-2)$-graph consisting of the edges of $\mathcal{K}_i$ involved in the configuration; also, let $e_i := |F_i|$ be the size and $d_i := e_i(r-2) - v(F_i)$ be the defect of F_i. Thus, $e_1 + e_2 = 5$.

Let d' be the number of edges in M between F_1 and F_2. By a calculation analogous to (4.3), we have that

$$d_1 + d_2 \geqslant 8 - 2d'. \tag{4.5}$$

Without loss of generality, assume that $e_1 \geqslant e_2$. Of course, $d' \leqslant e_2$. Since $3 \leqslant e_1 \leqslant 5$, we have by the large girth assumption that (4.2) holds, that is, $d_1 \leqslant 2e_1 - 3$.

If, $e_1 = 5$ then $e_2 = 0$ and so $d_2 = d' = 0$, but then (4.2) and (4.5) give a contradiction $8 \leqslant d_1 \leqslant 7$. If, $e_1 = 4$ then $e_2 = 1$ and so $d_2 = 0$ and $d' \leqslant 1$; however, then (4.2) and (4.5) give $6 \leqslant d_1 \leqslant 5$, a contradiction again. Finally, suppose that $e_1 = 3$. We have that $e_2 = 2$ and thus, $d_2 \leqslant 1$ and $d' \leqslant 2$. Now, (4.2) and (4.5) give that $3 \leqslant d_1 \leqslant 3$. Thus, we have equalities everywhere; in particular, $d_1 = 3$, $d_2 = 1$ and $d' = 2$. We see that the pair of edges of M coming from the two diamonds in F belongs to $\mathcal{C}_{3,3;2,1}$, a contradiction.

Now, suppose on the contrary that F contains a $(6r-11,6)$ configuration. For $i = 1, 2$, let F_i be the $(r-2)$-subgraph of $\mathcal{K}_i$ involved in it, with e_i and d_i denoting its size and defect. Without loss of generality, assume that $e_1 \geqslant e_2$. Let d' be the number of M-edges between F_1 and F_2. By a version of (4.3), we get that

$$d_1 + d_2 \geqslant 11 - 2d', \tag{4.6}$$

while the inequality in (4.2) remains unchanged.

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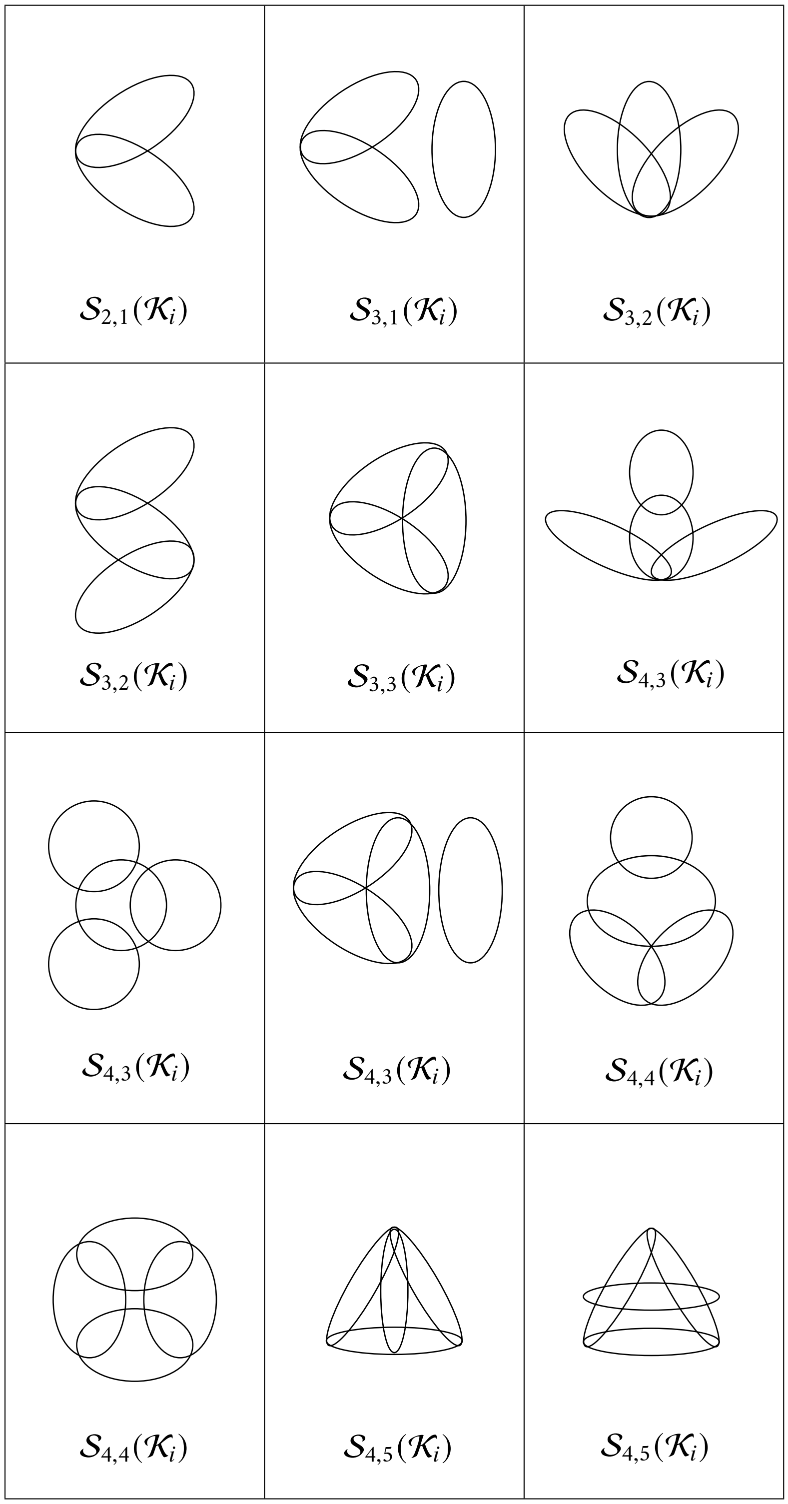

Figure 1: Examples of $\mathcal{S}_{e',d'}$-subgraphs (that is, having size e' and defect d') in the high-girth $(r-2)$-graph $\mathcal{K}_i$ for some pairs (e',d'). Since the hypergraph $\mathcal{K}_i$ is linear, each drawn intersection has size 1. For (e',d') in $\{(2,1),(3,1),(3,3)\}$, the family $\mathcal{S}_{e',d'}(\mathcal{K}_i)$ consists of a unique $(r-2)$-graph up to isomorphism. For (e',d') in $\{(3,2),(4,5)\}$, there are exactly two non-isomorphic examples. For the remaining pairs (e',d'), we provide a non-exhaustive list. .

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If $e_1 = 6$, then $e_2 = d_2 = d' = 0$, so (4.2) and (4.6) give that $11 \leqslant d_1 \leqslant 9$, a contradiction. If $e_1 = 5$, then $e_2 = 1$, $d_2 = 0$, $d' \leqslant 1$, and we get that $9 \leqslant d_1 \leqslant 7$, a contradiction. If $e_1 = 4$, then $e_2 = 2$, $d_2 \leqslant 1$, $d' \leqslant 2$, and we get that $6 \leqslant d_1 \leqslant 5$, a contradiction.

Finally, it remains to check the (slightly less straightforward) case when $e_1 = 3$. Here, we have $e_2 = 3$, $d_2 \leqslant 3$ and $d' \leqslant 3$. Thus, (4.2) and (4.6) give that $11-2d' \leqslant d_1+d_2 \leqslant 6$. We conclude that $d' = 3$ and thus, M fully matches F_1 and F_2. By symmetry, suppose that $d_1 \geqslant d_2$. Thus, $d_1 = 3$. Since $d_2 \geqslant 5 - d_1 \geqslant 2$ is positive, we can find $K_2^1, K_2^2 \in F_2$ that intersect. The $(r-2)$-graphs $F_1 \in \mathcal{S}_{3,3}(\mathcal{K}_1)$ and $\{K_2^1, K_2^2\} \in \mathcal{S}_{2,1}(\mathcal{K}_2)$ show that the edges of M containing K_2^1 and K_2^2 form a pair in $C_{3,3;2,1}$, a contradiction. ■

Next, we define the "exceptional" family. First, for $i = 1, 2$, let $\mathcal{S}'_{4,3}(\mathcal{K}_i)$ consist of those $(r-2)$-graphs in $\mathcal{S}_{4,3}(\mathcal{K}_i)$ that do not contain an isolated edge (equivalently, not containing an $\mathcal{S}_{3,3}$-subgraph). It is easy to show (see Claim 4.10) that $\mathcal{S}'_{4,3}(\mathcal{K}_i)$ consists precisely of subgraphs of $\mathcal{K}_i$ whose edge set can be ordered as $\{X_1, \dots, X_4\}$ such that $|X_i \cap (\bigcup_{j=1}^{i-1} X_j)| = 1$ for each $i \in [2,4]$, that is, it consists of "loose subtrees" with 4 edges. Also, let the conflict family $C'_{4,3;3,3}$ be obtained by taking full H-matchings of some element of $\mathcal{S}_{3,3}(\mathcal{K}_i)$ into $\mathcal{S}'_{4,3}(\mathcal{K}_{3-i})$ for $i = 1, 2$. Clearly, $C'_{4,3;3,3} \subseteq C_{4,3;3,3}$.

The final family C of the conflicts that we are going to use is

$$C := C_{3,3;2,1} \cup C_{4,4;3,2} \cup C_{4,5;2,1} \cup C_{5,7;2,1} \cup C'_{4,3;3,3}. \tag{4.7}$$

Basically, our definition of the conflict family C is motivated by the proof of Claim 4.9 below: for each still possible way of having a $(7r-12,7)$-configuration in $F(M)$, we add further conflicts that rule it out. We do not try to take minimal possible conflict families, but rather the ones that are easy to describe. Note that we cannot take the full family $C_{4,3;3,3}$ as the upper bound of Claim 4.11 on the expected number of conflicts fails for it; fortunately, its smaller subfamily $C'_{4,3;3,3}$ suffices.

Claim 4.9 *If a matching M in H does not contain any element of C as a subset, then the r-graph $F = F(M)$ defined by* (4.1) *contains no $(7r-12,7)$-configuration.*

Proof of Claim 4.9 Suppose on the contrary that F contains a $(7r-12,7)$-configuration. For $i = 1, 2$, let F_i be the $(r-2)$-subgraph of $\mathcal{K}_i$ involved in it, with e_i and d_i denoting its size and defect. Thus, $e_1 + e_2 = 7$. Without loss of generality, assume that $e_1 \geqslant e_2$. Let d' be the number of the diamonds involved. By a calculation analogous to (4.3), we have

$$d_1 + d_2 \geqslant 12 - 2d'. \tag{4.8}$$

First, we rule out the easy cases when $e_1 \geqslant 6$. If $e_1 = 7$, then $e_2 = d_2 = d' = 0$ but (4.2) and (4.8) give that $12 \leqslant d_1 \leqslant 11$, a contradiction. If $e_1 = 6$, then $e_2 = 1$, $d_2 = 0$ and $d'_1 \leqslant 1$, so we get $10 \leqslant d_1 \leqslant 9$, a contradiction again.

Thus, $e_1 \leqslant 5$. Since $e_2 \geqslant 2$, the large girth assumption gives similarly to (4.2) that

$$d_2 \leqslant 2e_2 - 3. \tag{4.9}$$

Suppose that $e_1 = 5$. Then $e_2 = 2$ and $d' \leqslant 2$. Our bounds (4.2), (4.8) and (4.9) give that $d_1 \leqslant 7$, $d_1 + d_2 \geqslant 8$ and $d_2 \leqslant 1$, respectively. Thus, we have equalities everywhere.

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In particular, $F_1 \in \mathcal{S}_{5,7}(\mathcal{K}_1)$, $F_2 \in \mathcal{S}_{2,1}(\mathcal{K}_2)$, $d' = 2$ and the pair of the involved M-edges belongs to $\mathcal{C}_{5,7;2,1}$, a contradiction.

So, assume for the rest of the proof that $e_1 = 4$. Then, $d' \leqslant e_2 = 3$. By (4.2), (4.8) and (4.9), we have that $d_1 \leqslant 5$, $d_1 + d_2 \geqslant 12 - 2d' \geqslant 6$ and $d_2 \leqslant 3$. It follows that $d' \geqslant 2$.

First, suppose that $d' = 2$. Thus, we must have that $d_1 = 5$ and $d_2 = 3$. Let $K_2^1, K_2^2 \in F_2$ be the two edges matched by M. Since $F_2 \in \mathcal{S}_{3,3}(\mathcal{K}_2)$, we have that $\{K_2^1, K_2^2\} \in \mathcal{S}_{2,1}(\mathcal{K}_2)$. Thus, we have a conflict from $\mathcal{C}_{4,5;2,1}$, a contradiction.

Thus, $d' = 3$, that is, F_2 is fully matched into F_1 by M.

Suppose first that $d_1 + d_2 = 6$. The case $(d_1, d_2) = (4, 2)$ is impossible as then we get a conflict from $\mathcal{C}_{4,4;3,2}$. Next, suppose that $(d_1, d_2) = (3, 3)$. To avoid a conflict from $\mathcal{C}'_{4,3;3,3}$, it must be the case that F_1 contains an $\mathcal{S}_{3,3}$-subgraph, say with edges K_1^1, K_1^2, K_1^3. The matching M matches at least two of these three edges, say to $K_2^1, K_2^2 \in F_2$. Since every two edges of F_2 intersect, we have that $\{K_2^1, K_2^2\} \in \mathcal{S}_{2,1}(\mathcal{K}_2)$ and thus M contains a conflict from $\mathcal{C}_{3,3;2,1}$, a contradiction. The only possible remaining case $(d_1, d_2) = (5, 1)$ gives that $F_2 \in \mathcal{S}_{3,1}(\mathcal{K}_2)$ is fully matched into $F_1 \in \mathcal{S}_{4,5}(\mathcal{K}_1)$ by M. But the two intersecting edges $K_2^1, K_2^2 \in F_2$ form an element of $\mathcal{S}_{2,1}(\mathcal{K}_1)$ fully matched into F_1, that is, M contains a conflict from $\mathcal{C}_{4,5;2,1}$, a contradiction.

