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Lambert series of logarithm, the derivative of Deninger's function $R(z)$, and a mean value theorem for $\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)$

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Dedicated to Christopher Deninger on account of his 65th birthday

Abstract. An explicit transformation for the series $\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1}$, or equivalently, $\sum_{n=1}^{\infty}d(n)\log(n)e^{-ny}$ for $\mathrm{Re}(y)>0$, which takes y to $1/y$, is obtained for the first time. This series transforms into a series containing the derivative of $R(z)$, a function studied by Christopher Deninger while obtaining an analog of the famous Chowla–Selberg formula for real quadratic fields. In the course of obtaining the transformation, new important properties of $\psi_1(z)$ (the derivative of $R(z)$) are needed as is a new representation for the second derivative of the two-variable Mittag-Leffler function $E_{2,b}(z)$ evaluated at $b=1$, all of which may seem quite unexpected at first glance. Our transformation readily gives the complete asymptotic expansion of $\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1}$ as $y\to 0$ which was also not known before. An application of the latter is that it gives the asymptotic expansion of $\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}\,dt$ as $\delta\to 0$.

1 Introduction

Eisenstein series are the building blocks of modular forms and thus lie at the heart of the theory. In the case of the full modular group $\mathrm{SL}_2(\mathbb{Z})$, the Eisenstein series of even integral weight $k\geq 2$ are given by

$$E_k(z):=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n}, \tag{1.1}$$

where $q=e^{2\pi iz}$ with $z\in\mathbb{H}$ (the upper-half plane) and B_k are the Bernoulli numbers. They satisfy the modular transformations

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$$
\begin{aligned}
&E_k(z+1) = E_k(z) \qquad (k \geq 2),\\
&E_k\left(\frac{-1}{z}\right) = z^k E_k(z) \quad (k > 2), \qquad E_2\left(\frac{-1}{z}\right) = z^2 E_2(z) + \frac{6z}{\pi i}.
\end{aligned} \tag{1.2}
$$

The series on the right-hand side of (1.1) is an example of what is known as a *Lambert series* whose general form is

$$
\sum_{n=1}^{\infty} \frac{a(n) q^n}{1 - q^n} = \sum_{n=1}^{\infty} \frac{a(n)}{e^{ny} - 1} = \sum_{n=1}^{\infty} (1 * a)(n) e^{-ny}, \tag{1.3}
$$

where $q = e^{-y}$ with $\mathrm{Re}(y) > 0$, and $a(n)$ is an arithmetic function with $(1 * a)(n) = \sum_{d|n} a(d)$ as the Dirichlet convolution.

For $\mathrm{Re}(s) > 1$, the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Ramanujan [9, pp. 275–276], [52, pp. 319–320, Formula (28)], [53, p. 173, Chapter 14, Entry 21(i)] derived a beautiful transformation involving the Lambert series associated with $a(n) = n^{-2m-1}, m \in \mathbb{Z}\backslash\{0\}$, and the odd zeta value $\zeta(2m+1)$, namely, for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ with $\alpha\beta = \pi^2$,

$$
\begin{aligned}
\alpha^{-m}\left\{\frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\alpha n} - 1}\right\} &= (-\beta)^{-m}\left\{\frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\beta n} - 1}\right\}\\
&\quad - 2^{2m} \sum_{j=0}^{m+1} \frac{(-1)^j B_{2j} B_{2m+2-2j}}{(2j)!(2m+2-2j)!} \alpha^{m+1-j} \beta^j.
\end{aligned} \tag{1.4}
$$

Along with the transformations of the Eisenstein series in (1.2), this formula also encapsulates the transformations of the corresponding Eichler integrals of the Eisenstein series as well as the transformation property of the Dedekind eta-function. The literature on this topic is vast with many generalizations and analogs for other L-functions, for example, [4, 6, 7, 12, 20–24, 32, 41, 42, 44]. See also the recent survey article [10].

Recently, Kesarwani, Kumar, and the second author [23, Theorem 2.4] obtained a new generalization of (1.4), namely, for $\mathrm{Re}(y) > 0$ and any complex a such that $\mathrm{Re}(a) > -1$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_a(n) e^{-ny} + \frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a} \operatorname{cosec}\left(\frac{\pi a}{2}\right) + 1\right)\zeta(-a) - \frac{1}{y}\zeta(1-a)\\
&= \frac{2\pi}{y \sin\left(\frac{\pi a}{2}\right)} \sum_{n=1}^{\infty} \sigma_a(n)\left(\frac{(2\pi n)^{-a}}{\Gamma(1-a)} {}_1F_2\left(1; \frac{1-a}{2}, 1 - \frac{a}{2}; \frac{4\pi^4 n^2}{y^2}\right) - \left(\frac{2\pi}{y}\right)^a \cosh\left(\frac{4\pi^2 n}{y}\right)\right),
\end{aligned} \tag{1.5}
$$

where ${}_1F_2(a; b, c; z)$ is the generalized hypergeometric function

$$
{}_1F_2(a; b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!},
$$

where $z \in \mathbb{C}$ and $(a)_n = a(a+1)\ldots(a+n-1)$. They [23, Theorem 2.5] also analytically continued this result to $\mathrm{Re}(a) > -2m-3, m \in \mathbb{N} \cup \{0\}$, and in this way, they were able to get as corollaries not only Ramanujan's formula (1.4) and the transformation formula for the Dedekind eta-function but also new transformations when a is an even integer. Moreover, they showed [23, Equation (2.19)] that letting $a \to 0$ in (1.5) gives the following transformation of Wigert [63, p. 203, Equation (A)]:

(1.6)
$$\sum_{n=1}^{\infty} \frac{1}{e^{ny}-1} = \frac{1}{4} + \frac{\gamma - \log(y)}{y} + \frac{2}{y}\sum_{n=1}^{\infty}\left\{\log\left(\frac{2\pi n}{y}\right) - \frac{1}{2}\left(\psi\left(\frac{2\pi i n}{y}\right) + \psi\left(-\frac{2\pi i n}{y}\right)\right)\right\},$$

where $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function commonly known as the digamma function. Wigert [63, p. 203] called this transformation *"la formule importante"* (an important formula). Indeed, it is important, for, if we let $y \to 0$ in any angle $|\arg(y)| \le \lambda$, where $\lambda < \pi/2$, it gives the complete asymptotic expansion upon using (3.13), that is,

$$\sum_{n=1}^{\infty} d(n)e^{-ny} \sim \frac{1}{4} + \frac{(\gamma - \log(y))}{y} - \sum_{n=1}^{\infty} \frac{B_{2n}^2 y^{2n-1}}{(2n)(2n)!}, \tag{1.7}$$

where γ is Euler's constant, which, in turn, readily implies [59, p. 163, Theorem 7.15]

$$\sum_{n=1}^{\infty} d(n)e^{-ny} = \frac{1}{4} + \frac{\gamma - \log y}{y} - \sum_{n=0}^{N-1} \frac{B_{2n+2}^2}{(2n+2)(2n+2)!} y^{2n+1} + O(|y|^{2N}). \tag{1.8}$$

The study of the moments

$$M_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \tag{1.9}$$

is of fundamental importance in the theory of the Riemann zeta function. It is conjectured that $M_k(T) \sim C_k T \log^{k^2}(T)$ as $T \to \infty$ for positive constants C_k although for $k = 1$ and 2, this has been proved by Hardy and Littlewood [33] and Ingham [35], respectively. Such results are known as mean value theorems for the zeta function. The importance of the study of moments lies, for example, in the fact that the estimate $M_k(T) = O_{k,\varepsilon}(T^{1+\varepsilon})$ for every natural number k is equivalent to the Lindelöf hypothesis [57] (see also [34]).

Another set of mean value theorems which plays an important role in the theory is the one concerning the asymptotic behavior of the smoothly weighted moments, namely,

$$\int_0^\infty \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} e^{-\delta t}\, dt \tag{1.10}$$

as $\delta \to 0$. The relation between the two types of moments in (1.9) and (1.10) is given by a result [59, p. 159] which states that if $f(t) \ge 0$ for all t and for a given positive m,

$$\int_0^\infty f(t)e^{-\delta t}\, dt \sim \frac{1}{\delta}\log^m\left(\frac{1}{\delta}\right)$$

as $\delta \to 0$, then

$$\int_0^T f(t)\, dt \sim T\log^m(T)$$

as $T \to \infty$. For an excellent survey on the moments, we refer the reader to [38].

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The asymptotic expansion of $\sum_{n=1}^{\infty} d(n)e^{-ny}$ as $y \to 0$ in (1.8) allows us to obtain the asymptotic estimate for the smoothly weighted second moment, namely, as $\delta \to 0$, for every natural number N, we have [59, p. 164, Theorem 7.15(A)]

$$\int_0^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 e^{-\delta t}\,dt = \frac{\gamma - \log(2\pi\delta)}{2\sin(\delta/2)} + \sum_{n=0}^{N} c_n\delta^n + O(\delta^{N+1}), \tag{1.11}$$

where the c_n are constants and the constant implied by the big-O depends on N. A simple proof of (1.11) was given by Atkinson [3]. In fact, (1.7) gives the complete asymptotic expansion for this moment.