Thus we can assume that $d_1 + d_2 \geqslant 7$. If $d_1 = 4$, then $d_2 = 3$ and $F_2 \in \mathcal{S}_{3,3}(\mathcal{K}_2)$. Among the three M-matches of F_2 in F_1, some two, say $K_1^1, K_1^2 \in F_1$, must intersect. (Indeed, otherwise these three edges contribute 0 to the defect of F_1 while the remaining edge can contribute at most 3, a contradiction to $F_1 \in \mathcal{S}_{4,4}(\mathcal{K}_1)$.) But then $\{K_1^1, K_1^2\} \in \mathcal{S}_{2,1}(\mathcal{K}_1)$ and $F_2 \in \mathcal{S}_{3,3}(\mathcal{K}_2)$ give a conflict from $\mathcal{C}_{3,3;2,1}$, a contradiction. Thus it remains to consider the case when $d_1 = 5$. Note that all pairs of edges in $F_1 \in \mathcal{S}_{4,5}(\mathcal{K}_1)$ except at most one pair intersect, and that $d_2 \geqslant 7 - d_1 = 2$. Thus, out of the three M-edges between F_1 and F_2, we can pick two such that their two endpoints on each side intersect. Among the two remaining edges of F_1, at least one intersects both of these F_1-endpoints in two distinct vertices. Thus, we get two edges in M between some sets in $\mathcal{S}_{3,3}(\mathcal{K}_1)$ and $\mathcal{S}_{2,1}(\mathcal{K}_2)$, which is a conflict from $\mathcal{C}_{3,3;2,1}$. This final contradiction proves that $F = F(M)$ is indeed $(7r - 12, 7)$-free. ■

Next, we need to bound from above the number of choices of $F_i \in \mathcal{S}_{e',d'}(\mathcal{K}_i)$ for each involved pair (e', d'). For this, we would like to construct each such F_i from the empty $(r-2)$-graph by iteratively adding edges or pairs of edges at each step. Let $F' \subseteq F_i$ be the currently constructed subgraph. A j*-attachment* for $j = 0, 1, 2$ occurs when we add one new edge that shares exactly j vertices with $V(F')$. To make a 3*-attachment*, we add two new edges K, K' such that each of the three intersections $K \cap K'$, $K \cap V(F')$ and $K' \cap V(F')$ consists of exactly one vertex and these three vertices are pairwise distinct, that is,

$$|K \cap K'| = |K \cap V(F')| = |K' \cap V(F')| = 1 \quad \text{and} \quad K \cap K' \cap V(F') = \emptyset.$$

If we construct F_i this way, then we let a_j be the number of j-attachments for $0 \leqslant j \leqslant 3$; note that then $a_0 + a_1 + a_2 + 2a_3 = e'$ and $a_1 + 2a_2 + 3a_3 = d'$.

***Claim* 4.10** *Let $i = 1$ or 2. Then, the following holds for every family $\mathcal{S}$ listed in Table 1.*

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$\mathcal{S}$	a_0	a_1	a_2	a_3	$\lvert\mathcal{S}\rvert$
$\mathcal{S}_{2,1}(\mathcal{K}_i)$	1	1	0	0	$O(\beta^2 m^3)$
$\mathcal{S}_{3,1}(\mathcal{K}_i)$	2	1	0	0	$O(\beta^3 m^5)$
$\mathcal{S}_{3,2}(\mathcal{K}_i)$	1	2	0	0	$O(\beta^3 m^4)$
$\mathcal{S}_{3,3}(\mathcal{K}_i)$	1	0	0	1	$O(\beta^3 m^3)$
$\mathcal{S}_{4,4}(\mathcal{K}_i)$	1	1	0	1	$O(\beta^4 m^4)$
$\mathcal{S}_{4,5}(\mathcal{K}_i)$	1	0	1	1	$O(\beta^3 m^3)$
$\mathcal{S}_{5,7}(\mathcal{K}_i)$	1	0	2	1	$O(\beta^3 m^3)$
$\mathcal{S}'_{4,3}(\mathcal{K}_i)$	1	3	0	0	$O(\beta^4 m^5)$

Table 1: The values for Claim 4.10

(i) Every $F_i \in \mathcal{S}$ can be constructed using the above attachment operations starting from the empty graph such that the corresponding vector (a_0, a_1, a_2, a_3) is exactly as stated in Table 1.

(ii) The size of $\mathcal{S}$ is at most the expression given in the last column of the table.

Proof of Claim 4.10 Let us prove the first part for F_i from a "regular" family $\mathcal{S} = \mathcal{S}_{e',d'}(\mathcal{K}_i)$. The cases $e' \leqslant 3$ are straightforward to check, so assume that $e' \geqslant 4$.

Let $e' = 4$. Take any $F_i = \{K^1, \ldots, K^4\}$ in $\mathcal{S}_{4,d'}(\mathcal{K}_i)$.

Let $d' = 4$. Suppose first that some three edges have defect 3, say $\{K^1, K^2, K^3\} \in \mathcal{S}_{3,3}(\mathcal{K}_i)$. We can build this subgraph using a 0-attachment and a 3-attachment. The remaining edge K^4 contributes exactly 1 to the defect, so its addition is a 1-attachment, giving the desired. So suppose F_i has no $\mathcal{S}_{3,3}$-subgraph. Add one by one some two intersecting edges, say K^1 and K^2, doing a 0-attachment and a 1-attachment. By the $\mathcal{S}_{3,3}$-freeness, each of the remaining edges K^3 and K^4 intersects $K^1 \cup K^2$ in at most one vertex. The above three intersections contribute at most 3 to the total defect while $|K^3 \cap K^4| \leqslant 1$ contributes at most 1, so all these intersections are non-empty (and of size exactly 1). Thus we can add $\{K^3, K^4\}$ as one 3-attachment.

Let $d' = 5$. Start with an intersecting pair of edges, say K^1 and K^2. Observe that at least one of the remaining edges, say K^3, intersects K^1 and K^2 at two different vertices (as otherwise the defect would be at most 4 by the linearity of F). We can construct $\{K^1, K^2, K^3\} \in \mathcal{S}_{3,3}(\mathcal{K}_i)$ using a 0-attachment and a 3-attachment. The addition of the remaining edge adds 2 to the defect and thus is a 2-attachment. This gives the vector $(1, 0, 1, 1)$, as desired.

So suppose that $(e', d') = (5, 7)$. Take any $F_i = \{K^1, \ldots, K^5\}$ in $\mathcal{S}_{5,7}(\mathcal{K}_i)$. By the linearity of F, there are at least $d' = 7$ pairs of intersecting edges so, by Mantel's theorem, there are three pairwise intersecting edges, say K^1, K^2, K^3. These edges contribute at most 3 to the defect. The pair of the remaining two edges contributes $|K^4 \cap K^5| \leqslant 1$ to the defect, so there are at least $d' - 3 - 1 = 3$ further intersections. Thus, at least one of the edges K^4, K^5, say K^4, satisfies

$$|K^4 \cap (K^1 \cup K^2 \cup K^3)| \geqslant 2. \tag{4.10}$$

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It follows that some three of $K^1, \dots, K^4$ have defect 3, say $\{K^1, K^2, K^3\} \in \mathcal{S}_{3,3}(\mathcal{K}_i)$. By the same argument as before, we can assume that (4.10) holds. Note that we must have equality in (4.10) as otherwise the union of $K^1, \dots, K^4$ has at most $4(r-4)+2$ vertices, contradicting the high-girth assumption on $\mathcal{K}_i$. Thus, the defect of $\{K^1, \dots, K^4\}$ is exactly 5 and

$$|K^5 \cap (K^1 \cup \cdots \cup K^4)| = d' - 5 = 2.$$

We conclude that we can build F_i using a 0-attachment and a 3-attachment (to construct $\{K^1, K^2, K^3\} \in \mathcal{S}_{3,3}(\mathcal{K}_i)$) and two 2-attachments (to add K^4 and then K^5), as desired.

Let us turn to F_i from the "exceptional" family $\mathcal{S} = \mathcal{S}'_{4,3}(\mathcal{K}_i)$. We know that F_i has no isolated edge. Starting with any edge $K^1 \in F_i$ (a 0-attachment), add some edge intersecting it (a 1-attachment), say K^2. At least one of the two remaining edges, say K^3, has non-empty intersection with $K^1 \cup K^2$ (as otherwise $d' \leqslant 2$, a contradiction). This intersection consists of exactly one vertex (as otherwise the defect of $\{K^1, K^2, K^3\}$ is already 3, a contradiction). Again, by $d' = 3$, the remaining edge K^4 shares exactly one vertex with $K^1 \cup K^2 \cup K^3$. So when we add K^3 and then K^4, we use two 1-attachments, giving $(a_0, a_1, a_2, a_3) = (1, 3, 0, 0)$, as desired.

Let us turn to the second part, namely bounding the number of choices of $F_i \in \mathcal{S}$. We build each such F_i as in Part (i). Given a partially built $F' \subseteq F_i$, there are by Claim 4.5 at most $O(\beta m^2)$, $O(\beta m)$, $O(1)$ and $O(\beta^2 m)$ ways to do a j-attachment for $j = 0, 1, 2, 3$ respectively. For example, every 3-attachment can be obtained by taking some $u, v \in V(F')$, having at most $\binom{3(r-2)}{2} = O(1)$ choices, then choosing some $w \in N_{G_i}(u) \cap N_{G_i}(v)$, having $O(\beta^2 m)$ choices, and then taking (if they exist) the unique two edges of $\mathcal{K}_i$ that contain the pairs uw and vw. Thus, the number of $F_i \in \mathcal{S}$ that give a fixed vector (a_0, a_1, a_2, a_3) is at most

$$O\left((\beta m^2)^{a_0} \cdot (\beta m)^{a_1} \cdot (\beta^2 m)^{a_3}\right) = O\left(\beta^{a_0+a_1+2a_3}\, m^{2a_0+a_1+a_3}\right). \tag{4.11}$$

By the first part, this directly gives an upper bound on $|\mathcal{S}|$ stated in the table. ∎

Recall that we defined $d = \alpha^{(r-2)^2} t$.

Claim 4.11 *For every quadruple (e_1, d_1, e_2, d_2) such that $C_{e_1,d_1;e_2,d_2}$ appears in the right-hand side of (4.7), the expected value of $|C_{e_1,d_1;e_2,d_2}|$ over the random graph G_3 is at most $O(d^{e_2} t)$. Also, the expectation of $|C'_{4,3;3,3}|$ is $O(d^3 t)$.*

Proof of Claim 4.11 We can bound the expectation of $|C_{e_1,d_1;e_2,d_2}|$ from above by

$$\sum_{i=1}^{2} |\mathcal{S}_{e_1,d_1}(\mathcal{K}_i)| \cdot |\mathcal{S}_{e_2,d_2}(\mathcal{K}_{3-i})| \cdot \alpha^{\ell},$$

where ℓ is the smallest number of pairs in $A_1 \times A_2$ that are covered by a full matching of some element of $\mathcal{S}_{e_2,d_2}(\mathcal{K}_{3-i})$ into some element of $\mathcal{S}_{e_1,d_1}(\mathcal{K}_i)$. Using the upper bounds coming from Claim 4.5 and recalling that $\beta = \alpha^{3/4}$, $t = \Theta(\beta m^2)$ and $d = \alpha^{(r-2)^2} t$, we

2024/12/30 08:04

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obtain the following estimates:

$$\mathbb{E}\left[\,|C_{3,3;2,1}|\,\right] = O\left(\beta^3 m^3 \cdot \beta^2 m^3 \cdot \alpha^{2(r-2)^2-1}\right) = O(\beta^2\alpha^{-1}\cdot d^2 t) = O(d^2 t),$$
$$\mathbb{E}\left[\,|C_{4,4;3,2}|\,\right] = O\left(\beta^4 m^4 \cdot \beta^3 m^4 \cdot \alpha^{3(r-2)^2-2}\right) = O(\beta^3\alpha^{-2}\cdot d^3 t) = O(d^3 t),$$
$$\mathbb{E}\left[\,|C_{4,5;2,1}|\,\right] = O\left(\beta^3 m^3 \cdot \beta^2 m^3 \cdot \alpha^{2(r-2)^2-1}\right) = O(\beta^2\alpha^{-1}\cdot d^2 t) = O(d^2 t),$$
$$\mathbb{E}\left[\,|C_{5,7;2,1}|\,\right] = O\left(\beta^3 m^3 \cdot \beta^2 m^3 \cdot \alpha^{2(r-2)^2-1}\right) = O(\beta^2\alpha^{-1}\cdot d^2 t) = O(d^2 t).$$

Finally, the obvious adaptation of this argument to the "exceptional" family $C'_{4,3;3,3}$ gives that

$$\mathbb{E}\left[\,|C'_{4,3;3,3}|\,\right] = O\left(\beta^4 m^5 \cdot \beta^3 m^3 \cdot \alpha^{3(r-2)^2-2}\right) = O(\beta^3\alpha^{-2}\cdot d^3 t) = O(d^3 t).$$

This finishes the proof of Claim 4.11. ■

Fix G_3 such that for each of the five families from Claim 4.11 its size is at most, say, 10 times the expected value, $|H| = \Theta(td)$ and $|G_3| \leqslant 2\alpha m^2$. This is possible since the last two properties hold with high probability by the Bounded Difference Inequality (Lemma 4.2) and the union bound.

Finally, let us describe how to construct a matching in H avoiding all conflicts listed in Claim 4.9. We shall use the probabilistic deletion method. Pick every edge of H randomly with probability μ/d, where μ is a sufficiently small constant which depends on the implicit constants in the asymptotic notations but not on α, that is, $1/r \gg \mu \gg \alpha$. Clearly, the expected number of chosen edges is $(\mu/d)|H| = \Theta(\mu t)$ and the expected number of pairs of edges which overlap is $O(d^2 t\cdot(\mu/d)^2) = O(\mu^2 t)$. By Claim 4.11, the expected number of elements of C all of whose edges have been chosen is at most

$$O\left(d^2 t\cdot(\mu/d)^2 + d^3 t\cdot(\mu/d)^3\right) = O(\mu^2 t).$$

Let M be obtained from the μ-random subset of $E(H)$ by removing edges which overlap with some other chosen edge or participate in a conflict from C with all of its edges being chosen. By construction, M is a matching in H that avoids all conflicts. Also,

$$\mathbb{E}\left[\,|M|\,\right] \geqslant \Omega(\mu t) - O(\mu^2 t) = \Omega(\mu t).$$

Take an outcome such that $|M|$ is at least its expectation and define $F = F(M)$ by (4.1).

Let us check that the obtained r-graph F satisfies all stated properties listed in Lemma 4.4. The first two, namely Parts (a) and (b), follow from Claims 4.7, 4.8 and 4.9. Also, the size of F is $2\,|M| = \Omega_\mu(t) = \Omega_\mu(\alpha^{3/4}m^2)$, proving Part (c).

Let us turn to Part (d). Since F consists of $|F|/2$ diamonds that do not share any pairs, we have that $|P_1(F)| = (2\binom{r}{2}-1)|F|/2$. Since $|G_3| \leqslant 2\alpha m^2$, it is enough to show that every pair $xy \in P_{\leqslant 3}(F)\setminus P_1(F)$ is an edge of G_3. Since F has no $(3r-4,3)$-configuration, we have that $xy \in P_{\bar{1}2}(F)$. Since every pair of cliques in $\mathcal{K}_1\cup\mathcal{K}_2$ shares at most one vertex, some diamond coming from $\{K_1,K_2\} \in M$ 2-claims the pair xy. Thus $xy \in \{ab : a \in K_1,\ b \in K_2\} \subseteq E(G_3)$, as desired. This finishes the proof of Lemma 4.4. ■

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Proof of the lower bounds in Theorem 1.1 and 1.2 with $r \geqslant 4$ Let F be the r-graph given by Lemma 4.4. Thus F is $\mathcal{G}_k^{(r)}$-free for $k \in \{5, 7\}$, $|F| = \Omega_r(\alpha^{3/4}m^2)$ and

$$|P_{\leqslant 3}(F)| \leqslant \frac{r^2 - r - 1}{2}|F| + 2\alpha m^2.$$

By Theorem 4.1, for each $k \in \{5, 7\}$, we have that

$$\liminf_{n\to\infty} n^{-2} f^{(r)}(n; k(r-2)+2, k) \geqslant \frac{|F|}{2\,|P_{\leqslant 3}(F)|} \geqslant \frac{1}{r^2 - r + 1 + O(\alpha^{1/4})}.$$

The lower bound $\frac{1}{r^2-r+1}$ in Theorems 1.1 and 1.2 follows by taking $\alpha \to 0$. ∎

4.2 Lower bounds in Theorems 1.3 and 1.4

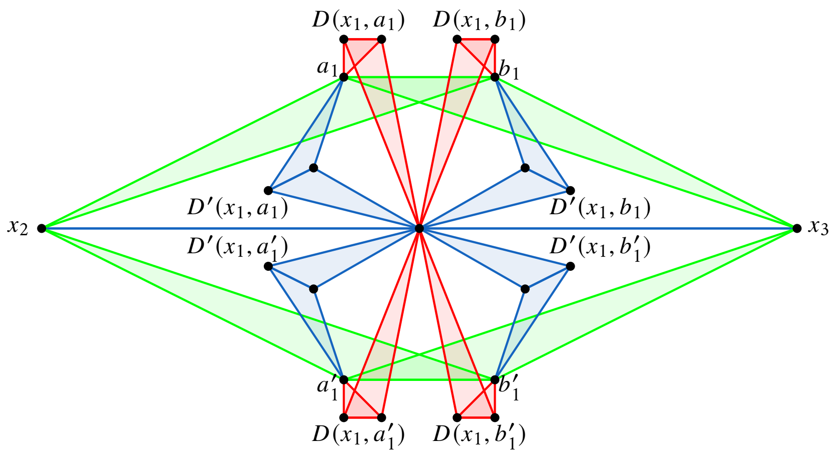

Figure 2: The figure depicts the subgraph of the 3-graph F_{63} "lying" on the pair x_2x_3. The central vertex in the figure is x_1, and the green diamonds correspond to D_1 and D_1'. Copies of the same construction "lie" on the pairs x_1x_2 and x_2x_3 in F_{63}..