The primary goal of this paper is to give a nontrivial application of (1.5). Note that the complex variable a in (1.5) enables differentiation of (1.5) with respect to a, which is not possible in (1.4) or other such known results. Indeed, it is an easy affair to check that differentiating the series on the left-hand side of (1.5) with respect to a and then letting $a = 0$ gives $\sum_{n=1}^{\infty} \frac{\log(n)}{e^{ny}-1}$, which, in view of (1.3), satisfies

$$\sum_{n=1}^{\infty} \frac{\log(n)}{e^{ny}-1} = \sum_{n=1}^{\infty} \log\Big(\prod_{d|n} d\Big)e^{-ny} = \frac{1}{2}\sum_{n=1}^{\infty} d(n)\log(n)e^{-ny}.$$

Here, in the last step, we used an elementary result $\prod_{d|n} d = n^{d(n)/2}$ (see, for example, [2, Exercise 10, p. 47]).

What is surprising though is, differentiating the right-hand side of (1.5) with respect to a and then letting $a = 0$ leads to an explicit and interesting series involving a well-known special function. This special function deserves a separate mention and hence after its brief introduction here, the literature on it is discussed in detail in Section 3.

In a beautiful paper [17], Deninger comprehensively studied the function $R : \mathbb{R}^+ \to \mathbb{R}$ uniquely defined by the difference equation

$$R(x+1) - R(x) = \log^2(x) \qquad (R(1) = -\zeta''(0)),$$

and with the requirement that R be convex in some interval (A, ∞), $A > 0$.

The function $R(x)$ is an analog of $\log(\Gamma(x))$ in view of the fact that the latter satisfies the difference equation $f(x+1) - f(x) = \log(x)$ with the initial condition $f(1) = 0$. As noted in [17, Remark 2.4], R can be analytically continued to

$$\mathbb{D} := \mathbb{C}\backslash\{x \in \mathbb{R} |\ x \le 0\}.$$

The special function which appears in our main theorem, that is, in Theorem 1.1, is $\psi_1(z)$, which is essentially the derivative of Deninger's function $R(z)$ (see (3.5)). For $z \in \mathbb{D}$, it is given by

$$\psi_1(z) = -\gamma_1 - \frac{\log(z)}{z} - \sum_{n=1}^{\infty}\left(\frac{\log(n+z)}{n+z} - \frac{\log(n)}{n}\right), \tag{1.12}$$

where γ_1 is the first Stieltjes constant.

We are now ready to state the main result of our paper which transforms the Lambert series of logarithm into an infinite series consisting of $\psi_1(z)$.

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Theorem 1.1 *Let $\psi_1(z)$ be given in (1.12). Then, for $Re(y) > 0$,*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1} &= -\frac{1}{4}\log(2\pi)+\frac{1}{2y}\log^2(y)-\frac{\gamma^2}{2y}+\frac{\pi^2}{12y}\\ &\quad -\frac{2}{y}(\gamma+\log(y))\sum_{n=1}^{\infty}\left\{\log\left(\frac{2\pi n}{y}\right)-\frac{1}{2}\left(\psi\left(\frac{2\pi in}{y}\right)+\psi\left(-\frac{2\pi in}{y}\right)\right)\right\}\\ &\quad +\frac{1}{y}\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi in}{y}\right)+\psi_1\left(-\frac{2\pi in}{y}\right)-\frac{1}{2}\left(\log^2\left(\frac{2\pi in}{y}\right)+\log^2\left(-\frac{2\pi in}{y}\right)\right)+\frac{y}{4n}\right\}. \end{aligned} \tag{1.13}$$

Equivalently,

$$\begin{aligned}&y\sum_{n=1}^{\infty}\frac{\gamma+\log(ny)}{e^{ny}-1}-\frac{1}{4}y\log(y)+y\left(\frac{1}{4}\log(2\pi)-\frac{\gamma}{4}\right)+\frac{1}{2}\log^2(y)-\frac{\gamma^2}{2}-\frac{\pi^2}{12}\\ &=\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi in}{y}\right)+\psi_1\left(-\frac{2\pi in}{y}\right)-\frac{1}{2}\left(\log^2\left(\frac{2\pi in}{y}\right)+\log^2\left(-\frac{2\pi in}{y}\right)\right)+\frac{y}{4n}\right\}.\end{aligned} \tag{1.14}$$

Remark 1.1 That the series

$$\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi in}{y}\right)+\psi_1\left(-\frac{2\pi in}{y}\right)-\frac{1}{2}\left(\log^2\left(\frac{2\pi in}{y}\right)+\log^2\left(-\frac{2\pi in}{y}\right)\right)+\frac{y}{4n}\right\}$$

converges absolutely is clear from (5.13).

The exact transformation in (1.13) is an analog of Wigert's result (1.6). This is evident from the fact that

$$\log\left(\tfrac{2\pi n}{y}\right)-\tfrac{1}{2}\left(\psi\left(\tfrac{2\pi in}{y}\right)+\psi\left(-\tfrac{2\pi in}{y}\right)\right)=-\tfrac{1}{2}\left\{\psi\left(\tfrac{2\pi in}{y}\right)+\psi\left(-\tfrac{2\pi in}{y}\right)-\left(\log\left(\tfrac{2\pi in}{y}\right)+\log\left(\tfrac{-2\pi in}{y}\right)\right)\right\},$$

which should be compared with the summand of the second series on the right-hand side of (1.13).

Thus, (1.13) allows us to transform $\sum_{n=1}^{\infty}d(n)\log(n)e^{-ny}$ into series having a constant times $1/y$ in the arguments of the functions in their summands. This "modular" behavior has an instant application: it gives the complete asymptotic expansion of $\sum_{n=1}^{\infty}d(n)\log(n)e^{-ny}$ as $y\to 0$, which is given in the following result.

Theorem 1.2 *Let A denote the Glaisher–Kinkelin constant defined by* [27, 28, 43] *and [61, p. 461, Equation (A.7)]*

$$\log(A):=\lim_{n\to\infty}\left\{\sum_{k=1}^{n}k\log(k)-\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\log(n)+\frac{n^2}{4}\right\}.$$

As $y\to 0$ in $|\arg(y)|<\pi/2$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1} &\sim \frac{1}{2y}\log^2(y)+\frac{1}{y}\left(\frac{\pi^2}{12}-\frac{\gamma^2}{2}\right)-\frac{1}{4}\log(2\pi)+\frac{y}{12}\left(\log A-\frac{1}{12}\right)\\ &\quad +\sum_{k=2}^{\infty}\frac{B_{2k}y^{2k-1}}{k}\left\{\frac{B_{2k}}{2(2k)!}\left(\gamma-\sum_{j=1}^{2k-1}\frac{1}{j}+\log(2\pi)\right)+\frac{(-1)^k\zeta'(2k)}{(2\pi)^{2k}}\right\}.\end{aligned} \tag{1.15}$$

As seen earlier, Wigert's result (1.8) is useful in getting the asymptotic estimate for the smoothly weighted second moment of the zeta function on the critical line given

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in (1.11). It is now natural to ask whether our result in Theorem 1.2 has an application in the theory of moments. Indeed, (1.15) implies the following result.

Theorem 1.3 *As $\delta \to 0, |\arg(\delta)| < \pi/2$,*

$$\int_0^\infty \zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}\,dt = -\frac{1}{4\sin\left(\frac{\delta}{2}\right)}\left(\log^2(2\pi\delta)+\frac{\pi^2}{6}-\gamma^2\right)+\sum_{k=0}^{2m-2}(d_k+d_k'\log(\delta))\delta^k + O\left(\delta^{2m-1}\log(\delta)\right), \tag{1.16}$$

where d_k and d_k' are effectively computable constants and the constant implied by the big-O depends on m.

In fact, one can obtain the complete asymptotic expansion of the left-hand side of (1.16) using (1.15).

Mean value theorems involving the derivatives of the Riemann zeta function have been well-studied. For example, Ingham [35] (see also Gonek [29, Equation (2)][1]) showed that

$$\int_0^T \zeta^{(\mu)}\left(\frac{1}{2}+it\right)\zeta^{(\nu)}\left(\frac{1}{2}-it\right)dt \sim \frac{(-1)^{\mu+\nu}\;T}{\mu+\nu+1}\log^{\mu+\nu+1}(T)$$

as $T \to \infty$, where $\mu, \nu \in \mathbb{N}\cup\{0\}$ (see also [60, p. 102]). In particular, for $\mu = 1$ and $\nu = 0$, we have

$$\int_0^T \zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)dt \sim \frac{-T}{2}\log^2(T)$$

as $T \to \infty$.

The proof of Theorem 1.1 is quite involved and first requires establishing several new results that may seem quite unexpected at first glance. These results are important in themselves and may have applications in other areas. Hence, this paper is organized as follows.

We collect frequently used results in the next section. In Section 3, we first prove Theorem 3.2 which gives an asymptotic expansion of $\psi_1(z)$ followed by Theorem 3.3, a Kloosterman-type result for $\psi_1(z)$. Theorem 3.4 is the highlight of this section and is an analog of (2.2) established in [22, Theorem 2.2]. Section 4 is devoted to obtaining a new representation for the second derivative of the two-variable Mittag-Leffler function $E_{2,b}(z)$ at $b = 1$. In Section 5, we prove our main results, that is, Theorems 1.1–1.3. Finally, we conclude the paper with some remarks and directions for future research.

2 Preliminaries

Stirling's formula in a vertical strip $\alpha \le \sigma \le \beta$, $s = \sigma + it$ states that [16, p. 224]

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{2.1}$$

uniformly as $|t| \to \infty$.

[1]There is a slight typo in the asymptotic formula on the right-hand side of this equation in that $(-1)^{\mu+\nu}$ is missing.

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We will also need the following result established in [22, Theorem 2.2] which is valid for $\mathrm{Re}(w) > 0$:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{t\cos(t)}{t^2+n^2w^2}\,dt=\frac{1}{2}\left\{\log\left(\frac{w}{2\pi}\right)-\frac{1}{2}\left(\psi\left(\frac{iw}{2\pi}\right)+\psi\left(-\frac{iw}{2\pi}\right)\right)\right\}. \tag{2.2}$$

Watson's lemma is a very useful result in the asymptotic theory of Laplace integrals $\int_0^{\infty} e^{-zt}f(t)\,dt$. This result typically holds for $|\arg(z)| < \pi/2$. However, with additional restrictions on f, Watson's lemma is known to hold for extended sectors. For the sake of completeness, we include it here in the form given in [58, p. 14, Theorem 2.2].

Theorem 2.1 *Let f be analytic inside a sector $D : \alpha < \arg(t) < \beta$, where $\alpha < 0$ and $\beta > 0$. For each $\delta \in \left(0, \frac{1}{2}\beta - \frac{1}{2}\alpha\right)$, as $t \to 0$ in the sector $D_\delta : \alpha + \delta < \arg(t) < \beta - \delta$, we have*

$$f(t) \sim t^{\lambda-1}\sum_{n=0}^{\infty} a_n t^n,$$

where $\mathrm{Re}(\lambda) > 0$. Suppose there exists a real number σ such that $f(t) = O\left(e^{\sigma|t|}\right)$ as $t \to \infty$ in D_δ. Then the integral

$$F_\lambda(z) := \int_0^{\infty} e^{-zt}f(t)\,dt \tag{2.3}$$

or its analytic continuation, has the asymptotic expansion

$$F_\lambda(z) \sim \sum_{n=0}^{\infty} a_n\frac{\Gamma(n+\lambda)}{z^{n+\lambda}}$$

as $|z| \to \infty$ in the sector

$$-\beta-\frac{\pi}{2}+\delta < \arg(z) < -\alpha+\frac{\pi}{2}-\delta.$$

The many-valued functions $t^{\lambda-1}$ and $z^{n+\lambda}$ have their principal values on the positive real axis and are defined by continuity elsewhere.

Actually, we will be using an analog of the above theorem where the integrand in (2.3) has a logarithmic factor.

We next give Dirichlet's test for uniform convergence of definite integrals [55, p. 261] which will be used in the course of proving Theorem 4.4.

Theorem 2.2 *If $g(x, y)$ is continuous on $\{(x, y)|c \le x, m \le y \le n\}$ and $|\int_c^X g(x, y)\,dx| < K$ for all $X \ge c$ and all y on $[m, n]$, if $f(x, y)$ is a decreasing function of x for $x \ge c$ and each fixed y on $[m, n]$, and if $f(x, y)$ tends to zero uniformly in y as $x \to \infty$, then $\int_c^{\infty} f(x, y)g(x, y)\,dx$ converges uniformly on $m \le y \le n$.*

Another result which will be needed in the sequel is the following [55, p. 260].

Theorem 2.3 *Let $\frac{\partial}{\partial y}h(x, y)$ and $h(x, y)$ are continuous on $[c, \infty) \times [m, n]$, if $\int_c^{\infty} h(x, y)\,dx$ converges for at least one y_0 in $[m, n]$, and if $\int_c^{\infty} \frac{\partial}{\partial y}h(x, y)\,dx$ converges*

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uniformly on $[m,n]$, *then* $\int_c^\infty h(x,y)\,dx$ *converges uniformly on* $[m,n]$, *and*

$$\frac{d}{dy}\int_c^\infty h(x,y)\,dx = \int_c^\infty \frac{\partial}{\partial y}h(x,y)\,dx. \tag{2.4}$$

We will also be using Parseval's theorem given next. Let $\mathfrak{F}(s) = M[f;s]$ and $\mathfrak{G}(s) = M[g;s]$ denote the Mellin transforms of functions $f(x)$ and $g(x)$, respectively, and let $c = \mathrm{Re}(s)$. If $M[f;1-c-it] \in L(-\infty,\infty)$ and $x^{c-1}g(x) \in L[0,\infty)$, then Parseval's formula [49, p. 83] is given by

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{(c)} \mathfrak{F}(1-s)\mathfrak{G}(s)\,ds, \tag{2.5}$$

where the vertical line $\mathrm{Re}(s) = c$ lies in the common strip of analyticity of the Mellin transforms $\mathfrak{F}(1-s)$ and $\mathfrak{G}(s)$, and, here and throughout the sequel, we employ the notation $\int_{(c)}$ to denote the line integral $\int_{c-i\infty}^{c+i\infty}$.

3 New results on $\psi_1(z)$

In [19], Dilcher studied in detail, the generalized gamma function $\Gamma_k(z)$ which relates to the Stieltjes constant γ_k, $k \geq 0$, defined by[2]

$$\gamma_k := \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{\log^k(j)}{j} - \frac{\log^{k+1}(n)}{k+1}\right), \tag{3.1}$$

in a similar way as the Euler gamma function $\Gamma(z)$ relates to the Euler constant $\gamma = \gamma_0$. Using [19, Equation (2.1)] and [17, Equation (2.3.1)], we see that Dilcher's $\Gamma_1(z)$ is related to Deninger's $R(z)$ by[3]

$$\log(\Gamma_1(z)) = \frac{1}{2}(R(z) + \zeta''(0)). \tag{3.2}$$

As mentioned by Deninger in [17, Remark 2.4], contrary to Euler's Γ, the function $\exp(R(x))$, or equivalently $\Gamma_1(x)$, where $x > 0$, cannot be meromorphically continued to the whole complex plane. But $\Gamma_1(z)$ is analytic in $z \in \mathbb{D}$. It is this $\exp(R(x))$ that Languasco and Righi [45] call as the *Ramanujan–Deninger gamma function*. They have also given a fast algorithm to compute it.

Dilcher also defined the generalized digamma function $\psi_k(z)$ as the logarithmic derivative of $\Gamma_k(z)$. His Proposition 10 from [19] implies that for $z \in \mathbb{D}$,

$$\psi_k(z) = -\gamma_k - \frac{\log^k(z)}{z} - \sum_{n=1}^{\infty}\left(\frac{\log^k(n+z)}{n+z} - \frac{\log^k(n)}{n}\right), \tag{3.3}$$

where $k \in \mathbb{N}\cup\{0\}$. Its special case $k=1$ has already been given in (1.12). The function $\psi_k(z)$ occurs in Entry 22 of Chapter 8 in Ramanujan's second notebook, see [8]. It is also used by Ishibashi [36] to construct the kth order Herglotz function which, in turn, plays an important role in his evaluation of the Laurent series coefficients of a

[2] Note that Deninger's definition of γ_1 in [17, p. 174] involves an extra factor of 2 which is not present in conventional definition of γ_1, that is, in the $k=1$ case of (3.1).

[3] By analytic continuation, Equation (2.3.1) from [17] is valid for $z \in \mathbb{D}$.

zeta function associated with an indefinite quadratic form. As noted by Ishibashi and Kanemitsu [37, p. 78],

$$\psi_k(z) = \frac{1}{k+1} R'_{k+1}(z),$$

where $R_k(z)$ is defined by [17, p. 173]

$$R_k(z) = (-1)^{k+1}\left(\frac{\partial^k}{\partial s^k}\zeta(0,z) - \zeta^{(k)}(0)\right), \tag{3.4}$$

and $R_2(z) = R(z)$ of Deninger. Thus,

$$\psi_1(z) = \frac{1}{2}R'(z), \tag{3.5}$$

which is also implied by (3.2). The function $\psi_k(z)$ is related to the Laurent series coefficients $\gamma_k(z)$ of the Hurwitz zeta function $\zeta(s,z)$ known as the *generalized Stieltjes constants*. To see this, from [5, Theorem 1], note that if

$$\zeta(s,z) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k(z)}{k!}(s-1)^k, \tag{3.6}$$

then[4]

$$\gamma_k(z) = \lim_{n\to\infty}\left(\sum_{j=0}^{n} \frac{\log^k(j+z)}{j+z} - \frac{\log^{k+1}(n+z)}{k+1}\right) \tag{3.7}$$

so that $\gamma_k(1) = \gamma_k$. Then from (3.1), (3.3), (3.7) and the fact [19, Lemma 1] that

$$\lim_{n\to\infty}\left(\log^{k+1}(n+z) - \log^{k+1}(n)\right) = 0 \qquad (z \in \mathbb{D}),$$

it is not difficult to see that

$$\psi_k(z) = -\gamma_k(z), \tag{3.8}$$

which was also shown by Shirasaka [56, p. 136]. Further properties and applications of $\psi_k(z)$ are derived in [18].