Proof of the lower bound in Theorem 1.3 We define a 3-graph F_{63} on 63 vertices with 61 edges as follows. Let T be the 3-graph which is obtained from an edge $x_1x_2x_3$ by adding, for every $i \in [3]$, two diamonds $D_i = \{a_ib_ix_s, a_ib_ix_t\}$ and $D_i' = \{a_i'b_i'x_s, a_i'b_i'x_t\}$ where $\{s, t\} = [3] \setminus \{i\}$. We say that these 6 diamonds are of *level 1*. Then, consider the following 12 pairs, which do not belong to $P_1(T)$, namely

$$x_ia_i, x_ib_i, x_ia_i', x_ib_i', \quad \text{for } i \in [3]. \tag{4.12}$$

Let F_{63} be obtained from T by adding, for every such pair xy, two vertex-disjoint diamonds $D(x, y)$ and $D'(x, y)$ $\overline{1}2$-claiming xy, calling these 24 diamonds *of level 2*. Thus, F_{63} has $3 + 6 \cdot 2 + 24 \cdot 2 = 63$ vertices and $1 + 6 \cdot 2 + 24 \cdot 2 = 61$ edges.

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To show that F_{63} is $\mathcal{G}_6^{(3)}$-free, we first prove the following claim.

***Claim* 4.12** *For every subgraph $G \subseteq F_{63}$, there is an integer $t \geqslant 1$ and a sequence $\emptyset = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_t = G$ where, for every $i \in [0, t-1]$, $G_{i+1} \setminus G_i$ is either a single edge that shares at most one vertex with G_i, or a diamond that shares at most two vertices with G_i so that if a pair is shared, then this pair is $\overline{1}2$-claimed by $G_{i+1} \setminus G_i$.*

Proof of Claim 4.12 We start from G and consecutively delete single edges or diamonds with the required property, thus implicitly defining the sequence $G_1, G_2, \ldots, G_t$ in the reverse order. Suppose that G contains an edge X in some diamond $\{X, Y\}$ of level 2. If $Y \notin G$, then $G \setminus \{X\}$ and X share at most one vertex. Otherwise, $\{X, Y\}$ attaches to $G \setminus \{X, Y\}$ via at most two vertices and if there are exactly two shared vertices, then this pair is $\overline{1}2$-claimed by $\{X, Y\}$. Thus, the edge X (in the first case) and the diamond $\{X, Y\}$ (in the second case) can be deleted from $G = G_t$ to obtain G_{t-1}. Repeating this step, one can delete all edges of G from diamonds of level 2. After this, all edges in G from diamonds of level 1 can be deleted in a similar way, finishing the proof. ∎

Note that, for any sequence returned by Claim 4.12, the first non-empty r-graph G_1 contains two more vertices than edges and each new attachment cannot decrease this difference. It immediately follows that F_{63} is $(j+1, j)$-free for every j and, in particular, for $j \in [3, 5]$. Furthermore, if there exists some subgraph $G \subseteq F_{63}$ with 6 edges and at most 8 vertices, then the sequence $(G_i)_{i=0}^{r}$ returned by Claim 4.12 satisfies by the parity of $|G|$ that $t = 3$ and that each of $G_1, G_2 \setminus G_1$ and $G_3 \setminus G_2$ is a diamond. Let us argue that this is impossible. If the diamond G_1 is of level 1, say D_1, then the only possibility for the diamond $G_2 \setminus G_1$ is D_1', but then no other diamond D of F_{63} connects to $G_2 = D_1 \cup D_1'$ via a pair $\overline{1}2$-claimed by D, a contradiction. Similarly, if G_1 is of level 2, say $D(x_1, a_1)$, then $G_2 \setminus G_1$ must be $D'(x_1, a_1)$ and, again, there is no suitable choice for the third diamond $G_3 \setminus G_2$. We conclude that F_{63} is $(8, 6)$-free and thus $\mathcal{G}_6^{(3)}$-free.

Note that the only $(4, 2)$-configurations in F are the 6 diamonds of level 1 ($\overline{1}2$-claiming only the pairs already 1-claimed by the central edge $x_1x_2x_3$) and the 24 diamonds of level 2 ($\overline{1}2$-claiming the 12 pairs in (4.12)). Also, every $(5, 3)$-configuration in F_{63} consists of the central edge $x_1x_2x_3$ plus a diamond of level 1 (with both of its $\overline{1}3$-claimed pairs being already 2-claimed by diamonds of level 2). It follows that $P_{\leqslant 3}(F_{63})$ consists of $P_1(F_{63})$ and the 12 pairs in (4.12). Thus $|P_{\leqslant 3}(F_{63})| = 3+(6+24)\cdot 5+12 = 165$. Using Theorem 4.1, we derive that

$$\liminf_{n\to\infty} n^{-2} f^{(3)}(n; 8, 6) \geqslant \frac{|F_{63}|}{2\,|P_{\leqslant 3}(F_{63})|} = \frac{61}{330},$$

as desired. ∎

Proof of the lower bound in Theorem 1.4 For $r \geqslant 4$, the lower bound on $f^{(r)}(n; 6r-10, 6)$ follows by Theorem 4.1 with the r-graph F being a single edge, for which $P_{\leqslant 3}(F) = P_1(F)$ has size $\binom{r}{2}$. ∎

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5 Upper Bounds

5.1 Some common definitions and results

Recall that for an r-graph F and a pair uv, the set $C_F(uv)$ consists of all integers $j \geqslant 0$ such that F has j edges that together with uv include at most $rj - 2j + 2$ vertices. Note that, by definition, 0 always belongs to $C_G(uv)$, which is notationally convenient in the statement of the following easy but very useful observation.

***Lemma* 5.1** *For any $\mathcal{F}^{(r)}(rk - 2k + 2, k)$-free r-graph G, any $uv \in \binom{V(G)}{2}$ and any edge-disjoint subgraphs $F_1, \ldots, F_s \subseteq G$, the sum-set $\sum_{i=1}^{s} C_{F_i}(uv) = \left\{\sum_{i=1}^{s} m_i : m_i \in C_{F_i}(uv)\right\}$ does not contain k.*

Proof If some sequence of m_i's sums up to exactly k, then the corresponding k edges of G are all distinct and span at most $2 + \sum_{i=1}^{s}(r-2)m_i = rk - 2k + 2$ vertices, a contradiction. ■

As we mentioned in the introduction, our proof strategy to bound the size of an $(rk - 2k + 2, k)$-free r-graph G from above is to analyse possible isomorphism types of the parts of some partition of $E(G)$ which is obtained from the trivial partition into single edges by iteratively applying some merging rules. Unfortunately, we did not find a single rule that works in all cases that are studied here. In fact, for every new pair (r, k) resolved in this paper except for $(3, 5)$, we build the final partition in stages (with each stage having a different merging rule) as the intermediate families are also needed in our analysis. Let us now develop some general notation and prove some basic results related to merging.

Let G be an arbitrary r-graph. When dealing with a partition $\mathcal{P}$ of $E(G)$, we will view each element $F \in \mathcal{P}$ as an r-graph whose vertex set is the union of the edges in F. Let $A, B \subseteq \mathbb{N}$ be any (not necessarily disjoint) sets of positive integers. For two subgraphs $F, H \subseteq G$, if they are edge disjoint and there is a pair uv such that $A \subseteq C_F(uv)$ and $B \subseteq C_H(uv)$, then we say that F and H are *$(A|B)$-mergeable (via uv)*. Note that this relation is not symmetric in F and H: the first (resp. second) r-graph A-claims (resp. B-claims) the pair uv. When the ordering of the two r-graphs does not matter, we use the shorthand *$A|B$-mergeable* to mean $(A|B)$-mergeable or $(B|A)$-mergeable. For a partition $\mathcal{P}$ of $E(G)$, its *$A|B$-merging* is the partition $\mathcal{M}_{A|B}(\mathcal{P})$ of $E(G)$ obtained from $\mathcal{P}$ by iteratively and as long as possible taking a pair of distinct $A|B$-mergeable parts in the current partition and replacing them by their union. Note that, if F and H are $A|B$-mergeable via uv, then

$$A \cup B \subseteq C_{F \cup H}(uv) \quad \text{and} \quad A + B \subseteq C_{F \cup H}(uv). \tag{5.1}$$

The first inclusion implies that, in particular, the order of merging operations does not affect the partition $\mathcal{M}_{A|B}(\mathcal{P})$. Note that the final partition $\mathcal{M}_{A|B}(\mathcal{P})$ is a coarsening of $\mathcal{P}$ and contains no $A|B$-mergeable pairs of r-graphs. When $\mathcal{P}$ is clear from the context, we may refer to the elements of $\mathcal{M}_{A|B}(\mathcal{P})$ as *$A|B$-clusters*. Likewise, a subgraph F of G that can appear as a part in some intermediate stage of the $A|B$-merging process starting with $\mathcal{P}$ is called a *partial $A|B$-cluster* and we let $\mathcal{M}'_{A|B}(\mathcal{P})$ denote the set of all

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partial $A|B$-clusters. In other words, $\mathcal{M}'_{A|B}(\mathcal{P})$ is the smallest family of r-graphs which contains $\mathcal{P}$ as a subfamily and is closed under taking the union of $A|B$-mergeable elements. The monotonicity of the merging rule implies that $\mathcal{M}_{A|B}(\mathcal{P})$ is exactly the set of maximal (by inclusion) elements of $\mathcal{M}'_{A|B}(\mathcal{P})$ and that the final partition $\mathcal{M}_{A|B}(\mathcal{P})$ does not depend on the order in which we merge parts.

In the frequently occurring case when $A=\{1\}$ and $B=\{j\}$, we abbreviate $(\{1\}|\{j\})$ to (j) and $\{1\}|\{j\}$ to j in the above nomenclature. Thus, (j)-mergeable (resp. j-mergeable) means $(\{1\}|\{j\})$-mergeable (resp. $\{1\}|\{j\}$-mergeable).

As an example, let us look at the following merging rule that is actually used as the first step in each of our proofs of the upper bounds. Namely, given G, let

$$\mathcal{M}_1 := \mathcal{M}_{\{1\}|\{1\}}(\mathcal{P}_{\text{trivial}})$$

be the 1-merging of the trivial partition $\mathcal{P}_{\text{trivial}}$ of G into single edges. We call the elements of $\mathcal{M}_1$ 1-*clusters*. Here is an alternative description of $\mathcal{M}_1$. Call a subgraph $F\subseteq G$ *connected* if for any two edges $X,Y\in F$ there is a sequence of edges $X_1=X,X_2,\dots,X_m=Y$ in F such that, for every $i\in[m-1]$, we have $|X_i\cap X_{i+1}|\geqslant 2$. Then, 1-clusters are exactly maximal connected subgraphs of G (and partial 1-clusters are exactly connected subgraphs).

We will also use (often without explicit mention) the following result, which is a generalisation of the well-known fact that we can remove edges from any connected 2-graph one by one, down to any given connected subgraph, while keeping the edge set connected. The assumption (5.2) states that, roughly speaking, the merging process cannot create any new mergeable pairs.

***Lemma* 5.2** (*Trimming Lemma*) *Fix an r-graph G, a partition $\mathcal{P}$ of $E(G)$ and sets $A,B\subseteq\mathbb{N}$.*

$$\begin{gathered}\textit{Suppose that, for all } (A|B)\textit{-mergeable (and thus edge-disjoint) } F,H\in\mathcal{M}'_{A|B}(\mathcal{P}),\\ \textit{there exist } (A|B)\textit{-mergeable } F',H'\in\mathcal{P} \textit{ such that } F'\subseteq F \textit{ and } H'\subseteq H.\end{gathered}\tag{5.2}$$

Then, for every partial $A|B$-cluster $F_0\subseteq F$, there is an ordering $F_1,\dots,F_s$ of the elements of $\mathcal{P}$ that lie inside $F\setminus F_0$ such that, for every $i\in[s]$, $\bigcup_{j=0}^{i-1}F_j$ and F_i are $A|B$-mergeable (and, in particular, $\bigcup_{j=0}^{i}F_j$ is a partial $A|B$-cluster for every $i\in[s]$).

Proof Suppose that the lemma is false. For every $H\in\mathcal{M}'_{A|B}(\mathcal{P})$, denote by $|H|_{\mathcal{P}}$ the number of elements of $\mathcal{P}$ contained in H. Choose a counterexample (F_0,F) with $m:=|F|_{\mathcal{P}}-|F_0|_{\mathcal{P}}$ smallest possible. Note that $m>0$ as otherwise $F_0=F$ and the conclusion of the lemma vacuously holds. Consider some $A|B$-merging process which leads to F; let H be the first occurring partial $A|B$-cluster that shares at least one edge with each of F_0 and $F\setminus F_0$. Since the final partial $A|B$-cluster F satisfies both these properties, H exists. As F_0 and $F\setminus F_0$ are unions of some elements of $\mathcal{P}$, we have $H\notin\mathcal{P}$. So, let the merging process for F build H as the union of $(A|B)$-mergeable partial $A|B$-clusters $H_A,H_B\subsetneq H$. By the definition of H, one of H_A and H_B, say H_A, shares an edge with F_0 but not with $F\setminus F_0$ while the opposite holds for H_B. Then, $H_A\subseteq F_0$ and $H_B\subseteq F\setminus F_0$.

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Since the partial $A|B$-clusters H_A and H_B are $(A|B)$-mergeable, the assumption of the lemma implies that there are $(A|B)$-mergeable $H'_A \subseteq H_A$ and $H'_B \subseteq H_B$ with $H'_A, H'_B \in \mathcal{P}$. Then, H'_B, which is edge-disjoint from F_0, is $A|B$-mergeable with $F_0 \supseteq H_A$ as well. The minimality of m guarantees an ordering $F_2, \ldots, F_m$ of the elements of $\mathcal{P}$ that lie inside $F \setminus (F_0 \cup H'_B)$ satisfying the statement of the claim for the pair $(F_0 \cup H'_B, F)$. However, then the ordering $H'_B, F_2, \ldots, F_m$ satisfies the statement of the claim for (F_0, F), so this pair cannot be a counterexample, on the contrary to our assumption. ■

In the special case $A = B = \{1\}$ (when partial clusters are just connected subgraphs), the assumption of Lemma 5.2 is vacuously true. Since we are going to use its conclusion quite often, we state it separately.

***Corollary* 5.3** *For every pair $F_0 \subseteq F$ of connected r-graphs, there is an ordering $X_1, \ldots, X_s$ of the edges in $F \setminus F_0$ such that, for every $i \in [s]$, the r-graph $F_0 \cup \{X_1, \ldots, X_i\}$ is connected.* ■

We say that an r-graph is a 1-*tree* if it contains only one edge. For $i \geqslant 2$, we recursively define an i-*tree* as any r-graph that can be obtained from an $(i-1)$-tree T by adding a new edge that consists of a pair ab in the 2-shadow $P_1(T)$ of T and $r-2$ new vertices (not present in T). Clearly, every i-tree is connected. Like the usual 2-graph trees, i-trees are the "sparsest" connected r-graphs of given size. Any i-tree T satisfies

$$|P_1(T)| = i\binom{r}{2} - i + 1 \quad \text{and} \quad |P_{\bar{1}2}(T)| \geqslant (i-1)(r-2)^2. \tag{5.3}$$

(Recall that $P_{\bar{1}2}(T)$ is the set of pairs which are 2-claimed but not 1-claimed by T.) Note that the second inequality in (5.3) is equality if, for example, T is an i-*path*, that is, we can order the edges of T as $X_1, \ldots, X_i$ so that, for each $j \in [i-1]$, the intersection $X_{j+1} \cap (\cup_{s=1}^{j} X_s)$ consists of exactly one pair of vertices and this pair belongs to $P_1(X_j) \setminus P_1(\cup_{s=1}^{j-1} X_s)$.

The following result shows that the 1-clusters of any $\mathcal{G}_k^{(r)}$-free graph have a very simple structure: namely, they are all small trees.

***Lemma* 5.4** *With the above notation, if G is $\mathcal{G}_k^{(r)}$-free, then every $F \in \mathcal{M}_1$ is an m-tree for some $m \in [k-1]$.*

Proof Since F is connected, Corollary 5.3 gives an ordering $X_1, \ldots, X_m$ of its edge set such that, for each $i \in [2, m]$, the i-th edge X_i shares at least two vertices with some earlier edge. By induction, the number of vertices spanned by $\{X_1, \ldots, X_i\}$ is at most $i(r-2)+2$ for each $i \in [m]$. Since F is $(k(r-2)+2, k)$-free, we have $m < k$. Furthermore, for each $i \in [2, m]$, the edge X_i has at least $r-2$ vertices not present in the previous edges by the $(i(r-2)+1, i)$-freeness of G. It follows that F is an m-tree, as desired. ■

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5.2 Upper bound for $(r,k) = (3,5)$

We begin with the case $(r,k) = (3,5)$, which is simpler but still embodies some key ideas that also apply to higher uniformities.

Proof of the upper bound of Theorem 1.1 for $r = 3$ By Lemma 3.1, it is enough to prove that $|G| \leqslant n^2/5$ for every 3-graph on n vertices which is $\mathcal{G}_5^{(3)}$-free, that is, contains no $(7,5)$, $(5,4)$ and $(4,3)$-configurations. Recall that $\mathcal{M}_1$ denotes the partition of $E(G)$ into 1-clusters. By Lemma 5.4, each 1-cluster is an i-tree with $i \leqslant 4$ edges.

For an i-tree F, consider the difference $2\,|P_1(F)| - 5\,|F| = 2(2i+1) - 5i = 2 - i$, which is non-negative when $i \in \{1,2\}$. For $i \in \{3,4\}$, we would like to take an extra pair (in addition to those in $P_1(F)$) into account to make the difference non-negative. The following combinatorial lemma suffices for this.

***Claim* 5.5** *For every i-tree $F \in \mathcal{M}_1$ with $i \in \{3,4\}$, there is a pair which is $\overline{1}2$-claimed by F but neither 1-claimed nor 2-claimed by $G \setminus F$.*

Proof of Claim 5.5 Note that no pair xy can be 2-claimed by both F and $G \setminus F$, for otherwise we can find a 3-subtree in F (by Corollary 5.3) 3-claiming xy. This would mean that $5 \in C_G(xy)$ by (5.1), which contradicts Lemma 5.1.

If $i = 3$, then F $\overline{1}2$-claims at least 2 pairs and at least one of them is not 1-claimed by $G \setminus F$: indeed, if they are 1-claimed by different edges, then we get a $(7,5)$-configuration, and if they are 1-claimed by the same edge, then we get a $(5,4)$-configuration, a contradiction in either case. If $i = 4$, then F $\overline{1}2$-claims at least 3 pairs and, in fact, none can be 1-claimed by $G \setminus F$ (as otherwise we would have a $(7,5)$-configuration). ∎

Now, for each i-tree $F \in \mathcal{M}_1$, define $P'_1(F)$ to be $P_1(F)$ if $i \in \{1,2\}$, and to be $P_1(F)$ plus the pair returned by Claim 5.5 if $i \in \{3,4\}$.

The sets $P'_1(F)$ for $F \in \mathcal{M}_1$ are pairwise disjoint and satisfy $2\,|P'_1(F)| \geqslant 5\,|F|$. Thus,

$$|G| = \sum_{F \in \mathcal{M}_1} |F| \leqslant \frac{2}{5} \sum_{F \in \mathcal{M}_1} |P'_1(F)| \leqslant \frac{2}{5}\binom{n}{2} < \frac{n^2}{5},$$

giving the desired. ∎

5.3 Upper bounds of Theorems 1.1 and 1.2

In this section, we shall prove the remaining upper bounds of Theorems 1.1 and 1.2, that is, the cases when $k = 5$ and $r \geqslant 4$, or $k = 7$ and $r \geqslant 3$. First, let us present some definitions and auxiliary results that are common to the proofs of both theorems.

Let $k \in \{5,7\}$ and let G be an arbitrary $\mathcal{G}_k^{(r)}$-free r-graph. Recall that $\mathcal{M}_1$ denotes the partition of $E(G)$ into 1-clusters.

We say that a subgraph $H \subseteq G$ 2^+-*claims* a pair $uv \in \binom{V(G)}{2}$ if H has a subtree T with 3 edges that $\{2,3\}$-claims uv (which, by Corollary 5.3, is equivalent to T 2-claiming the pair uv). Let us say that two edge-disjoint subgraphs $F, H \subseteq G$ are (2^+)-*mergeable (via*

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a pair uv) if uv is 1-claimed by F and is 2^+-claimed by H. If the order of F and H does not matter, we just say 2^+*-mergeable*. Let $\mathcal{M}_{2^+}$ be the partition of $E(G)$ obtained from $\mathcal{M}_1$ by iteratively and as long as possible taking two 2^+-mergeable elements F and H and replacing them by $F \cup H$. Let $\mathcal{M}'_{2^+}$ be the smallest family of subgraphs of G that contains all 1-clusters and is closed under 2^+-merging. We call the elements of $\mathcal{M}'_{2^+}$ *partial* 2^+*-clusters*. By the monotonicity of the merging rule, the family $\mathcal{M}_{2^+}$ consists of the maximal elements of $\mathcal{M}'_{2^+}$ and does not depend on the order of the merging steps.

Note that the relations of being (2^+)-mergeable and $(\{1\}|\{2,3\})$-mergeable, while having many similarities, differ in general (e.g. a subgraph with 3 edges 3-claiming a pair need not be a tree). As far as we see, the latter relation can also be used to prove the upper bounds for $k \in \{5,7\}$ but the former is more convenient for us to work with.

If a partial 2^+-cluster F 2^+-claims a pair uv as witnessed by a 3-tree $T \subseteq F$, then trivially T is a subgraph of one of the 1-clusters in F and this 1-cluster 2^+-claims the pair uv. Since the analogous statement for 1-claimed pairs is trivial, we have the analogue of Assumption (5.2) for (2^+)-mergeability (with $\mathcal{P} = \mathcal{M}_1$). The proof of Lemma 5.2 with the obvious modifications works for partial 2^+-clusters. We will need only the following special case.

***Lemma* 5.6** *For every $F \in \mathcal{M}'_{2^+}$ and $T \in \mathcal{M}_1$ such that $T \subseteq F$, there is an ordering $T_1, \ldots, T_t$ of all* 1*-clusters in F such that $T_1 = T$ and, for every $i \in [t-1]$, T_{i+1} and $\bigcup_{j=1}^{i} T_j$ are* 2^+*-mergeable. (In particular, $\bigcup_{j=1}^{i} T_j \in \mathcal{M}'_{2^+}$ for every $i \in [t]$.)* ∎

We say that an r-graph F $\overline{1}2^+$*-claims* a pair uv if it 2^+-claims but not 1-claims uv, and denote the set of all such pairs by $P_{\overline{1}2^+}(F)$. Note that, if we build 2^+-clusters by starting with 1-clusters and merge some (2^+)-mergeable F and H via uv, then the pair uv is $\overline{1}2^+$-claimed by H. An integer sequence $(e_1, \ldots, e_t)$ is called a *composition* of a partial 2^+-cluster $F \in \mathcal{M}'_{2^+}$ if there is a sequence $(T_1, \ldots, T_t)$ as in Lemma 5.6 with $|T_i| = e_i$ for each $i \in [t]$. Of course, every two compositions of the same 2^+-cluster are permutations of each other. *The (non-increasing) composition* of F is the non-increasing reordering of $(e_1, \ldots, e_t)$; in general, there need not be a sequence of iterative legal merges realising it.

Proof of the upper bound of Theorem 1.1 for $r \geqslant 4$ By Lemma 3.1, it is enough to bound the size of any $\mathcal{G}_5^{(r)}$-free r-graph G on $[n]$ from above. As before, $\mathcal{M}_1$ (resp. $\mathcal{M}_{2^+}$) denotes the set of 1-clusters (resp. 2^+-clusters) of G. By Lemma 5.4, each 1-cluster is an i-tree with $i \in [4]$.

Let every 2^+-cluster $F \in \mathcal{M}_{2^+}$ assign weight 1 to every pair in $P_1(F)$ and in $P_{\overline{1}2^+}(F)$. Note that every pair xy receives weight at most 1. Indeed, suppose for contradiction that xy receives weight 1 from two different 2^+-clusters F and H. Then, xy must be $\overline{1}2^+$-claimed by both (as otherwise F and H would be merged together). However, this implies that $\{2,3\} \subseteq C_F(xy) \cap C_H(xy)$, which contradicts Lemma 5.1.

Thus, in order to prove the theorem, it is enough to show that, for every $F \in \mathcal{M}_{2^+}$, we have $\lambda(F) \geqslant 0$ where

$$\lambda(F) := 2\left(|P_1(F)| + |P_{\overline{1}2^+}(F)|\right) - (r^2 - r - 1)|F|.$$

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Indeed, we will then have that

$$|G| = \sum_{F \in \mathcal{M}_{2^+}} |F| \leqslant \frac{2}{r^2-r-1}\left(|P_1(F)| + |P_{\overline{1}2^+}(F)|\right) \leqslant \frac{2}{r^2-r-1}\binom{n}{2}. \qquad (5.4)$$

***Claim* 5.7** *Every $F \in \mathcal{M}_{2^+}$ has at most four edges.*

Proof of Claim 5.7 Suppose for the sake of contradiction that $|F| \geqslant 5$ (and thus $|F| \geqslant 6$ since F contains at most $(r-2)|F|+2$ vertices and G is $(5r-8,5)$-free). Let F be obtained by merging 1-clusters $F_1, \ldots, F_m \in \mathcal{M}_1$ in this order as in Lemma 5.6. Let us stop the merging process when we reach a partial 2^+-cluster containing at least 5 (and thus at least 6) edges. Suppose that we have merged $F_1, \ldots, F_s$ until this point. By Lemma 5.4, we have $|F_s| \leqslant 4$ and thus $s \geqslant 2$.

We claim that the last tree F_s has exactly three edges. As G is $(5r-8,5)$-free, it holds that $|\bigcup_{i=1}^{s-1} F_i| \neq 5$. Thus, F_s has at least two edges. In fact, F_s cannot have exactly two edges as otherwise the subgraph $\bigcup_{i=1}^{s-1} F_i$ of size 4 would 4-claim a pair in $P_1(F_s)$, contradicting Lemma 5.1. Also, the tree F_s cannot have 4 edges as otherwise any 2^+-merging involving it would lead to a $(5r-8,5)$-configuration and we would have $s=1$, a contradiction.

Next, we can build the partial 2^+-cluster $\bigcup_{i=1}^{s} F_i$ by starting with $H_1 := F_s$ as in Lemma 5.6; let us stop here at the first moment when the current partial 2^+-cluster H, say composed of $H_1, \ldots, H_t \in \{F_1, \ldots, F_s\}$ in this order, has at least five edges. As before, we have that $|H_t| = 3$. By $(5r-8,5)$-freeness, the sizes of the 1-clusters in H in the order of merging are either $(3,3)$ or $(3,1,3)$. In the former (resp. latter) case, we can remove an edge from one (resp. each) of the 3-trees H_1 and H_t so that the remaining subgraph is a diamond $\overline{1}2$-claiming the pair along which this tree attaches to the rest of H. This way, we can find a sequence of trees of sizes $(3,2)$ or $(2,1,2)$ with each one containing some 2 previously used vertices. This gives a forbidden 5-edge configuration, proving that $|F| \leqslant 4$ for every $F \in \mathcal{M}_{2^+}$. ■

It follows that, if $F \in \mathcal{M}_{2^+}$ is not a tree, then F is made of a single edge and a 3-tree which must share exactly two vertices (for otherwise a $(4r-7,4)$-configuration appears). In this case, we have that $|P_1(F)| = 4\binom{r}{2} - 2$ and $|P_{\overline{1}2^+}(F)| \geqslant 2(r-2)^2 - 1$. Thus,

$$\lambda(F) \geqslant 2\left(4\binom{r}{2} - 2 + 2(r-2)^2 - 1\right) - (r^2-r+1)\cdot 4 = 4r^2 - 16r + 6,$$

which is positive for $r \geqslant 4$.

Let $F \in \mathcal{M}_{2^+}$ be an i-tree. If $i \geqslant 3$, then $|P_{\overline{1}2^+}(F)| \geqslant (i-1)(r-2)^2$ and we have

$$\begin{aligned} \lambda(F) &\geqslant 2\left(i\binom{r}{2} - i + 1 + (i-1)(r-2)^2\right) - (r^2-r-1)i \\ &= i\left(2r^2 - 8r + 7\right) - \left(2r^2 - 8r + 6\right). \end{aligned}$$

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For $r \geqslant 4$, this expression is monotone increasing in i and thus is at least its value when $i = 3$, which is $4r^2 - 16r + 15 \geqslant 15$. Finally, if $i = 1, 2$, then $\lambda(F)$ is respectively $2\binom{r}{2} - (r^2 - r - 1) = 1$ and $2\left(2\binom{r}{2} - 1\right) - 2(r^2 - r - 1) = 0$.

Hence, for all $F \in \mathcal{M}_{2^+}$, we have that $\lambda(F) \geqslant 0$, which finishes the proof of the theorem by (5.4). ■

Proof of the upper bound of Theorem 1.2 By Lemma 3.1, it is sufficient to bound the size of any $\mathcal{G}_7^{(r)}$-free r-graph G of order n from above. As before, $\mathcal{M}_1$ (resp. $\mathcal{M}_{2^+}$) is the set of all 1-clusters (resp. 2^+-clusters) of G. By Lemma 5.4, each element of $\mathcal{M}_1$ is an i-tree with $i \in [6]$.