We thus see that the literature on $R(z)$, that is, $R_2(z)$, and, in general, on $R_k(z)$, is growing fast. In the words of Ishibashi [36, p. 61], *"Deninger proved several analytic properties of $R_2(x)$ in order to familiarize and assimilate it as one of the most commonly used number-theoretic special functions,. . . ."* Languasco and Righi [45] have also given a fast algorithm to compute $\psi_1(x), x > 0$. The papers of Berndt [5], Blagouchine [11], Coffey [15], Chatterjee and Khurana [14] (and also the references therein) on the generalized Stieltjes constants, which, in view of (3.8), are nothing but $-\psi_k(z)$, are also vast sources of information on them. However, none of the studies prior to our current work is devoted to applications of these constants in obtaining a "modular" transformation for the Lambert series of logarithm and its application in the theory of the moments of the Riemann zeta function given in Theorems 1.1 and 1.3, respectively.

[4] It is to be noted that Berndt includes the factor $\frac{(-1)^k}{k!}$ in the definition of $\gamma_k(z)$ and does not have it in the summand of (3.6).

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Our first result of this section gives the asymptotic expansion of $\psi_1(z)$ for $z \in \mathbb{D}$. To accomplish it, we require a generalization of Watson's lemma which allows for a logarithmic factor in the integrand. Such an expansion seems to have been first obtained by Jones [40, p. 439] (see also [65, Equations (4.14) and (4.15)]). Though we will be using the same expansion, it is useful to rigorously derive it as a special case of a more general result due to Wong and Wyman [65, Theorem 4.1] given in the following theorem. We note in passing that Riekstins [54] has also obtained asymptotic expansions of integrals involving logarithmic factors.

Theorem 3.1 *For $\gamma \in \mathbb{R}$, define a function*

$$F(z) := \int_0^{\infty e^{i\gamma}} f(t)e^{-zt}dt.$$

Assume that $F(z)$ exists for some $z = z_0$. If:

(1) *For each integer $N \in \mathbb{N} \cup \{0\}$,*

$$f(t) = \sum_{n=0}^{N} a_n t^{\lambda_n - 1} P_n(\log t) + o(t^{\lambda_N - 1}(\log t)^{m(N)}),$$

as $t \to 0$ along $\arg(t) = \gamma$.

(2) *$P_n(\omega)$ is a polynomial of degree $m = m(n)$.*

(3) *$\{\lambda_n\}$ is a sequence of complex numbers, with $\Re(\lambda_{n+1}) > \Re(\lambda_n), \Re(\lambda_0) > 0$, for all n such that n and $n+1$ are in $\mathbb{N} \cup \{0\}$.*

(4) *$\{a_n\}$ is a sequence of complex numbers.*

Then as $z \to \infty$ in $S(\Delta)$

$$F(z) \sim \sum_{n=0}^{N} a_n P_n(D_n)[\Gamma(\lambda_n)z^{-\lambda_n}] + o\left(z^{-\lambda_N}(\log z)^{m(N)}\right),$$

where $S(\Delta) : |\arg(ze^{i\gamma})| \leqq \frac{\pi}{2} - \Delta$, and D_n is the operator $D_n := \frac{d}{d\lambda_n}$. This result is uniform in the approach of $z \to \infty$ in $S(\Delta)$.

Theorem 3.2 *Fix any $\Delta > 0$. Then as $z \to \infty$ with $|\arg(z)| \le \pi - \Delta$,*

$$\psi_1(z) \sim \frac{1}{2}\log^2(z) - \frac{1}{2z}\log z + \sum_{k=1}^{\infty} \frac{B_{2k}}{2kz^{2k}} \left(\sum_{j=1}^{2k-2} \frac{1}{j} + \frac{1}{2k-1} - \log z \right). \tag{3.9}$$

Proof We first prove the result for $|\arg(z)| < \pi/2$ and then extend it to $|\arg(z)| \le \pi - \Delta$. To that end, we begin with the analog for $R(z)$ of Plana's integral for $\log(\Gamma(z))$, namely, for $\mathrm{Re}(z) > 0$, we have [17, Equation (2.12)]

$$R(z) = -\zeta''(0) - 2\int_0^{\infty} \left((z-1)e^{-t} + \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} \right) \frac{\gamma + \log(t)}{t}\, dt. \tag{3.10}$$

Differentiating (3.10) under the integral sign with respect to z and using (3.5), we see that

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$$\psi_1(z) = -\int_0^\infty \left(e^{-t} - \frac{te^{-zt}}{1-e^{-t}}\right)\frac{(\gamma+\log(t))}{t}dt$$
$$= -\int_0^\infty \left(e^{-t} - e^{-zt}\right)(\gamma+\log(t))\frac{dt}{t} - \int_0^\infty e^{-zt}\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)(\gamma+\log(t))dt$$
$$= \frac{1}{2}\log^2(z) - \int_0^\infty e^{-zt}\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)(\gamma+\log(t))dt, \tag{3.11}$$

where we used the fact that for $\mathrm{Re}(z) > 0$,

$$\int_0^\infty (e^{-t} - e^{-zt})(\gamma+\log(t))\frac{dt}{t} = -\frac{1}{2}\log^2(z),$$

which follows from [17, Equation (2.13)][5]

$$\int_0^\infty (e^{-\beta t} - e^{-\alpha t})(\gamma+\log(t))\frac{dt}{t} = \frac{1}{2}\left(\log^2(\beta) - \log^2(\alpha)\right).$$

Thus

$$\psi_1(z) = \frac{1}{2}\log^2(z) - \gamma(\psi(z) - \log(z)) - \int_0^\infty e^{-zt}\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)\log(t)\,dt, \tag{3.12}$$

where we employed the well-known result [31, p. 903, Formula 8.361.8] that for $\mathrm{Re}(z) > 0$,

$$\psi(z) = \log(z) + \int_0^\infty e^{-zt}\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)dt.$$

We now find the asymptotic expansion of the integral on the right-hand side of (3.12), that is, of

$$I := \int_0^\infty e^{-zt} f(t)\,dt,$$

where

$$f(t) := \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)\log(t),$$

by applying Theorem 3.1. To that end, observe that for $|t| < 2\pi$,

$$f(t) = \log(t)\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)$$
$$= -\frac{\log(t)}{t}\left(\frac{te^t}{e^t-1} - 1\right)$$
$$= -\frac{\log(t)}{t}\left(\sum_{n=0}^\infty \frac{B_n(1)t^n}{n!} - 1\right)$$
$$= \log(t)\sum_{n=0}^\infty \frac{(-1)^n B_{n+1}t^n}{(n+1)!},$$

[5] Deninger requires it with $\alpha > 0, \beta > 0$, however, it is easily seen to hold for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ as well.

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where $B_n(x)$ are Bernoulli polynomials. Thus, with $P_n(x) = x$, $\lambda_n = n+1$, $a_n = \frac{(-1)^n B_{n+1}}{(n+1)!}$, $n \geq 0$, all of the hypotheses of Theorem 3.1 are satisfied, and hence

$$I \sim \sum_{n=0}^{\infty} \frac{(-1)^n B_{n+1}}{(n+1)z^{n+1}} \left(\psi(n+1) - \log(z)\right)$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{n z^n} \left(-\gamma + \sum_{k=1}^{n-1} \frac{1}{k} - \log(z)\right),$$

where we used the elementary fact $\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}$. Inserting this asymptotic expansion of I in (3.12) along with that of $\psi(z)$, namely, for $|\arg z| \leq \pi - \Delta$, (with $\Delta > 0$),

$$\psi(z) \sim \log(z) - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n z^{2n}}, \tag{3.13}$$

as $z \to \infty$, we arrive at

$$\psi_1(z) \sim \frac{1}{2}\log^2(z) - \gamma\left(-\frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{nz^n}\left(-\gamma + \sum_{j=1}^{n-1} \frac{1}{j} - \log(z)\right)$$
$$= \frac{1}{2}\log^2(z) + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{nz^n}\left(\sum_{j=1}^{n-1} \frac{1}{j} - \log(z)\right)$$
$$= \frac{1}{2}\log^2(z) - \frac{1}{2z}\log z + \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}\left(\sum_{j=1}^{2n-2} \frac{1}{j} + \frac{1}{2n-1} - \log(z)\right)$$

using the well-known facts $B_1 = -1/2$ and $B_{2n-1} = 0, n > 1$. This proves (3.9) for $|\arg(z)| \leq \pi/2 - \Delta$, where $\Delta > 0$.

To extend it to $|\arg(z)| \leq \pi - \Delta$, we use the analog of Theorem 2.1 containing a logarithmic factor in the integrand of the concerned integral, which practically changes none of the hypotheses in the statement of Theorem 2.1 and its proof[6] since $\log(t) = O(t^{\epsilon})$ as $t \to \infty$ for any $\epsilon > 0$. We apply it with $\alpha = -\pi/2$ and $\beta = \pi/2$. It shows that the expansion in (3.9) holds for $|\arg(z)| \leq \pi - \Delta$. ■

Remark 3.1 The result in the above theorem is not new. In fact, in [15, Proposition 3], this has been done for $\psi_k(z)$ for all $k \in \mathbb{N}$ in view of (3.8). We derive the asymptotic expansion for $\psi_1(z)$ right from scratch only to make this paper self-contained. Also, it is to be noted that in [15], the result has been proved only for $|\arg(z)| < \pi/2$ whereas we crucially require the result to hold for $|\arg(z)| \leq \pi - \Delta$ for $\Delta > 0$. We also note that one could also obtain this asymptotic behavior beginning with the Euler–Maclaurin summation formula.