Let each 2^+-cluster $F \in \mathcal{M}_{2^+}$ assign weight 1 to each pair it 1-claims and weight $1/2$ to each pair it $\overline{1}2^+$-claims. Let us show that every pair $xy \in \binom{V(G)}{2}$ receives weight at most 1. If a pair of vertices is 1-claimed by some 2^+-cluster, then it cannot be 1-claimed or 2^+-claimed by another 2^+-cluster as otherwise this would violate the merging rules for $\mathcal{M}_1$ or $\mathcal{M}_{2^+}$. Furthermore, a pair of vertices xy cannot be $\{2, 3\}$-claimed by three 2^+-clusters by Lemma 5.1. So xy indeed receives weight at most 1.

Thus, in order to prove the upper bound, it is sufficient to show that each 2^+-cluster F satisfies that

$$\lambda(F) := 2w(F) - (r^2 - r - 1)|F| \geqslant 0, \tag{5.5}$$

where $w(F) := |P_1(F)| + \frac{1}{2}|P_{\overline{1}2^+}(F)|$ is the total weight assigned by F to the vertex pairs. Indeed, we would then be done since

$$|G| = \sum_{F \in \mathcal{M}_{2^+}} |F| \leqslant \sum_{F \in \mathcal{M}_{2^+}} \frac{2}{r^2 - r - 1}\, w(F) \leqslant \frac{2}{r^2 - r - 1}\binom{n}{2}.$$

To show (5.5), we first prove the following claim.

***Claim* 5.8** *For each $F \in \mathcal{M}_{2^+}$, we have $|F| \leqslant 6$.*

Proof of Claim 5.8 Suppose for contradiction that $F \in \mathcal{M}_{2^+}$ has at least 7 edges. Since F is $(7r - 12, 7)$-free, we have $|F| \geqslant 8$. Let F be obtained by merging m distinct 1-clusters $T_1, \ldots, T_m \in \mathcal{M}_1$ in this order as in Lemma 5.6.

Let $s \in [m]$ be the first index satisfying $|\bigcup_{i=1}^{s} T_i| \geqslant 8$. Then, $|\bigcup_{i=1}^{s-1} T_i| \leqslant 6$ as F is $(7r - 12, 7)$-free. Hence, $|T_s| \geqslant 2$. It is impossible that T_s and $T_1 \cup \ldots \cup T_{s-1}$ are (2^+)-mergeable as otherwise we could trim edges from T_s using Corollary 5.3 to get a $(7r - 12, 7)$-configuration in F, a contradiction. Thus, T_s 2^+-claims some pair xy 1-claimed by $\bigcup_{i=1}^{s-1} T_i$; in particular, $|T_s| \geqslant 3$. Let D be the diamond in T_s 2-claiming xy. We note that $|(\bigcup_{i=1}^{s-1} T_i) \cup D| \geqslant 8$ since otherwise we could trim edges in $T_s \setminus D$ using Corollary 5.3 to obtain a $(7r - 12, 7)$-configuration. Thus, $|\bigcup_{i=1}^{s-1} T_i| = 6$. Let $T'_1 := T_s$ and let T'_2 be any 1-cluster in $\{T_1, \ldots, T_{s-1}\}$ which is 2^+-mergeable with T'_1. By Lemma 5.6, we can obtain the partial 2^+-cluster $\bigcup_{i=1}^{s-1} T_i$ by starting with T'_2 and 2^+-merging the remaining 1-clusters one at a time, say $T'_3, \ldots, T'_s$ in this order. Let $t \in [s]$ be the smallest index such that $T'_1 \cup \cdots \cup T'_t$ has at least 7 edges. By the same argument as before, we derive that $|\bigcup_{i=1}^{t-1} T'_i| = 6$ and $|T'_t| \geqslant 3$. Also, we can trim edges one by one from each of the "pendant" 1-clusters T'_1 and T'_t down to a diamond so that each intermediate subgraph is always a tree that shares 2 vertices with the partial 2^+-cluster

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$F' := \bigcup_{i=2}^{t-1} T_i'$. Since $|F'| \leqslant 6-|T_1'| \leqslant 3$, we must encounter a $(7r-12, 7)$-configuration inside $\bigcup_{i=1}^{t} T_i'$ in this process, a contradiction. ■

Now, we prove (5.5) for every 2^+-cluster $F \in \mathcal{M}_{2^+}$.

Take any 2^+-merging sequence $T_1, \ldots, T_m$ for F as in Lemma 5.6. For $i \in [m]$, let $e_i := |T_i|$. Thus, $(e_1, \ldots, e_m)$ is a composition of F. Then,

$$|P_{\overline{1}2^+}(F)| \geqslant 1 - m + \sum_{e_i \geqslant 3} (e_i - 1)(r-2)^2,$$

since each T_i shares exactly two vertices with $\bigcup_{j=1}^{i-1} T_j$ (as otherwise there would be an $(r\ell - 2\ell + 1, \ell)$-configuration in G for some $\ell \in [2, 6]$). Thus, by (5.3), we have that

$$\begin{aligned}\lambda(F) &\geqslant 2\sum_{i=1}^{m}\left(e_i\binom{r}{2} - e_i + 1\right) + 1 - m + \sum_{e_i \geqslant 3}(e_i - 1)(r-2)^2 - (r^2 - r - 1)\sum_{i=1}^{m} e_i \\ &= 1 + (r-2)^2 \sum_{e_i \geqslant 3}(e_i - 1) - \sum_{i=1}^{m}(e_i - 1). \end{aligned} \tag{5.6}$$

Our goal is to show that (5.6) is non-negative. Let us denote by $x = x(F)$ the number of diamonds in the merging sequence of F, that is, the number of $i \in [m]$ with $e_i = 2$. Note that $x \leqslant 1$: indeed, if $m \geqslant 2$, then $\max(e_1, e_2) \geqslant 3$ (in order for the first merging to occur) and Claim 5.8 implies that $x \leqslant \lfloor \frac{6-3}{2} \rfloor = 1$. Since the contribution of each $e_i \neq 2$ to the right-hand side of (5.6) is non-negative, the expression there is at least $1 - x \geqslant 0$, as desired. ■

5.4 Upper bounds for $k = 6$

Here we set $k = 6$. We will continue using the definitions of Section 5.1 for a given $\mathcal{G}_6^{(r)}$-free r-graph G. In particular, recall that $\mathcal{M}_1$ denotes the set of 1-clusters of G. However, unlike in the cases $k = 5, 7$, diamonds in the final partition would have to assign some positive weight to $\overline{1}2$-claimed pairs as otherwise the best we could hope for would be only $|G| \leqslant (\frac{1}{r^2-r-1} + o(1))n^2$, which is strictly larger than the desired upper bound. This brings extra challenges to proving that each pair of vertices receives weight at most 1. We resolved this by using a different merging rule. Namely, in addition to the partition $\mathcal{M}_1$ of $E(G)$ into 1-clusters, we will also use the partition

$$\mathcal{M}_2 := \mathcal{M}_{\{1\}|\{2\}}(\mathcal{M}_1),$$

which is obtained from the partition $\mathcal{M}_1$ by iteratively and as long as possible combining (2)-mergeable pairs, that is, two current parts such that the first 1-claims and the second 2-claims the same pair. Also, we define $\mathcal{M}_2'$ to consist of all subgraphs of G that may appear at any stage of this process, calling the elements of $\mathcal{M}_2$ (resp. $\mathcal{M}_2'$) *2-clusters* (resp. *partial 2-clusters*).

Let us observe some basic properties of (partial) 2-clusters. If some two partial 2-clusters F and H are (2)-mergeable via a pair uv then it holds that $uv \notin P_1(H)$ (as otherwise 1-clusters of F and H 1-claiming the pair uv would have been merged when

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constructing $\mathcal{M}_1$). Likewise, if a partial 2-cluster F 2-claims uv then the pair uv is 2-claimed by one of the 1-clusters that make F. Thus, Assumption (5.2) of Lemma 5.2 holds and the conclusion of the lemma applies here.

For $F \in \mathcal{M}_2'$ which is made by merging 1-clusters $F_1, \ldots, F_m$ in this order as in Lemma 5.2, we call the sequences of sizes $(|F_1|, \ldots, |F_m|)$ a *composition* of F. Its non-increasing reordering is called *the (non-increasing) composition* of F.

Also, recall that, by Lemma 5.1, it holds for any edge-disjoint subgraphs $F_1, \ldots, F_s \subseteq G$ and any $xy \in \binom{V(G)}{2}$ that

$$6 \notin \sum_{i=1}^{s} C_{F_i}(xy). \tag{5.7}$$

Next, in Lemma 5.9 below, we derive some combinatorial properties that every partial 2-cluster has to satisfy and that will suffice for our estimates. (An exact description of all possible partial 2-clusters is possible, with some extra work.) Since the proof of the lemma does not introduce any new ideas in addition to the ones seen before, the reader may skip it in the first reading.

***Lemma* 5.9** *Let $r \geqslant 3$ and G be an arbitrary $\mathcal{G}_6^{(r)}$-free r-graph. Let a partial 2-cluster F be obtained by merging the elements $T_1, \ldots, T_m$ of $\mathcal{M}_1$ in this order as in Lemma 5.2. Let $(e_1, \ldots, e_m)$ be the composition of F, that is, the non-increasing reordering of $(|T_1|, \ldots, |T_m|)$. Then, each of the following statements holds.*

(a) If F has at least 7 edges, then $(e_1, \ldots, e_m)$ is either $(3, 2, 2)$ or $(2, \ldots, 2, 1)$ with at most $r(r-1)$ entries equal to 2. If, moreover, T_1 is the unique 1-cluster of size different from 2 (that is, $|T_1| = 1$ or 3) then, for each $i \in [2, m]$, $H_i := \bigcup_{j=1}^{i-1} T_j$ and T_i are (2)-mergeable via some pair $xy \in P_{\overline{1}2}(T_i)$ while no other pair in $P_{\overline{1}2}(T_i)$ is 1-claimed or 2-claimed by H_i.

(b) It holds that $|P_{\overline{1}2}(F)| \geqslant 1 - m + \sum_{i=1}^{m}(e_i - 1)(r-2)^2$.

(c) If $(e_1, \ldots, e_m) = (2, 1, 1, 1)$ then no pair in $P_{\overline{1}2}(F)$ is 1-claimed or 2-claimed by $G \setminus F$.

Proof Suppose first that the partial 2-cluster F has at least 7 edges. We prove the first two claims of the lemma for this F simultaneously.

Let $s \in [m]$ be the first index such that $|\bigcup_{i=1}^{s} T_i| \geqslant 7$. Then, $|H_s| \leqslant 5$ as F is $(6r-10, 6)$-free, and hence, $|T_s| \geqslant 2$. If T_s and H_s are (2)-mergeable, then by Corollary 5.3 we can remove some edges from T_s to get a $(6r-10, 6)$-configuration inside $H_s \cup T_s$, a contradiction. Hence, T_s must $\overline{1}2$-claim some pair xy 1-claimed by H_s. Let $D \subseteq T_s$ be the (unique) diamond $\overline{1}2$-claiming xy. Note that $|H_s \cup D| \geqslant 7$ as otherwise Corollary 5.3 implies that we could remove some edges from $T_s \setminus D$ one by one to obtain a $(6r-10, 6)$-configuration. Thus, $|H_s| = 5$.

Let $T_1' := T_s$ and let $T_2' \in \{T_1, \ldots, T_{s-1}\}$ be a 1-cluster 2-mergeable with T_s. Let $(T_2', \ldots, T_s')$ be the ordering of $\{T_1, \ldots, T_{s-1}\}$ returned by Lemma 5.2 for the partial 2-clusters $T_2' \subseteq \bigcup_{i=1}^{s-1} T_i$. By the choice of T_2', for each $i = 2, \ldots, s$ the 1-cluster T_i' is 2-mergeable with the partial 2-cluster $\bigcup_{j=1}^{i-1} T_j'$. Let $t \in [s]$ be the first index such that $|\bigcup_{i=1}^{t} T_i'| \geqslant 7$. Set $H' := \bigcup_{i=1}^{t-1} T_i'$. By the same argument as in the previous paragraph, we have that $|H'| = 5$ and there is a diamond $D' \subseteq T'_t$ such that H' and D' are (2)-mergeable.

Thus, we have a partial 2-cluster $F' := \bigcup_{i=1}^{t} T'_i$ with at least 7 edges built via the sequence $(T'_1, T'_2, \dots, T'_t)$ so that the first 1-cluster $T'_1 = T_s$ (resp. the last 1-cluster T'_t) can be merged with the rest through only one pair, which is $\overline{1}2$-claimed by the diamond $D \subseteq T_s$ (resp. $D' \subseteq T'_t$). Here we have the freedom to trim one or both of these two clusters, leaving any number of edges in each except exactly 1 edge. It routinely follows that $|T'_1| = |T'_t| = 2$ (that is, $T'_1 = D$ and $T'_t = D'$). For example, if $(|T'_1|, |T'_t|) = (3,3)$, then we can trim exactly one edge (from T'_1), two edges (one from each of T'_1 and T'_t) or three edges (all of T'_1), with one of these operations leaving a forbidden subgraph of F' with exactly 6 edges.

Thus, the 1-clusters T'_i with $2 \leqslant i \leqslant t-1$ contain exactly 3 edges in total. This leaves us with the following possibilities for the sequence $(e'_1, \dots, e'_t)$ of the encountered sizes $e'_i := |T'_i|$.

The first case that $(e'_1, \dots, e'_t) = (2,1,1,1,2)$ is in fact impossible because one of the 1-trees can be removed so that the remaining r-graph is a partial 2-cluster with exactly 6 edges, a contradiction.

If $(e'_1, \dots, e'_t) = (2,3,2)$ then, in order to avoid a forbidden configuration, the following statements must hold: T'_2 is a 3-path, the diamonds T'_1 and T'_3 $\overline{1}2$-claim pairs that are 1-claimed by the opposite end-edges of the 3-path T'_2 but not by the middle edge, $|V(T'_1) \cap V(T'_2)| = |V(T'_3) \cap V(T'_2)| = 2$, $|V(T'_1) \cap V(T'_3)| \leqslant 1$, no further 1-cluster can be merged with $F' = T'_1 \cup T'_2 \cup T'_3$ (so $F' = F$ by Lemma 5.2), and therefore F satisfies Part (a) of the lemma. Part (b) now easily follows.

Finally, suppose that $(e'_1, \dots, e'_t)$ is $(2,1,2,2)$ or $(2,2,1,2)$, with the single-edge 1-cluster being $\{X\}$. By two applications of Lemma 5.2 (for $\{X\} \subseteq F'$ and for $F' \subseteq F$), we can additionally assume that F' is made of $T_1 = \{X\}$ and three diamonds T_2, T_3 and T_4 (and thus F can be obtained from F' by iteratively merging 1-clusters $T_5, \dots, T_m$ in this order). Call a 1-cluster T_i for $i \geqslant 2$ of *type ab* if the merging chain connects it to X via a pair $\{a,b\} \subseteq X$. (Note that the vertices a, b are not necessarily in T_i: for example, T_i can merge with a diamond 2-claiming ab.) By convention, we assume that the 1-tree T_1 is of all $\binom{r}{2}$ types. Observe that at least two of the initial diamonds T_2, T_3, T_4 must be of different types (as otherwise $T_2 \cup T_3 \cup T_4$ would be a $(6r-10, 6)$-configuration).