Our next result is a new analog of Kloosterman's result for $\psi(x)$ [59, pp. 24–25].

Theorem 3.3 *Let $|\arg(z)| < \pi$. Let ψ_1 be defined in (1.12). For $0 < c = Re(s) < 1$,*

$$\psi_1(z+1) - \frac{1}{2}\log^2(z) = \frac{1}{2\pi i}\int_{(c)} \frac{\pi\zeta(1-s)}{\sin(\pi s)}\left(\gamma - \log(z) + \psi(s)\right) z^{-s}\, ds. \tag{3.14}$$

[6] Temme [58, p. 15] refers to Olver [47, p. 114] for a proof.

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Proof We first prove the result for $z > 0$ and later extend it to $|\arg(z)| < \pi$ by analytic continuation. Using (3.11), we have

$$\psi_1(z) - \frac{1}{2}\log^2(z) = \int_0^\infty e^{-zt}(\gamma + \log(t))\, dt + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-zt}(\gamma + \log(t))\, dt$$

$$= -\frac{\log(z)}{z} + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-zt}(\gamma + \log(t))\, dt, \tag{3.15}$$

where in the last step, we employed [31, p. 573, Formula 4.352.1]

$$\int_0^\infty t^{s-1} e^{-zt}(\gamma + \log(t))\, dt = \frac{\Gamma(s)}{z^s}(\gamma - \log(z) + \psi(s)) \qquad (\mathrm{Re}(s) > 0) \tag{3.16}$$

with $s = 1$ and the fact that $\psi(1) = -\gamma$. We now evaluate the integral on the right-hand side of (3.15) by means of Parseval's formula for Mellin transforms [49, p. 83, Equation (3.1.13)]. For $0 < \mathrm{Re}(s) < 1$, we have [59, p. 23, Equation (2.7.1)]

$$\int_0^\infty t^{s-1}\left(\frac{1}{e^t - 1} - \frac{1}{t}\right) dt = \Gamma(s)\zeta(s). \tag{3.17}$$

Therefore, along with (3.16) and the equation given above, for $0 < c = \mathrm{Re}(s) < 1$, Parseval's formula (2.5) implies

$$\int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-zt}(\gamma + \log(t))\, dt = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)\zeta(s)\frac{\Gamma(1-s)}{z^{1-s}}(\gamma - \log(z) + \psi(1-s))\, ds$$

$$= \frac{1}{2\pi i z}\int_{(c)} \frac{\pi\zeta(s)}{\sin(\pi s)}(\gamma - \log(z) + \psi(1-s))\, z^s\, ds$$

$$= \frac{1}{2\pi i}\int_{(c')} \frac{\pi\zeta(1-s)}{\sin(\pi s)}(\gamma - \log(z) + \psi(s))\, z^{-s}\, ds, \tag{3.18}$$

where $0 < c' < 1$. In the second step, we used the reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, $s \notin \mathbb{Z}$, and in the last step, we replaced s by $1 - s$.

Now, (3.14) follows by substituting (3.18) in (3.15) and using the fact [19, Equation (8.3)]

$$\psi_1(z) = \psi_1(z+1) - \frac{\log(z)}{z}. \tag{3.19}$$

This proves (3.14) for $z > 0$. The result is easily seen to be true for any complex z such that $|\arg(z)| < \pi$ by analytic continuation with the help of (1.12), elementary bounds on the Riemann zeta function, Stirling's formula (2.1) and the corresponding estimate for $\psi(s)$. ∎

In the next theorem, we give a closed-form evaluation of an infinite series of integrals in terms of the digamma function and $\psi_1(z)$. This theorem is an analog of (2.2) and will play a fundamental role in the proof of Theorem 1.1.

Theorem 3.4 *For* $\mathrm{Re}(w) > 0$, *we have*

$$4\sum_{m=1}^\infty \int_0^\infty \frac{u\cos(u)\log(u/w)}{u^2 + (2\pi m w)^2}\, du = \psi_1(iw) - \frac{1}{2}\log^2(iw) + \psi_1(-iw) - \frac{1}{2}\log^2(-iw) + \frac{\pi}{2w}$$

$$+ \gamma\left(\psi(iw) + \psi(-iw) - 2\log(w)\right). \tag{3.20}$$

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Proof Using Theorem 3.3, once with $z = iw$, and again with $z = -iw$ and adding the respective sides, for $0 < c = \mathrm{Re}(s) < 1$, we obtain

$$\begin{aligned}
&\psi_1(iw+1) - \frac{1}{2}\log^2(iw) + \psi_1(-iw+1) - \frac{1}{2}\log^2(-iw) \\
&= \frac{1}{2\pi i}\int_{(c)} \frac{\pi\zeta(1-s)}{\sin(\pi s)}\left(\gamma - \log(iw) + \psi(s)\right)(iw)^{-s} ds \\
&\quad + \frac{1}{2\pi i}\int_{(c)} \frac{\pi\zeta(1-s)}{\sin(\pi s)}\left(\gamma - \log(-iw) + \psi(s)\right)(-iw)^{-s} ds \\
&= \frac{1}{2\pi i}\int_{(c)} \frac{2\pi\zeta(1-s)}{\sin(\pi s)}\left(\gamma + \psi(s)\right)\cos\left(\frac{\pi s}{2}\right) w^{-s}\, ds - I_2, \qquad (3.21)
\end{aligned}$$

where

$$I_2 := \frac{1}{2\pi i}\int_{(c)} \frac{\pi\zeta(1-s)}{\sin(\pi s)}\left(e^{-\frac{i\pi s}{2}}\log(iw) + e^{\frac{i\pi s}{2}}\log(-iw)\right) w^{-s}\, ds. \qquad (3.22)$$

Next, using the series expansions of the exponential functions $e^{\frac{i\pi s}{2}}$ and $e^{-\frac{i\pi s}{2}}$ and splitting them according to n even and n odd, it is easily seen that

$$e^{-\frac{i\pi s}{2}}\log(iw) + e^{\frac{i\pi s}{2}}\log(-iw) = 2\log(w)\cos\left(\frac{\pi s}{2}\right) + \pi\sin\left(\frac{\pi s}{2}\right). \qquad (3.23)$$

Hence, from (3.21)–(3.23), we see that

$$\psi_1(iw+1) - \frac{1}{2}\log^2(iw) + \psi_1(-iw+1) - \frac{1}{2}\log^2(-iw) =: J_1 - J_2, \qquad (3.24)$$

where

$$\begin{aligned}
J_1 &:= \frac{1}{2\pi i}\int_{(c)} \frac{\pi\zeta(1-s)}{\sin\left(\frac{\pi s}{2}\right)}\left(\gamma + \psi(s) - \log(w)\right) w^{-s}\, ds, \\
J_2 &:= \frac{1}{2\pi i}\int_{(c)} \frac{\pi^2\zeta(1-s)}{2\cos\left(\frac{\pi s}{2}\right)} w^{-s}\, ds.
\end{aligned}$$

Using the functional equation of $\zeta(s)$ [59, p. 13, Equation (2.1.1)]

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{1}{2}\pi s\right), \qquad (3.25)$$

with s replaced by $1-s$, we have

$$J_2 = \frac{\pi^2}{2\pi i}\int_{(c)} \Gamma(s)\zeta(s)(2\pi w)^{-s}\, ds = \pi^2\left(\frac{1}{e^{2\pi w}-1} - \frac{1}{2\pi w}\right), \qquad (3.26)$$

where the last step follows from (3.17). Again using (3.25) with s replaced by $1-s$, we observe that

$$J_1 = \frac{1}{2\pi i}\int_{(c)} 2\pi\Gamma(s)\zeta(s)\cot\left(\frac{\pi s}{2}\right)\left(\gamma + \psi(s) - \log(w)\right)(2\pi w)^{-s}\, ds.$$

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Now, it is important to observe that shifting the line of integration from $\mathrm{Re}(s) = c$, $0 < c < 1$ to $\mathrm{Re}(s) = d$, $1 < d < 2$ does not introduce any pole of the integrand. So consider the rectangular contour $[c - iT, d - iT], [d - iT, d + iT], [d + iT, c + iT]$, and $[c + iT, c - iT]$. By Cauchy's residue theorem and the fact that the integral along the horizontal segments of the contour tend to zero as $T \to \infty$, as can be seen from (2.1), elementary bounds of the zeta function and the fact [29, Equation (15)] $\psi(s) = \log(s) + O(1/|s|)$, it is seen that

$$\begin{aligned} J_1 &= \frac{1}{2\pi i}\int_{(d)} 2\pi\Gamma(s)\zeta(s)\cot\left(\frac{\pi s}{2}\right)(\gamma + \psi(s) - \log(w))(2\pi w)^{-s}\,ds \\ &= \sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{(d)} 2\pi\Gamma(s)\cot\left(\frac{\pi s}{2}\right)(\gamma + \psi(s) - \log(w))(2\pi m w)^{-s}\,ds \\ &= 2\pi\sum_{m=1}^{\infty}\left\{\frac{(\gamma - \log(w))}{2\pi i}\int_{(d)}\Gamma(s)\cot\left(\frac{\pi s}{2}\right)(2\pi m w)^{-s}\,ds\right. \\ &\qquad \left. + \frac{1}{2\pi i}\int_{(d)}\Gamma(s)\psi(s)\cot\left(\frac{\pi s}{2}\right)(2\pi m w)^{-s}\,ds\right\}, \end{aligned} \tag{3.27}$$

where in the first step, we used the series representation for $\zeta(s)$ and then interchanged the order of summation and integration which is valid because of absolute and uniform convergence. We now find convenient representations for the two line integrals.