Let us continue denoting $H_i := T_1 \cup \dots \cup T_{i-1}$ for $i \in [m-1]$. In order to finish Part (a), it remains to prove the following.

***Claim* 5.10** *For every $i \in [2,m]$, T_i is a diamond that $\overline{1}2$-claims some previously 1-claimed pair $x_i y_i \in P_1(H_i)$, and no pair in $P_{\overline{1}2}(T_i) \setminus \{x_i y_i\}$ is 1-claimed or 2-claimed by H_i. Also, $m \leqslant 1 + 2\binom{r}{2}$.*

Proof of Claim 5.10 For the first part, we use induction on $i \in [m]$. To begin with, it is easy to check that the configuration on T_1, T_2, T_3, T_4 satisfies the statement of the claim. Let $i \in [5,m]$ and let the 1-cluster T_i be of type ab. Note that we have at most two 1-clusters of each type among the diamonds $T_2, \dots, T_{i-1}$ (otherwise, the first three diamonds of any given type would form a forbidden 6-edge configuration). If some edge $e' \in T_i$ 1-claims a pair $\overline{1}2$-claimed by H_i, then by keeping only the edge e' in T_i and

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removing one by one the diamonds of types different from ab, we can reach a partial 2-cluster with exactly 6 edges, a contradiction. So let the diamond $D_i \subseteq T_i$ $\overline{1}2$-claim a pair $x_i y_i \in P_1(H_i)$. We know that there is at most one previous diamond T_j of the same type as T_i (otherwise D_i with two such diamonds would form a $(6r-10, 6)$-configuration). It follows that $D_i = T_i$ as otherwise a forbidden 6-edge configuration would be formed by T_1, D_i, some suitable edge of $T_i \setminus D_i$ plus either the diamond T_j of type ab (if it exists) or a diamond $\overline{1}2$-claiming a pair in $P_1(T_1) \setminus \{ab\}$ (such diamond exists among T_2, T_3, T_4). If D_i contains some other vertex $z_i \notin \{x_i, y_i\}$ from an earlier 1-cluster of the same type ab, then some edge e' of D_i shares at least two vertices with H_i, again leading to a forbidden 6-vertex configuration in $H_i \cup \{e'\}$. Thus, we are done unless $P_{\overline{1}2}(T_i)$ contains a pair uv with both vertices in a 1-cluster of some different type $a'b'$ (where $\{u, v\}$ may possibly intersect $\{x_i, y_i\}$). If each of the types ab and $a'b'$ contains an earlier diamond (which then must be unique), then these two diamonds and T_i form a configuration on 6 edges and at most $6r-10$ vertices; otherwise, we have in total at most two diamonds of Type ab or $a'b'$ (including T_i) and these diamonds together with T_1 have 5 edges and at most $5(r-2)+1$ vertices, a contradiction.

Finally, the inequality $m \leqslant 1 + 2\binom{r}{2}$ follows from the observation made earlier that for every type $ab \in \binom{e}{2}$ there are at most 2 diamonds among $T_2, \ldots, T_m$ of this type. ∎

Claim 5.10 implies that the addition of each new diamond gives $(r-2)^2 - 1$ new $\overline{1}2$-claimed pairs, from which Part (b) follows in the case $|F| \geqslant 7$.

Now, suppose that $|F| \leqslant 6$, and thus $|F| \leqslant 5$. When we construct F by merging 1-clusters one by one as in Lemma 5.2, each new 1-cluster shares exactly 2 vertices with the current partial 2-cluster (since G is $\mathcal{G}_6^{(r)}$-free). Thus, Part (b) follows.

Finally, if $(e_1, \ldots, e_m) = (2, 1, 1, 1)$, then F consists of a diamond D with three single edges X_1, X_2, X_3 attached along some pairs $\overline{1}2$-claimed by D. Take any pair $xy \in P_{\overline{1}2}(F)$; then, xy is $\overline{1}2$-claimed by D but not 1-claimed by any X_i. Note that, for an edge $X \in G \setminus F$ 1-claiming xy (resp. a diamond $D' \subseteq G \setminus F$ $\overline{1}2$-claiming xy), $F \cup \{X\}$ (resp. $(F \cup D') \setminus \{X_1\}$) is a $(6r-10, 6)$-configuration. Thus, $G \setminus F$ neither 1-claims nor 2-claims xy, finishing the proof of Part (c). ∎

5.4.1 Upper bound of Theorem 1.4

Here, we deal with the case $k = 6$ and $r \geqslant 4$.

Proof of the upper bound of Theorem 1.4 By Lemma 3.1, it is enough to upper bound the size of a $\mathcal{G}_6^{(r)}$-free r-graph G with n vertices. Recall that, by Lemma 5.4, each element of $\mathcal{M}_1$ is an i-tree with $i \in [5]$.

Now, we define the weights. A 2-cluster $F \in \mathcal{M}_2$ assigns weight 1 to each pair in $P_1(F)$ and weight $1/2$ to each pair in $P_{\overline{1}2}(F)$, except if the composition of F is $(2, 1, 1, 1)$ in which case every pair $\overline{1}2$-claimed by F receives weight 1 (instead of $1/2$). Let us show that every pair xy of vertices of G receives weight at most 1. This is clearly true if there is a 2-cluster F with the composition $(2, 1, 1, 1)$ such that $xy \in P_{\overline{1}2}(F)$: then, F gives weight 1 to xy and no other 2-cluster 1-claims or 2-claims xy by Part (c)

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of Lemma 5.9. Suppose that xy is not $\overline{1}2$-claimed by any 2-cluster with the composition $(2,1,1,1)$. If xy is 1-claimed by some 2-cluster F_1, then it cannot be 1-claimed or 2-claimed by another 2-cluster F_2 (as otherwise F_1 and F_2 would be merged). On the other hand, the pair xy can be 2-claimed by at most two 2-clusters by (5.7). In all cases, the pair xy receives weight at most 1.

Let us show that for each 2-cluster $F \in \mathcal{M}_2$, we have

$$\lambda(F) := 2w(F) - r(r-1)|F| \geqslant 0, \tag{5.8}$$

where $w(F)$ denotes the total weight assigned by the 2-cluster F. First, consider the exceptional case when F is composed of a diamond and 3 single edges. Here, $w(F)$ does not depend on how the three edges are merged with the diamond and we have

$$\lambda(F) = 2\left(5\binom{r}{2} + (r-2)^2 - 4\right) - r(r-1)\cdot 5 = 2((r-2)^2-4) \geqslant 0.$$

(Note that, if F gave weight of $1/2$ to each $\overline{1}2$-claimed pair, then (5.8) may be false for $r=4$, so some exceptional weight distribution is necessary.)

So let F be any other (non-exceptional) 2-cluster. For $j \in \mathbb{N}$, let n_j denote the number of 1-clusters in F with j edges. Thus, $n_j = 0$ for $j \geqslant 6$ by Lemma 5.4. We have

$$\begin{aligned}
\lambda(F) &= 2\,|P_1(F)| + |P_{\overline{1}2}(F)| - r(r-1)|F| \\
&\geqslant 2\sum_{j=1}^{5}\left(j\binom{r}{2} - j + 1\right)n_j + \left(1 - \sum_{j=1}^{5} n_j + \sum_{j=1}^{5}(j-1)(r-2)^2 n_j\right) - r(r-1)\sum_{j=1}^{5} j n_j \\
&= 1 + \sum_{j=1}^{5}\big((r^2-4r)(j-1) + 2j - 3\big)\, n_j,
\end{aligned}$$

where the inequality in the middle follows from Part (b) of Lemma 5.9. Since $r \geqslant 4$, the coefficient at n_j is at least $2j-3$. This is negative only if $j=1$. Thus, $\lambda(F) \geqslant 0$ unless $n_1 \geqslant 2$. By Part (a) of Lemma 5.9 (and since we have already excluded the exceptional $(2,1,1,1)$-case of Part (c)), this is only possible if F has the composition $(3,1,1)$ or $(2,1,1)$. The corresponding sequences of (n_1,n_2,n_3) are $(2,0,1)$ and $(2,1,0)$; thus, the corresponding values of $\lambda(F)$ are 2 and 0. Hence, $\lambda(F) \geqslant 0$ for every 2-cluster F, so the familiar double counting argument implies that $|G| \leqslant \binom{r}{2}^{-1}\binom{n}{2}$, proving the theorem. ∎

5.4.2 Upper bound of Theorem 1.3

In this section, we deal with the case $(r,k) = (3,6)$.

Proof of the upper bound of Theorem 1.3 By Lemma 3.1, it is enough to provide a uniform upper bound on the size of an arbitrary $\mathcal{G}_6^{(3)}$-free 3-graph G on $V := [n]$ from above.

As before, $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) denotes the partition of $E(G)$ into 1-clusters (resp. 2-clusters). Call edge-disjoint subgraphs $F, F' \subseteq G$ *(3^+)-mergeable (via $uv \in \binom{V}{2}$)* if they

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are $(\{1\}|\{3,4\})$ or $(\{1,2\}|\{3\})$-mergeable via uv, that is, if at least one of the following two conditions holds:

- $1 \in C_F(uv)$ and $\{3,4\} \subseteq C_{F'}(uv)$, or
- $\{1,2\} \subseteq C_F(uv)$ and $3 \in C_{F'}(uv)$.

If the order of F and F' does not matter, then we simply say that they are 3^+-*mergeable*. Let the partition $\mathcal{M}_{3^+}$ of $E(G)$ be obtained by starting with $\mathcal{M}_2$ and, iteratively and as long as possible, merging any two 3^+-mergeable parts. Also, let $\mathcal{M}'_{3^+}$ be the set of 3-graphs that could appear at some point of the above process. We refer to the elements of $\mathcal{M}_{3^+}$ (resp. $\mathcal{M}'_{3^+}$) as 3^+-*clusters* (resp. *partial* 3^+-*clusters*). By monotonicity, the partition $\mathcal{M}_{3^+}$ does not depend on the order in which we perform the merging steps.

Let us observe some basic properties of $\mathcal{M}_{3^+}$.

***Lemma* 5.11** *Suppose that the edge-disjoint partial* 3^+*-clusters* F *and* H *are* (3^+)*-mergeable via some pair* uv*. Then, there are* 2*-clusters* $F' \subseteq F$ *and* $H' \subseteq H$ *that are* (3^+)*-mergeable via* uv*.*

Proof Let $F' \subseteq F$ be the (unique) 2-cluster that 1-claims the pair uv. Let H'' be a $(5,3)$-configuration in H that 3-claims the pair uv. Note that the pair uv is not 2-claimed by H'' since otherwise the 2-cluster in H'' claiming this pair would have been merged with F'. Since $H'' \subseteq G$ is $\mathcal{G}_6^{(3)}$-free, H'' is either a 3-tree or the union of a single edge and a diamond that can be (2)-merged. Thus, H'' lies entirely inside some 2-cluster H'. Of course, $H' \subseteq H$. Let us show that F' and H' satisfy the lemma.

If $\{1,2\} \subseteq C_F(uv)$, as witnessed by an edge e and a diamond D in F, then e is an edge of D (as otherwise $D \cup \{e\} \cup H''$ would be a forbidden 6-edge configuration) and the lemma is satisfied (since F' must contain D as a subgraph).

So suppose that $4 \in C_H(uv)$. We are done if $4 \in C_{H'}(uv)$ so suppose otherwise. This assumption implies that no pair in $P_{\overline{1}2}(H'')$ can be 1-claimed by an edge from $G \setminus H''$. Furthermore, no pair in $P_1(H'')$ can be 2-claimed by a diamond D in $G \setminus H''$ as otherwise $D \cup H''$ together with an edge of F' 1-claiming the pair uv would form a $(8,6)$-configuration. Hence, $H' = H''$. Since $C_H(uv) \ni 4$ is strictly larger than $C_{H'}(uv)$, the 2-cluster H' is 3^+-mergeable with some other 2-cluster H''' in $H \setminus H'$ via some pair $u'v'$. Note that $3 \in C_{H'}(u'v')$ since H' is a $(5,3)$-configuration, so it 3-claims every pair of vertices it contains. By (5.7), $3 \notin C_{H'''}(u'v')$, and hence $1 \in C_{H'''}(u'v')$. It follows that $u'v' \neq uv$ as otherwise uv is 1-claimed by F and H''', contradicting the merging rule for $\mathcal{M}_1$. As H' has only 3 edges, the definition of 3^+-mergeability gives that $2 \in C_{H'''}(u'v')$. However, in that case the union of H', a diamond D in H''' that 2-claims the pair $u'v'$, and an edge in F' that 1-claims the pair uv forms a $(8,6)$-configuration, a contradiction. ■

Lemma 5.11 provides us with an analogue of Assumption (5.2) of Lemma 5.2 and the proof of Lemma 5.2 trivially adapts to 3^+-merging. We will need only the following special case.

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***Claim* 5.12** *For every partial 3^+-cluster F and any 2-cluster $F_0 \subseteq F$, there is an ordering $F_0, \ldots, F_s$ of the 2-clusters constituting F such that, for each $i \in [s]$, the 3-graphs F_i and $\bigcup_{j=0}^{i-1} F_j$ are 3^+-mergeable.* ■

Now, we consider the following two functions f and h from subsets of $[5]$ to the reals. Namely, for $A \subseteq [5]$, we define

$$f(A) := \begin{cases} 55/61 & \text{if } A = \{1\}, \\ 1 & \text{if } A = \{1, x\} \text{ for some } x \in \{2,3\}, \\ 55/61 & \text{if } A = \{1, x\} \text{ for some } x \in \{4,5\}, \\ 25/61 & \text{if } A = \{2\}, \\ 36/61 & \text{if } A = \{2,3\}, \\ 1 & \text{if } A = \{2,3,4\}, \\ 1/2 & \text{if } A = \{2,4\}, \\ 6/61 & \text{if } A = \{3\}, \\ 11/61 & \text{if } A = \{3,5\}, \\ 1 & \text{if } A = \{3,4\}, \\ 0 & \text{for all other sets } A \subseteq [5], \end{cases}$$

and

$$h(A) := \max\{f(A') : A' \subseteq A\}.$$

Then, the function h is clearly non-decreasing and satisfies that

$$h(A) > 0 \quad \Leftrightarrow \quad A \cap \{1,2,3\} \neq \emptyset. \tag{5.9}$$

In the sequel, we abbreviate $h(\{i_1, \ldots, i_s\})$ to $h(i_1, \ldots, i_s)$.

Define the weight attributed to a pair $uv \in \binom{V}{2}$ by a subgraph $F \subseteq G$ to be

$$w_F(uv) := h([5] \cap C_F(uv)).$$

Moreover, we set $w(uv) := \sum_{F \in \mathcal{M}_{3^+}} w_F(uv)$ to be the total weight received by a pair uv from all 3^+-clusters.

***Claim* 5.13** *For every $uv \in \binom{V}{2}$, it holds that $w(uv) \leqslant 1$.*

Proof of Claim 5.13 Fix uv and let $F_1, \ldots, F_s$ be all 3^+-clusters with $w_{F_i}(uv) > 0$. We have to show that $\sum_{i=1}^{s} w_{F_i}(uv) \leqslant 1$. Note that each $C_{F_i}(uv)$ intersects $\{1,2,3\}$ by (5.9). If $s = 1$, then we are done (since $h(A) \leqslant 1$ for every $A \subseteq [5]$), so assume that $s \geqslant 2$.

The following cases cover all possibilities up to a permutation of $F_1, \ldots, F_s$.

Case 1. Assume that $C_{F_1}(uv)$ contains 1.

Then, for every $j \in [2,s]$, we have that $1, 2 \notin C_{F_j}(uv)$ (as otherwise the corresponding 1-clusters of F_1 and F_j would be merged when building $\mathcal{M}_1$ or $\mathcal{M}_2$) and it follows from (5.9) that $3 \in C_{F_j}(uv)$. Since the subgraphs $F_1, \ldots, F_s \subseteq G$ are edge-disjoint, (5.7) implies that $s = 2$. Furthermore, since F_1 and F_2 are not (3^+)-mergeable, it holds

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that $4 \notin C_{F_2}(uv)$ and $2 \notin C_{F_1}(uv)$. By (5.7), $5 \notin C_{F_2}(uv)$ and $3 \notin C_{F_1}(uv)$. Thus,

$$w(uv) = w_{F_1}(uv) + w_{F_2}(uv) \leqslant h(1,4,5) + h(3) = \frac{55}{61} + \frac{6}{61} = 1.$$

Case 2. Assume that no $C_{F_i}(uv)$ contains 1 but $C_{F_1}(uv)$ contains 2.

Here, it is impossible to have distinct $i, j \in [2, s]$ with $2 \in C_{F_i}(uv) \cap C_{F_j}(uv)$ as otherwise the edge-disjoint subgraphs $F_1, F_i, F_j \subseteq G$ would contradict (5.7). We split this case into 2 subcases.

Case 2-1. Assume $2 \in C_{F_2}(uv)$.

Suppose first that $s \geqslant 3$. Then, for every $j \in [3, s]$, we have $2 \notin C_{F_j}(uv)$ by (5.7) and thus $3 \in C_{F_j}(uv)$ by (5.9). It follows from (5.7) that $s = 3$ and, moreover, $4 \notin C_{F_3}(uv)$ and $3, 4 \notin C_{F_1}(uv) \cup C_{F_2}(uv)$. Hence

$$w(uv) = w_{F_1}(uv) + w_{F_2}(uv) + w_{F_3}(uv) \leqslant 2\,h(2,5) + h(3,5) = 2 \cdot \frac{25}{61} + \frac{11}{61} = 1.$$

If $s = 2$, then (5.7) implies that $4 \notin C_{F_1}(uv) \cup C_{F_2}(uv)$ and $3 \notin C_{F_1}(uv) \cap C_{F_2}(uv)$. Therefore,

$$w(uv) = w_{F_1}(uv) + w_{F_2}(uv) \leqslant h(2,5) + h(2,3,5) = \frac{25}{61} + \frac{36}{61} = 1.$$

Case 2-2. Assume that $2 \notin C_{F_j}(uv)$ for all $j \in [2, s]$.

By (5.9), $C_{F_j}(uv)$ contains 3 for every $j \in [2, s]$. By (5.7), it holds that $s = 2$ and, moreover, $3 \notin C_{F_1}(uv)$ and $4 \notin C_{F_2}(uv)$. Thus, we have

$$w(uv) = w_{F_1}(uv) + w_{F_2}(uv) \leqslant h(2,4,5) + h(3,5) = \frac{1}{2} + \frac{11}{61} \leqslant 1.$$

Case 3. Assume that no $C_{F_i}(uv)$ contains 1 or 2.

By (5.9), we have $3 \in C_{F_i}(uv)$ for each $i \in [s]$. However, our assumption that $s \geqslant 2$ contradicts (5.7). This finishes the case analysis and the proof. ■

Now, let us show that, for every $F \in \mathcal{M}_{3^+}$, the total weight

$$w(F) := \sum_{uv \in \binom{V}{2}} w_F(uv)$$

assigned by F to different vertex pairs is at least $\frac{165}{61}|F|$.

First, we check this for the 3^+-clusters F consisting of a single 1-cluster.

***Claim* 5.14** *For all $F \in \mathcal{M}_1$, we have $w(F) \geqslant \frac{165}{61}|F|$.*

Proof of Claim 5.14 Recall that F is an i-tree with $i \leqslant 5$ by Lemma 5.4. Assume that $i \geqslant 2$ as otherwise $w(F) = 3h(1) = \frac{165}{61}$ and the claim holds.

Every pair in $P_1(F)$ is $\{1, 2\}$-claimed (in fact, $\{1, \dots, i\}$-claimed by Corollary 5.3) and, in particular, receives weight at least $h(1, 2) = 1$ from F. Moreover, since F is an

i-tree, it 1-claims $2i+1$ pairs. Then, if $i=2$, $w(F)=5\,h(1,2)+h(2)=5+\frac{25}{61}=\frac{165}{61}\cdot 2$, as desired.

Now, assume that $i\geqslant 3$. Then, each pair in $P_{\overline{1}2}(F)$ is $\{2,3\}$-claimed by Corollary 5.3 and receives weight at least $h(2,3)=\frac{36}{61}$. Moreover, $|P_{\overline{1}2}(F)|\geqslant i-1$, and if we exclude all pairs in $P_1(F)$ and some $i-1$ pairs in $P_{\overline{1}2}(F)$ then, regardless of the structure of F, there will remain at least $i-2$ pairs that are 3-claimed. Indeed, easy induction shows that any i-tree with $i\geqslant 3$ contains at least $2i-3$ different sub-paths of length 2 or 3 such that the opposite vertices of degree 1 in these paths give distinct pairs outside of $P_1(F)$ that are 3-claimed by F. Thus,

$$\begin{aligned} w(F) &\geqslant (2i+1)\,h(1,2)+(i-1)\,h(2,3)+(i-2)\,h(3)\\ &= (2i+1)+(i-1)\,\frac{36}{61}+(i-2)\,\frac{6}{61}\\ &= \frac{13-i}{61}+\frac{165}{61}\,i \;>\; \frac{165}{61}\,|F|, \end{aligned}$$

as required. ■

As a next step, we estimate the weight assigned by those 2-clusters that consist of more than one 1-cluster.

***Claim* 5.15** *For all $F\in\mathcal{M}_2\setminus\mathcal{M}_1$, we have $w(F)\geqslant\frac{165}{61}\,|F|$.*

Proof of Claim 5.15 Note that if $F,H\subseteq G$ are (2)-mergeable via a pair uv, then $\{1\}+C_H(uv)\subseteq C_{F\cup H}(uv)$ and $\{2\}+C_F(uv)\subseteq C_{F\cup H}(uv)$ holds. In particular, we conclude by (5.7) that $5\notin C_H(uv)$ and $4\notin C_F(uv)$.

Suppose first that $|F|\geqslant 7$. By Lemma 5.9(a), there are two cases to consider. First, let F be made from a 3-tree by (2)-merging two diamonds one by one. Note that F $\{1,2\}$-claims all 17 pairs in $P_1(F)$. Since each new diamond abx,aby attaches to the rest via its $\overline{1}2$-claimed pair xy, which is also $\{1,2\}$-claimed by the previous edges (in particular, one of these edges is xyz for some vertex $z\in V\setminus\{a,b\}$), this gives 2 further pairs $\{3,4\}$-claimed by F, namely za and zb (so 4 such pairs in total for the two diamonds). Thus,

$$w(F)\geqslant 17\,h(1,2)+4\,h(3,4)=17+4>\frac{165}{61}\cdot 7,$$

as desired. So, by Lemma 5.9(a), we can assume that F is made from a single edge e by iteratively (2)-merging $i\in[3,6]$ diamonds. Then, all $5i+3$ pairs in $P_1(F)$ are $\{1,3\}$-claimed. Also, F $\{3,5\}$-claims further $2i$ pairs. Indeed, each new diamond abx,aby 2-claims a pair xy 1-claimed by some previous edge xyz, and since $i\geqslant 3$, the pairs za and zb are $\{3,5\}$-claimed by the final 2-cluster F. Thus, we have

$$\begin{aligned} w(F) &\geqslant (5i+3)\,h(1,3)+2i\cdot h(3,5) \;=\; (5i+3)+2i\cdot\frac{11}{61}\\ &= \frac{18-3i}{61}+\frac{165}{61}\cdot(2i+1) \;\geqslant\; \frac{165}{61}\cdot|F|. \end{aligned}$$

Thus, suppose that $|F|\leqslant 6$. By $(8,6)$-freenees, we have that $|F|\leqslant 5$. First, consider the case that $F=F_1\cup F_2$ for (2)-mergeable $F_1,F_2\in\mathcal{M}_1$ via some pair uv (thus

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$1 \in C_{F_1}(uv) \setminus C_{F_2}(uv)$). Let F_1 be an i-tree and F_2 be a j-tree. Then, F has $i+j$ edges and 1-claims $2i+2j+2$ pairs. Note that $j \geqslant 2$ as F_2 has to contain a diamond. Also, since G is $\mathcal{G}_6^{(3)}$-free, the subgraphs F_1 and F_2 do not share any further vertices in addition to u and v.

Suppose first that $i = 1$. Every pair in $P_1(F)$ is $\{1,3\}$-claimed by F. If $j = 2$, then F 3-claims the remaining 2 pairs and we have

$$w(F) = 8\,h(1,3) + 2\,h(3) = 8 + 2 \cdot \frac{6}{61} = \frac{500}{61} > \frac{165}{61} \cdot 3.$$

If $j \geqslant 3$, then F $\{3,4\}$-claims at least 2 pairs not in $P_1(F)$. Hence,

$$w(F) \geqslant (2j+4)\,h(1,3) + 2\,h(3,4) = (2j+4) + 2 = \frac{(201-43j)}{61} + \frac{165}{61}\,(j+1),$$

which is at least $\frac{165}{61}\,(j+1)$ since $j = |F| - 1 \leqslant 4$, as desired.

Suppose that $i \geqslant 2$. Then, every pair in $P_1(F)$ is $\{1,2\}$-claimed by F. Also, given an edge uvx in F_1 and a diamond abu, abv in F_2, the 2-cluster F $\{3,4\}$-claims the pairs $ax, bx \notin P_1(F)$. Moreover, if $i = 2$, then the (unique) pair in $\binom{V(F_1)}{2} \setminus P_1(F_1)$ is $\{2,4\}$-claimed by F; combining this with the fact that $j = |F| - i \leqslant 3$ yields

$$\begin{aligned} w(F) \;&\geqslant\; (2j+6)\,h(1,2) + 2\,h(3,4) + h(2,4) \;=\; (2j+6) + 2 + \frac{1}{2} \\ &=\; \frac{377-86j}{122} + \frac{165}{61}\,(j+2) > \frac{165}{61}\,|F|. \end{aligned}$$

If $i = 3$, then $j = 2$. Again, F $\{1,2\}$-claims all 12 pairs in $P_1(F)$ and $\{3,4\}$-claims another 2 pairs, but it also $\{2,3\}$-claims at least 2 other pairs, namely the pairs 2-claimed but not 1-claimed by F_1. Hence, we have

$$w(F) \geqslant 12\,h(1,2) + 2\,h(3,4) + 2\,h(2,3) = 12 + 2 + 2 \cdot \frac{36}{61} = \frac{926}{61} > \frac{165}{61} \cdot 5.$$

Now, note that a 2-cluster made of at least four 1-clusters has at least 6 edges: indeed, this 2-cluster was obtained by doing at least 3 consecutive $\{1\}|\{2\}$-mergings, so some 1-cluster in it contains at least three edges or some two 1-clusters in it contain at least two edges each. Thus, it remains to consider the case when F is obtained by merging three trees $F_1, F_2, F_3 \in \mathcal{M}_1$. We cannot have $|F| \leqslant 4$ as then, the 2-cluster F would be made of at least two 1-trees and at most one 2-tree, which is impossible for 3-graphs. Hence, we obtain that $|F| = 5$ with the composition $(3,1,1)$ or $(2,2,1)$.

If F has composition $(3,1,1)$, then F $\{1,3\}$-claims all 13 pairs in $P_1(F)$ and $\{3,4\}$-claims at least 4 other pairs (namely, for each 1-tree xyz and the corresponding diamond aby, abz inside the 3-tree, the pairs ax and bx are $\{3,4\}$-claimed by F). We have that

$$w(F) \geqslant 13\,h(1,3) + 4\,h(3,4) = 13 + 4 = \frac{1037}{61} > \frac{165}{61} \cdot 5.$$

Suppose that F has composition $(2,2,1)$. Then, F $\{1,3\}$-claims all 13 pairs in $P_1(F)$. Also, F can be built from its 1-tree by attaching each new diamond abx, aby via a previously 1-claimed pair xy, say by $xyz \in F$; here each of the pairs az and bz is

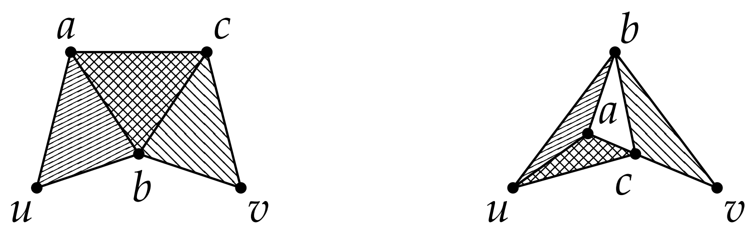

Figure 3: Configurations P_3 and C_3..

$\{3,5\}$-claimed by the final 2-cluster F. Thus, we have that

$$w(F) \geqslant 13\,h(1,3) + 4\,h(3,5) = 13 + 4\cdot\frac{11}{61} = \frac{837}{61} > \frac{165}{61}\cdot 5,$$

which finishes the case analysis and the proof. ∎

Finally, we prove the following claim.

***Claim* 5.16** *For all $F \in \mathcal{M}_{3^+} \setminus \mathcal{M}_2$, we have that $w(F) \geqslant \frac{165}{61}|F|$.*

Proof of Claim 5.16 Recall that if $F_1, F_2 \in \mathcal{M}_2$ are (3^+)-mergeable via uv, then $1 \in C_{F_1}(uv)$ and $3 \in C_{F_2}(uv)$ and, in addition, either $2 \in C_{F_1}(uv)$ or $4 \in C_{F_2}(uv)$. In particular, we have that $1,2 \notin C_{F_2}(uv)$ (as otherwise we would have already merged F_1 and F_2 when constructing $\mathcal{M}_2$); also, by (5.7) $5 \notin C_{F_2}(uv)$ and $3 \notin C_{F_1}(uv)$.

Let $F \in \mathcal{M}_{3^+} \setminus \mathcal{M}_2$ be made of $F_1, \ldots, F_s \in \mathcal{M}_2$ 3^+-merged in this order as in Claim 5.12. Assume without loss of generality that F_1 and F_2 are (3^+)-mergeable via some pair uv. Let $F' := F_1 \cup F_2$. As $1,2 \notin C_{F_2}(uv)$ and $3 \in C_{F_2}(uv)$, we know that there is a $(5,3)$-configuration in F_2 containing uv which is either a 3-path $P_3 = \{uab, abc, bcv\} \subseteq F_2$ or a 2-cluster $C_3 = \{aub, auc, bcv\} \subseteq F_2$ (which is the union of a 1-tree and a 2-tree that are (2)-mergeable), see Figure 3. Since $3 \notin C_{F_1}(uv)$ by (5.7), F_1 cannot be an i-tree for $i \geqslant 3$. We split the proof into 3 cases depending on whether F_1 is a 2-tree, a 1-tree or an element of $\mathcal{M}_2 \setminus \mathcal{M}_1$.

Case 1. Assume that F_1 is a 2-tree.