For $0 < \mathrm{Re}(s) = (c_1) < 2$ and $\mathrm{Re}(z) > 0$, we show that

$$\frac{1}{2\pi i}\int_{(c_1)}\Gamma(s)\cot\left(\frac{\pi s}{2}\right)z^{-s}\,ds = \frac{2}{\pi}\int_0^\infty \frac{u\cos(u)}{u^2 + z^2}\,du. \tag{3.28}$$

We first prove this result for $z > 0$ and then extend it by analytic continuation. We also first prove the result for $0 < \mathrm{Re}(s) = c_1 < 1$. To that end, using the double angle formulas for sine and cosine, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{(c_1)}\Gamma(s)\cot\left(\frac{\pi s}{2}\right)z^{-s}\,ds &= \frac{1}{2\pi i}\int_{(c_1)}\Gamma(s)\frac{(1 + \cos(\pi s))}{\sin(\pi s)}z^{-s}\,ds \\ &= \frac{1}{2\pi i}\int_{(c_1)}\Gamma(s)\mathrm{cosec}(\pi s)z^{-s}\,ds \\ &\quad + \frac{1}{2\pi i}\int_{(c_1)}\Gamma(s)\cot(\pi s)z^{-s}\,ds \\ &= -\frac{1}{\pi}e^{z}\mathrm{Ei}(-z) - \frac{1}{\pi}e^{-z}\mathrm{Ei}(z), \end{aligned} \tag{3.29}$$

where in the last step we used [13, p. 102, Formula 3.3.2.1], and where $\mathrm{Ei}(x)$ is the exponential integral defined [51, p. 788] for $z > 0$ by $\mathrm{Ei}(z) := \int_{-\infty}^{z} e^t/t\,dt$, or, as is seen from [39, p. 1], by $\mathrm{Ei}(-z) := -\int_z^\infty e^{-t}/t\,dt$. Equation (3.28) now follows upon using [50, p. 395, Formula 2.5.9.12]

$$e^{-z}\mathrm{Ei}(z) + e^{z}\mathrm{Ei}(-z) = -2\int_0^\infty \frac{u\cos(u)}{u^2 + z^2}\,du. \tag{3.30}$$

Since both sides are analytic for $\mathrm{Re}(z) > 0$, the result now follows by analytic continuation for $\mathrm{Re}(z) > 0$.

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Next, observe that shifting the line of integration from $\mathrm{Re}(s) = c_1$ to $\mathrm{Re}(s) = d$, where $1 < d < 2$, and using Cauchy's residue theorem does not change the evaluation of the integral on the left-hand side of (3.28). Hence, invoking (3.28) with $z = 2\pi m w$, we find that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{1}{2\pi i} \int_{(d)} \Gamma(s) \cot\left(\frac{\pi s}{2}\right) (2\pi m w)^{-s}\, ds &= \frac{2}{\pi} \sum_{m=1}^{\infty} \int_0^{\infty} \frac{u \cos(u)}{u^2 + (2\pi m w)^2}\, du \\ &= \frac{1}{\pi}\left(\log(w) - \frac{1}{2}\left(\psi(iw) + \psi(-iw)\right)\right), \qquad (3.31)\end{aligned}$$

where in the last step, we employed (2.2).

Here, it is important to note that the series representation of J_1 in (3.27) and the convergence of the series in (3.31) together imply that the series

$$\sum_{m=1}^{\infty} \frac{1}{2\pi i} \int_{(d)} \Gamma(s)\psi(s) \cot\left(\frac{\pi s}{2}\right) (2\pi m w)^{-s}\, ds$$

converges too. We now suitably transform the line integral in the above equation into an integral of a real variable by proving that for $0 < \mathrm{Re}(s) = c < 1$ and $\mathrm{Re}(z) > 0$, we have

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s)\psi(s) \cot\left(\frac{\pi s}{2}\right) z^{-s}\, ds = \frac{2}{\pi} \int_0^{\infty} \frac{u \cos(u) \log(u)}{u^2 + z^2}\, du + \frac{\pi}{2} e^{-z}. \qquad (3.32)$$

Again, we first prove this result for $z > 0$ and later extend it by analytic continuation to $\mathrm{Re}(z) > 0$. Note that from [46, p. 201, Equation (5.68)], for $0 < \mathrm{Re}(s) = c < 1$ and $x > 0$,

$$\frac{1}{2\pi i} \int_{(c)} \Gamma(s)\psi(s) \cos\left(\frac{\pi s}{2}\right) (xz)^{-s}\, ds = \cos(xz) \log(xz) + \frac{\pi}{2} \sin(xz). \qquad (3.33)$$

Multiply both sides by $\frac{x}{1+x^2}$ and integrate with respect to x from 0 to ∞ thereby obtaining

(3.34)
$$\int_0^{\infty} \frac{x}{1 + x^2} \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\psi(s) \cos\left(\frac{\pi s}{2}\right) (xz)^{-s}\, ds\, dx = \int_0^{\infty} \left(\cos(xz) \log(xz) + \frac{\pi}{2} \sin(xz)\right) \frac{x\, dx}{1 + x^2}.$$

On the left-hand side, an application of Fubini's theorem allows us to interchange the order of integration. On the right-hand side, we employ the change of variable $x = u/z$. Doing this and then using the well-known evaluation

$$\int_0^{\infty} \frac{x^{1-s}}{1 + x^2}\, dx = \frac{\pi}{2 \sin\left(\frac{\pi s}{2}\right)},$$

we get

(3.35)
$$\frac{\pi}{2} \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\psi(s) \cot\left(\frac{\pi s}{2}\right) z^{-s}\, ds = \int_0^{\infty} \frac{u \cos(u) \log(u)}{u^2 + z^2}\, du + \frac{\pi}{2} \int_0^{\infty} \frac{u \sin(u)}{u^2 + z^2}\, du.$$

From [26, p. 65, Formula (15)],

$$\int_0^{\infty} \frac{u \sin(u)}{u^2 + z^2}\, du = \frac{\pi}{2} e^{-z}. \qquad (3.36)$$

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Substituting (3.36) in (3.35) and multiplying both sides by $2/\pi$, we arrive at (3.32). The identity also holds, by analytic continuation, for $\mathrm{Re}(z) > 0$, since both sides are analytic in this region.

Now again, shifting the line of integration from $\mathrm{Re}(s) = c$ to $\mathrm{Re}(s) = d, 1 < d < 2$, noting that there is no pole of the integrand, applying Cauchy's residue theorem and making use of the fact that the integrals along the horizontal segments tend to zero as the height of the contour tends to ∞, we see that for $\mathrm{Re}(w) > 0$,

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\psi(s)\cot\left(\frac{\pi s}{2}\right)(2\pi m w)^{-s}\,ds = \frac{1}{2\pi i}\int_{(d)}\Gamma(s)\psi(s)\cot\left(\frac{\pi s}{2}\right)(2\pi m w)^{-s}\,ds.$$

Hence, from (3.32) with $z = 2\pi m w$,

$$\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{(d)}\Gamma(s)\psi(s)\cot\left(\frac{\pi s}{2}\right)(2\pi m w)^{-s}\,ds = \sum_{m=1}^{\infty}\left\{\frac{2}{\pi}\int_0^{\infty}\frac{u\cos(u)\log(u)}{u^2+(2\pi m w)^2}\,du + \frac{\pi}{2}e^{-2\pi m w}\right\}. \tag{3.37}$$

Therefore, substituting (3.37) and (3.31) in (3.27), we get

$$\begin{aligned} J_1 &= 2\pi\left[\frac{(\gamma-\log(w))}{2\pi}\left(2\log(w)-(\psi(iw)+\psi(-iw))\right)+\sum_{m=1}^{\infty}\left\{\frac{2}{\pi}\int_0^{\infty}\frac{u\cos(u)\log(u)}{u^2+(2\pi m w)^2}\,du+\frac{\pi}{2}e^{-2\pi m w}\right\}\right]\\ &= 2(\gamma-\log(w))\left(\log(w)-\frac{1}{2}(\psi(iw)+\psi(-iw))\right)+4\sum_{m=1}^{\infty}\int_0^{\infty}\frac{u\cos(u)\log(u)}{u^2+(2\pi m w)^2}\,du+\frac{\pi^2}{e^{2\pi w}-1}. \end{aligned} \tag{3.38}$$

Now, from (3.24), (3.26), and (3.38), we have

$$\begin{aligned} &\psi_1(iw+1)-\frac{1}{2}\log^2(iw)+\psi_1(-iw+1)-\frac{1}{2}\log^2(-iw)\\ &= 2(\gamma-\log(w))\left(\log(w)-\frac{1}{2}(\psi(iw)+\psi(-iw))\right)+4\sum_{m=1}^{\infty}\int_0^{\infty}\frac{u\cos(u)\log(u)}{u^2+(2\pi m w)^2}\,du+\frac{\pi^2}{e^{2\pi w}-1}\\ &\quad -\pi^2\left(\frac{1}{e^{2\pi w}-1}-\frac{1}{2\pi w}\right). \end{aligned}$$

Now, using (3.19) twice and using the elementary fact $\log(iw) - \log(-iw) = \pi i$ for $\mathrm{Re}(w) > 0$, we are led to

$$\begin{aligned} 4\sum_{m=1}^{\infty}\int_0^{\infty}\frac{u\cos(u)\log(u)}{u^2+(2\pi m w)^2}\,du &= \psi_1(iw)-\frac{1}{2}\log^2(iw)+\psi_1(-iw)-\frac{1}{2}\log^2(-iw)+\frac{\pi}{2w}\\ &\quad+(\gamma-\log(w))\left(\psi(iw)+\psi(-iw)-2\log(w)\right). \end{aligned} \tag{3.39}$$

Lastly, employing (2.2) again with w replaced by $2\pi w$, we see that

$$-4\log(w)\sum_{m=1}^{\infty}\int_0^{\infty}\frac{u\cos(u)}{u^2+(2\pi m w)^2}\,du = \log(w)\left(\psi(iw)+\psi(-iw)-2\log(w)\right). \tag{3.40}$$

Finally, adding the respective sides of (3.39) and (3.40), we are led to (3.20). This completes the proof. ■

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4 A new representation for $\frac{\partial^2}{\partial b^2}E_{2,b}(z)\Big|_{b=1}$

The two-variable Mittag-Leffler function $E_{\alpha,\beta}(z)$, introduced by Wiman [64], is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0). \tag{4.1}$$

There is an extensive literature on these Mittag-Leffler functions (see, for example, [30] and the references therein). Yet, closed-form expressions exist only for the first derivatives of $E_{\alpha,\beta}(z)$ with respect to the parameters α and β. The reader is referred to a recent paper of Apelblat [1] for a collection of such evaluations most of which involve the $\mathrm{Shi}(z)$ and $\mathrm{Chi}(z)$ functions defined in (4.3).

From (4.1), it is easy to see that [1, Equation (98)]

$$\frac{\partial^2}{\partial\beta^2}E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{\psi^2(\alpha k+\beta)-\psi'(\alpha k+\beta)}{\Gamma(\alpha k+\beta)}t^k. \tag{4.2}$$

However, there are no closed-form evaluations known for the above series. In what follows, we establish a new result which transforms $\frac{\partial^2}{\partial\beta^2}E_{2,\beta}(t)\Big|_{\beta=1}$ into a suitable integral which is absolutely essential in proving Theorem 1.1. Before we embark upon the proof though, we need a few lemmas concerning the *hyperbolic sine and cosine integrals* $\mathrm{Shi}(z)$ and $\mathrm{Chi}(z)$ defined by [48, p. 150, Equations (6.2.15) and (6.2.16)]

$$\mathrm{Shi}(z) := \int_0^z \frac{\sinh(t)}{t}\,dt, \quad \mathrm{Chi}(z) := \gamma+\log(z)+\int_0^z \frac{\cosh(t)-1}{t}\,dt. \tag{4.3}$$

The first lemma involving these functions was established in [23, Lemma 9.1]. Here, and throughout the sequel, we use the following notation for brevity.

$$(\sinh\mathrm{Shi}-\cosh\mathrm{Chi})(w) := \sinh(w)\,\mathrm{Shi}(w)-\cosh(w)\,\mathrm{Chi}(w).$$

Lemma 4.1 *Let* $\mathrm{Re}(w)>0$. *Then*

$$\int_0^\infty \frac{t\cos t\,dt}{t^2+w^2} = (\sinh\mathrm{Shi}-\cosh\mathrm{Chi})(w).$$

We require another result from [22, Lemma 3.2].

Lemma 4.2 *For Re*$(w)>0$,

$$\sum_{k=0}^{\infty} \frac{\psi(2k+1)}{\Gamma(2k+1)}w^{2k} = (\sinh\mathrm{Shi}-\cosh\mathrm{Chi})(w)+\log(w)\cosh(w).$$

We also need the following integral representation for the Mittag-Leffler function $E_{2,b}(z)$ which was obtained by Dzhrbashyan [25, p. 130, Equation (2.12)]. Since Dzhrbashyan's book [25] is not easily accessible and we could not find this result in any contemporary texts on these functions, for example, [30], we briefly sketch his proof of it.

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Lemma 4.3 *Let the two-variable Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined in (4.1). For $0 < Re(b) < 3$, and $0 \le \arg(z) < \pi$ or $-\pi < \arg(z) \le 0$, one has*[7]

(4.4)
$$E_{2,b}(z) = \frac{1}{2}z^{\frac{1}{2}(1-b)}\left\{e^{\sqrt{z}} + e^{\mp i\pi(1-b)-\sqrt{z}}\right\} + \frac{\sin(\pi b)}{2\pi}\int_0^\infty \frac{e^{\pm i\left(\sqrt{t}-\frac{\pi}{2}(1-b)\right)}}{t+z}t^{\frac{1}{2}(1-b)}\,dt,$$

where the upper or lower signs are taken, respectively, for $0 \le \arg(z) < \pi$ or $-\pi < \arg(z) \le 0$.

Proof For $\mathrm{Re}(\eta) > 0$, Hankel's representation for the reciprocal of $\Gamma(\eta)$ is given by [25, Equation (2.7')]

$$\frac{1}{\Gamma(\eta)} = \frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)} e^{s^{1/2}} s^{-(\eta+1)/2}\,ds, \tag{4.5}$$

where $\gamma(\varepsilon;\pi)$ represents the Hankel contour which runs from $-\infty, \arg(s) = -\pi$, encircles the origin in the form a circle with infinitesimal radius $\varepsilon > 0$ in the counter-clockwise direction and then terminates at $-\infty$, but now with $\arg(s) = +\pi$. This contour divides the complex plane into two unbounded regions $G^{(-)}(\varepsilon;\pi)$ and $G^{(+)}(\varepsilon;\pi)$, where $G^{(-)}(\varepsilon;\pi) = \{s : |\arg(s)| < \pi, |s| < \varepsilon\}$ and $G^{(+)}(\varepsilon;\pi) = \{s : |\arg(s)| < \pi, |s| > \varepsilon\}$.

Let $|\arg(z)| < \pi$ and $\mathrm{Re}(b) > 0$. Using (4.5) in the definition (4.1) of $E_{2,b}(z)$, after some simplification, we find that for $z \in G^{(-)}(\varepsilon;\pi)$, we have [25, Equation (2.4)],

$$E_{2,b}(z) = \frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)} \frac{e^{s^{1/2}} s^{(1-b)/2}}{s-z}\,ds.$$

Now, if $z \in G^{(+)}(\varepsilon;\pi)$, say, $\varepsilon < |z| < \varepsilon_1$, then using Cauchy's residue theorem (along the curve obtained by considering a straight line segment going from a point on $\gamma(\varepsilon;\pi)$ radially to $\gamma(\varepsilon_1;\pi)$ which misses z), we see that [25, Equation (2.9)],

$$\frac{1}{4\pi i}\int_{\gamma(\varepsilon_1;\pi)} \frac{e^{s^{1/2}} s^{(1-b)/2}}{s-z}\,ds - \frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)} \frac{e^{s^{1/2}} s^{(1-b)/2}}{s-z}\,ds = \frac{1}{2}z^{(1-b)/2}e^{z^{1/2}},$$

and hence, this implies that, for $z \in G^{(+)}(\varepsilon;\pi)$, we have [25, Equation (2.5)]

$$E_{2,b}(z) = \frac{1}{2}z^{(1-b)/2}e^{z^{1/2}} + \frac{1}{4\pi i}\int_{\gamma(\varepsilon;\pi)} \frac{e^{s^{1/2}} s^{(1-b)/2}}{s-z}\,ds. \tag{4.6}$$

For more details, see [25, p. 129]. Now, in addition to $\mathrm{Re}(b) > 0$, if we assume $\mathrm{Re}(b) < 3$, then we can let $\varepsilon \to 0$ in (4.6), thereby obtaining

$$E_{2,b}(z) = \frac{1}{2}z^{(1-b)/2}e^{z^{1/2}} + \frac{1}{4\pi i}\int_{\gamma(0;\pi)} \frac{e^{s^{1/2}} s^{(1-b)/2}}{s-z}\,ds.$$

[7] It is to be noted that Dzhrbashyan uses a different notation for the two-variable Mittag-Leffler function in his book, namely, $E_\rho(z;b) = \sum_{k=0}^{\infty} z^k/\Gamma(b+k\rho^{-1})$ (see [25, p. 117]). We have used the contemporary notation here.

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Since the integral over the Hankel contour can be written as sum of two integrals from 0 to ∞, we can rewrite the above equation in the form

$$(4.7)\qquad \frac{1}{2}z^{(1-b)/2}e^{z^{1/2}}+\frac{1}{2\pi}\int_0^\infty \frac{\sin\left(\sqrt{t}+\frac{\pi}{2}(1-b)\right)}{t+z}t^{\frac{1-b}{2}}\,dt=E_{2,b}(z).$$

This representation holds for any fixed z such that $|\arg(z)|<\pi$.