Then, $1,2 \in C_{F_1}(uv)$. By (5.7), we have $4,5 \notin C_{F_2}(uv)$, and recall that F_2 contains one of P_3 or C_3 as described above. Also, the union $F'' := F_1 \cup P_3$ or $F_1 \cup C_3$ is a $(7,5)$-configuration, so

$$\binom{V(F'')}{2} \cap P_1(G \setminus F'') = \emptyset. \tag{5.10}$$

Moreover, since $5 \notin C_{F_2}(uv)$, no pair in $V(P_3)$ (resp. $V(C_3)$) can be 2-claimed by $G \setminus F'$. It follows that the 2-cluster F_2 is equal to P_3 or C_3, and thus $F'' = F'$.

Suppose that F consists of $s \geqslant 3$ 2-clusters. Let the next 3^+-merging step (of F' and F_3) be via a pair $u'v'$. It follows from (5.10) (and from $F'' = F'$) that $u'v' \in$

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$P_1(F') \cap P_3(F_3)$. By (5.7), it holds that $3 \notin C_{F'}(u'v')$. As F_2 contains 3 edges, necessarily $u'v' \in P_1(F_1)$. Also, since $u'v'$ is in $P_1(F_1) \subseteq P_2(F_1)$, we see that F_1 alone and F_3 are (3^+)-mergeable via $u'v'$. Now, our previous argument about the structure of F_2 applies to F_3 as well and shows that F_3 is isomorphic to a copy of P_3 or C_3 3-claiming $u'v'$. Furthermore, the $(5,3)$-configurations F_2 and F_3 can share at most one vertex, so $\binom{V(F_2)}{2} \cap \binom{V(F_3)}{2} = \emptyset$. The very same reasoning applies in turn to each F_i with $i \in [4,s]$ with the analogous conclusion. Indeed, by Lemma 5.11, there is $j \in [1,i-1]$ such that F_i and F_j 3^+-merge. Together with the $(8,6)$-freeness and the fact that $F_1 \cup F_2, \dots, F_1 \cup F_{i-1}$ form $(7,5)$-configurations, F_i can only 3^+-merge with F_1. Let us bound from below the total weight assigned by F, being rather loose in our estimates. The five pairs in $P_1(F_1)$ are $\{1,2\}$-claimed (and we just ignore the remaining pair inside $V(F_1)$ which is 2-claimed). For each $i \in [2,s]$, if F_i is a 3-path, then it $\{1,3\}$-claims all seven pairs in its 2-shadow $P_1(F_i)$ and two further pairs inside $V(F_i)$ are $\{2,3,4\}$-claimed by $F_1 \cup F_i \subseteq F$ (for example, if $i=2$, then these are the pairs uc and av). All these pairs are unique to F_i and thus F_i contributes at least $7\,h(1,3) + 2\,h(2,3,4) = 9$ to $w(F)$. Also, if F_i is isomorphic to C_3, then it $\{1,3\}$-claims all 8 pairs in $P_1(F_i)$ and one further pair inside $V(F_i)$ is $\{3,4\}$-claimed by $F_1 \cup F_i$ (for example, if $i=2$ then it is the pair av). Thus, F_i contributes at least $8\,h(1,3) + h(3,4) = 9$ to $w(F)$. We conclude by $s \geqslant 2$ (which follows from $F \notin \mathcal{M}_2$) that

$$\begin{aligned} w(F) &\geqslant 5\,h(1,2) + 9(s-1) = 5 + 9(s-1) \\ &= \frac{54s-79}{61} + \frac{165}{61}\cdot(3s-1) > \frac{165}{61}|F|, \end{aligned}$$

as desired.

Case 2. Assume that F_1 is a 1-tree $\{uvw\}$.

As F_1 and F_2 are (3^+)-mergeable via uv, we have $3,4 \in C_{F_2}(uv)$. Thus, $|F_2| \geqslant 4$ and $|F_2| \notin \{5,6\}$ (otherwise, we would get a $(8,6)$-configuration). Also, $1,2 \notin C_{F_2}(uv)$ since $F_1, F_2 \in \mathcal{M}_2$ are distinct. Hence, as before, F_2 contains a copy of P_3 or C_3 3-claiming uv.

Suppose first that $|F_2| \geqslant 7$. Then, F_2 has the structure given by Lemma 5.9(a), consisting of a 1-tree or 3-tree with a number of diamonds merged in one by one. A $(5,3)$-configuration in F_2 that contains uv involves at most two 1-clusters of F_2, whose union F'' has at most 5 edges; moreover, if F'' consists of two 1-clusters, then these 1-clusters are 2-mergeable. The proof of Lemma 5.9(a) shows that, if we build $F_2 \in \mathcal{M}_2$ by starting with the partial 2-cluster F'' and attaching 1-clusters one by one, then we reach a $(7,5)$-configuration F''' just before the number of edges jumps over 6. However, F_1 shares at least 2 vertices with $F''' \supseteq F''$, so $F_1 \cup F'''$ is a $(8,6)$-configuration. This contradiction shows that $|F_2| = 4$.

Now, denote the 4-edge hypergraph F_2 by T_4 if F_2 consists of (a copy of) P_3 that 3-claims uv together with one more edge (thus, T_4 is a 4-tree or a 2-cluster made of a 3-tree and a 1-tree as its 1-clusters), and denote F_2 by C_4 if F_2 consists of (a copy of) C_3 that 3-claims uv with one more edge (thus, the 1-clusters of C_4 are two 2-trees, or a 3-tree and a 1-tree). Recall that $F' = F_1 \cup F_2$.

Suppose first that $s \geqslant 3$. Let F_3 and F' be merged via a pair $u'v'$. The 3-graphs F_3 and F' cannot be (3^+)-mergeable via $u'v'$ as otherwise an edge in F_3 containing $u'v'$

together with F' would be a $(8,6)$-configuration. So F' and F_3 are (3^+)-mergeable, that is, $1 \in C_{F'}(u'v')$ and $3 \in C_{F_3}(u'v')$. Then, $3 \notin C_{F'}(u'v')$. This greatly limits the number of possibilities for the pair $u'v'$ inside $P_1(F')$. If $F_2 = T_4$, one can easily notice that $u'v'$ cannot be in $P_1(F_2) \cup \{uv\}$ and thus $u'v' \in P_1(F_1) \setminus \{uv\}$. Hence, F_1 alone and F_3 are (3^+)-mergeable and the above analysis for F_2 applies to F_3 as well, showing that the 2-cluster F_3 is isomorphic to a copy of T_4 or C_4 3-claiming $u'v'$. If $F_2 = C_4$, then either $u'v' \in P_1(F_1) \setminus \{uv\}$ (and, again, F_3 is isomorphic to T_4 or C_4) or, for some vertices a, b, c, d, we have $F_2 = \{aub, auc, bcv, cdv\}$ and $u'v'$ is cd or dv; also, the argument of Case 1 (with $\{bcv, cdv\}$ playing the role of the 2-tree F_1 from Case 1) shows that F_3 rooted at $u'v'$ is isomorphic to P_3 or C_3, no other F_i with $i \geqslant 4$ can be 3^+-merged with F_3, and all pairs in $\binom{V(F_3)}{2}$ are unique to F_3. Using Lemma 5.11, the same argument applies to each new F_i with $i \geqslant 4$. To summarise, we obtained that each F_i for $i \geqslant 2$ is either some instance of T_4 or C_4 3^+-merged with the single edge F_1, or an instance of P_3 or C_3 3^+-merged with a copy of C_4 as specified above; also, the only pairs shared between these 2-clusters are the pairs along which these 3^+-mergings occur.

Now, assume the final F consists of one 1-tree, i copies T_4 or C_4, and j copies of P_3 or C_3. (Although we could say more about the structure of F, e.g. that $i \leqslant 3$ and $j \leqslant 2i$, these observations are not needed for our estimates.) Then, $|F| = 1 + 4i + 3j$. Each copy of T_4 (which is a 4-tree or a 3-tree 2-merged with a single edge) has, in addition to the pair via which it is merged with F_1, at least nine $\{1,3\}$-claimed pairs and other two $\{2,3,4\}$-claimed pairs. Thus, it contributes at least $9\,h(1,3)+2\,h(2,3,4) = 11$ to $w(F)$. Likewise, each copy of C_4 contributes at least $8\,h(1,3)+2\,h(1,2)+h(3,4) = 11$ to $w(F)$. Also, as in Case 1, each copy of P_3 or C_3 contributes at least 9 to $w(F)$. Additionally, we have 3 pairs in $P_1(F_1)$ which are $\{1,4\}$-claimed by $F_1 \cup F_2$. Hence, we get the required:

$$\begin{aligned} w(F) &\geqslant 3\,h(1,4) + 11i + 9j \;=\; 3 \cdot \frac{55}{61} + 11i + 9j \\ &= \frac{11i + 54j}{61} + \frac{165}{61}\,(1 + 4i + 3j) \;>\; \frac{165}{61}\,|F|. \end{aligned}$$

Case 3. Assume that $F_1 \in \mathcal{M}_2 \setminus \mathcal{M}_1$.

Here, F_1 is a 2-merging of at least two 1-clusters and thus has at least 3 edges. Let $T_1 \in \mathcal{M}_1$ be the 1-cluster in F_1 1-claiming uv (recall that uv is the pair via which F_1 and F_2 are (3^+)-mergeable). The tree T_1 has at most 2 edges as otherwise T_1 together with F_2 would contain an $(8,6)$-configuration, a contradiction. Also, T_1 cannot be a 1-tree as otherwise $F_1 \setminus T_1$ would contain a diamond 2-claiming a pair in $P_1(T_1)$ (since $F_1 \in \mathcal{M}_2 \setminus \mathcal{M}_1$), which implies that $3 \in C_{F_1}(uv)$, a contradiction. Therefore, T_1 is a 2-tree. As $F_1 \in \mathcal{M}_2 \setminus \mathcal{M}_1, T_1$ has to be 2-merged with some other 1-cluster $T_2 \subseteq F_1$. It is impossible that T_2 and T_1 are (2)-mergeable as otherwise T_1 plus an edge of T_2 would be a $(5,3)$-configuration containing uv in F_1, which contradicts $3 \notin C_{F_1}(uv)$. Thus, T_1 and T_2 are (2)-mergeable. Also, T_2 has at most 3 edges since trees with more edges would form an $(8,6)$-configuration with T_1. It is routine to check that we can assume (after swapping u and v if necessary) that, for some vertices $a, b \in V$, the 2-tree T_1 is $\{avu, avb\}$ and the pair 2-claimed by T_2 is ab or bv. Also, any further 2-merging involving $T_1 \cup T_2$ would cause an $(8,6)$-configuration. We conclude that $F_1 = T_1 \cup T_2$.

The same argument as in Case 1 shows that F_2 is given by P_3 (a 3-path) or C_3 (a diamond and a single edge), see Figure 3. Let $F' := F_1 \cup F_2$. We have 2 subcases depending on T_2.

Case 3-1. Assume that $|T_2| = 2$.

Suppose first that $s \geqslant 3$. If F_3 and F' are (3^+)-mergeable via some pair $u'v'$ then, by $(8, 6)$-freeness, $u'v'$ must be one of the two pairs $\overline{1}3$-claimed by F_1, and F_3 is a 1-tree; therefore, by Claim 5.12, we can reorder 2-clusters constituting F starting with F_3 and follow the analysis in the Case 2. Now assume that both pairs $\overline{1}3$-claimed by F_1 are not used for further 3^+-mergings. Thus, F' and F_3 are (3^+)-mergeable via $u'v'$. Then, since $3 \in C_{F_3}(u'v')$, $u'v'$ must be in $P_1(F') \setminus P_3(F') = \{au\}$ (recall that $3 \in C_{F_2}(uv) \subseteq C_{F'}(uv)$), F_3 rooted at $u'v' = au$ is isomorphic to P_3 or C_3; moreover no further 3^+-mergings are possible and thus $s = 3$. Thus, for both $s = 2$ and $s = 3$, we can assume the final 3^+-cluster F is made of F_1 and j copies of P_3 or C_3 where $j \in \{1, 2\}$. Then, $|F| = 4+3j$, F_1 contributes at least $8\,h(1, 3)+2\,h(1, 2)+2h(3, 4)+h(2, 4) = 10+2+\frac{1}{2} = \frac{25}{2}$ to $w(F)$. Hence, we have

$$w(F) \geqslant \frac{25}{2} + 9j = \frac{108j + 205}{122} + \frac{165}{61}(4 + 3j) > \frac{165}{61}|F|,$$

as desired.

Case 3-2. Assume that T_2 is a 3-tree.

Here $C_{F_1}(uv) \supseteq \{1, 2, 4, 5\}$ so F_2 has exactly 3 edges and $F' = F_1 \cup F_2$ is a $(10, 8)$-configuration. Suppose first that $s \geqslant 3$. Since no 3-claimed pair of F' can be 1-claimed by $G \setminus F'$, the 3-graphs F_3 and F' cannot be (3^+)-mergeable. So let F' and F_3 be (3^+)-mergeable via some pair $u'v'$. Since $3 \in C_{F_3}(u'v')$, $u'v'$ must be in $P_1(F') \setminus P_3(F') = \{au\}$; furthermore, F_3 rooted at $u'v' = au$ is isomorphic to P_3 or C_3, no further 3^+-mergings are possible and $s = 3$. Thus, for both $s = 2$ and $s = 3$, the final 3^+-cluster F consists of F_1 and j copies of P_3 or C_3 where $j \in \{1, 2\}$. Here, $|F| = 5 + 3j$. Note that F_1 contributes at least $10\,h(1, 3) + 2\,h(1, 2) + 4\,h(3, 4) = 12 + 4 = 16$ to the total weight. We have

$$w(F) \geqslant 16 + 9j = \frac{54j + 151}{61} + \frac{165}{61}(5 + 3j) > \frac{165}{61}|F|.$$

This finishes the proof of the claim. ■

Hence, by the previous claims, we conclude that

$$|G| = \sum_{F \in \mathcal{M}_{3^+}} |F| \leqslant \frac{61}{165} \sum_{F \in \mathcal{M}_{3^+}} w(F) = \frac{61}{165} \sum_{uv \in \binom{V(H)}{2}} w(uv) \leqslant \frac{61}{165}\binom{n}{2}.$$

This proves Theorem 1.3. ■

6 Concluding remarks

In this paper, we made progress on the Brown–Erdős–Sós Problem (the case of 3-uniform hypergraphs) with $k \in \{5, 6, 7\}$ edges and its extension to r-graphs. We note

https://doi.org/10.4153/S0008414X25000021 Published online by Cambridge University Press

that a further extension of the Brown–Erdős–Sós Problem proposed by Shangguan and Tamo [27] asks to determine whether the limits

$$\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$$

exist for all fixed r, t, k and, if so, to find their values. (The case that we studied here corresponds to $t = 2$.)

Let us briefly summarise what is known. The results in [4, 24] resolve the case $k = 2$. In [13, 15, 27], this problem was completely solved when $k \in \{3, 4\}$. In [8, 26], the existence of the limit was proved for $t = 2$. In [21], the value of the limit (and thus its existence) was established for even k when $r \gg k, t$ is sufficiently large. Also, it was proved in [21] that if $k \in \{5, 7\}$ then the limit exists for any r and t. Our results determine the limit values when $t = 2$ and $k \in \{5, 6, 7\}$.

It would be interesting to study the existence of limits for the remaining sets of parameters as well as their precise values.

Acknowledgments

We thank Felix Joos and Marcus Kühn for useful discussions. In particular, some starting ideas of the present paper originated from the joint work [15]. Also, we thank the anonymous referee for useful comments.

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