Next, we show that for $|\arg(z)|<\pi$, we have [25, Equation (2.15)]

$$(4.8)\qquad z^{(1-b)/2}e^{-z^{1/2}}+\frac{1}{2\pi i}\int_{\gamma(0;\pi)}\frac{e^{-s^{1/2}}s^{(1-b)/2}}{s-z}\,ds=0.$$

Let $\gamma_R(\varepsilon;\pi)$ denote the portion of the contour $\gamma(\varepsilon;\pi)$ belonging to the disk $|s|<R$. By Cauchy's residue theorem,

$$(4.9)\qquad \frac{1}{2\pi i}\int_{\gamma_R(\varepsilon;\pi)}\frac{e^{-s^{1/2}}s^{(1-b)/2}}{s-z}\,ds+\frac{1}{2\pi i}\int_{|s|=R}\frac{e^{-s^{1/2}}s^{(1-b)/2}}{s-z}\,ds=-e^{-z^{1/2}}z^{(1-b)/2}$$

if $\varepsilon<|z|<R$ and $|\arg(z)|<\pi$. Observe that the following estimate holds:

$$\left|\int_{|s|=R}\frac{e^{-s^{1/2}}s^{(1-b)/2}}{s-z}\,ds\right|\le\frac{R^{\frac{3}{2}-\frac{\mathrm{Re}(b)}{2}}}{R-|z|}\int_{-\pi}^{\pi}e^{-R^{1/2}\cos\left(\frac{\phi}{2}\right)+\frac{\phi}{2}\mathrm{Im}(b)}\,d\phi<\frac{2\pi R}{R-|z|}R^{-\frac{1}{2}\mathrm{Re}(b)}e^{\frac{\pi}{2}\mathrm{Im}(b)},$$

in the last step, we applied Jordan's inequality, namely, $\sin(u/2)\ge u/\pi$ for $u\in[0,\pi]$ after the change of variable $\phi=\pi-u$. Using the above estimate, and letting $\varepsilon\to0$, $R\to\infty$ in (4.9) leads us to (4.8), since b is a fixed complex number with $0<\mathrm{Re}(b)<3$.

Now, for $z\neq0$, (4.8) can be rewritten in the form

$$z^{(1-b)/2}e^{-z^{1/2}}+\frac{1}{2\pi i}\int_0^\infty\frac{\exp\left(-i\left(\sqrt{t}-\frac{\pi}{2}(1-b)\right)\right)-\exp\left(i\left(\sqrt{t}-\frac{\pi}{2}(1-b)\right)\right)}{t+z}t^{\frac{1-b}{2}}\,dt=0.$$

Now, multiplying both sides of the above equation by $\frac{1}{2}e^{-i\pi(1-b)}$ if $0\le\arg(z)<\pi$, and by $\frac{1}{2}e^{i\pi(1-b)}$ if $-\pi<\arg(z)\le0$, and then adding the corresponding sides of the respective resulting equations to those of (4.7), we arrive at (4.4). ■

We are now ready to prove our result on evaluating the second derivative of the Mittag-Leffler function $E_{2,b}(z)$ with respect to the parameter b at $b=1$.

Theorem 4.4 *Let $1\le b\le 3/2$ and $w>0$. Let the two-variable Mittag-Leffler function $E_{2,b}(z)$ be defined in (4.1). Then*

$$(4.10)\qquad \frac{\partial^2}{\partial b^2}E_{2,b}(w^2)\bigg|_{b=1}=\log^2(w)\cosh(w)+2\int_0^\infty\frac{u\cos(u)\log(u)\,du}{u^2+w^2}.$$

Proof Assume $z>0$. We would like to differentiate (4.4) twice with respect to b. This, in turn, requires differentiating the integral in (4.4) under the integral sign twice with respect to b. (Note that since $\arg(z)=0$, we can take the plus sign in the exponent of the exponential function inside the integrand.) To justify this, one needs to invoke Theorem 2.3 twice. We justify this only for the first derivative with respect to b. The verification for the second derivative can be done similarly.

To that end, let

$$\mathcal{I}(b) := \int_0^\infty \frac{e^{i\sqrt{t}}\ t^{\frac{1}{2}(1-b)}}{t+z}\,dt.$$

Observe that the convergence of $\mathcal{I}(b)$ for any $b \in [1, 3/2]$ results from (4.4) itself. Thus, we can apply (2.4) provided, we show that

$$\mathcal{J}(b) := \int_0^\infty \frac{\partial}{\partial b}\frac{e^{i\sqrt{t}}\ t^{\frac{1}{2}(1-b)}}{t+z}\,dt = -\frac{1}{2}\int_0^\infty \frac{e^{i\sqrt{t}}\ t^{\frac{1}{2}(1-b)}\log(t)}{t+z}\,dt$$

converges uniformly in $[1, 3/2]$. Now, write the above integral as the sum of two integrals—one from 0 to 1, and the second from 1 to ∞. Then in the first, replace t by $1/t$ and in the second, employ the change of variable $t = x^2$ to get

$$\begin{aligned}\mathcal{J}(b) &= \frac{1}{2}\int_1^\infty \frac{e^{\frac{i}{\sqrt{t}}}t^{\frac{1}{2}(b-3)}\log(t)}{1+tz}\,dt - 2\int_1^\infty \frac{e^{ix}x^{(2-b)}\log(x)}{x^2+z}\,dx\\ &=: I_2 + I_3.\end{aligned}$$

We now show that both I_2 and I_3 converge uniformly in $[1, 3/2]$, whence we will be done. We first handle I_2. Let $f(t,b) = \frac{t^{1/2}}{1+zt}$ and $g(t,b) = t^{\frac{b}{2}-2}e^{i/\sqrt{t}}\log(t)$.

Suppose first $z \in [1,\infty)$. Then $f(t,b)$ not only tends to zero as $t \to \infty$ but is also decreasing on $[1,\infty)$. Further,

$$\begin{aligned}\left|\int_1^X t^{\frac{b}{2}-2}e^{i/\sqrt{t}}\log(t)\,dt\right| &\le \int_1^X t^{\frac{b}{2}-2}\log(t)\,dt = \frac{X^{\frac{b}{2}-1}\log(X)}{\frac{b}{2}-1} - \frac{(X^{\frac{b}{2}-1}-1)}{(\frac{b}{2}-1)^2}\\ &\le 16.\end{aligned}$$

Hence, the hypotheses of Dirichlet's test for uniform convergence given in Theorem 2.2 are satisfied, and I_2 converges uniformly on $[1, 3/2]$.

Now, if $z \in (0,1)$, then $f(t,b)$ increases on $[1, 1/z]$ and decreases on $[1/z, \infty)$. Thus, in this case, one needs to further split I_2 as

$$I_2 = \int_1^{1/z} f(t,b)g(t,b)\,dt + \int_{1/z}^\infty f(t,b)g(t,b)\,dt.$$

The integral from $1/z$ to ∞ converges uniformly by Dirichlet's test similarly as was just shown above. For the integral from 1 to $1/z$, we apply Weierstrass M-test [62, p. 289, Theorem 7*]. Together, we see that for any fixed $z > 0$, I_2 converges uniformly on $[1, 3/2]$.

We now focus on I_3 and begin by rewriting it as

$$I_3 = -2\int_1^\infty \frac{x^{(2-b)}\cos x\log x}{x^2+z}\,dx - 2i\int_1^\infty \frac{x^{(2-b)}\sin x\log x}{x^2+z}\,dx.$$

We show the uniform convergence of only the integral involving $\cos(x)$. That for the second integral can be shown similarly.

To that end, we again apply Dirichlet's test (Theorem 2.2) with $f(x,b) = \frac{x^{2-b}\log x}{x^2+z}$ and $g(x,b) = \cos x$. To verify that its hypotheses are satisfied, first observe that $f(x,b)$ tends to zero as $x \to \infty$. However, $f(x,b)$ is *not* decreasing throughout the interval

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$[1,\infty)$. But if we write $f(x,b)=\frac{\log(x)}{\sqrt{x}}\frac{x^{\frac{5}{2}-b}}{x^2+z}$, by elementary calculus, it is not difficult to see that if $C_z:=\max\{e^2,\sqrt{3z}\}$, then $f(x,b)$ is decreasing in $[C_z,\infty)$, since $\frac{\log(x)}{\sqrt{x}}$ is positive and decreasing in (e^2,∞) and $\frac{x^{\frac{5}{2}-b}}{x^2+z}$ is positive and decreasing in $(\sqrt{3z},\infty)$.

Hence, we write

$$\int_1^\infty \frac{x^{(2-b)}\cos x\log x}{x^2+z}\,dx=\int_1^{C_z}\frac{x^{(2-b)}\cos x\log x}{x^2+z}\,dx+\int_{C_z}^\infty\frac{x^{(2-b)}\cos x\log x}{x^2+z}\,dx=J_2+J_3(\text{say}).$$

Using Weierstrass test for J_2 with $M(x)=\frac{2}{z}x\log x$ justifies its uniform convergence. Finally, for J_3, we have $\left|\int_{C_z}^X\cos x\,dx\right|\le 2$. Hence, by another application of Dirichlet's test, J_3 converges uniformly. Therefore, I_3 converges uniformly too. Combining all of the above facts, we conclude that $\mathcal{J}(b)$ converges uniformly for all $b\in[1,3/2]$.

Differentiating (4.4) twice with respect to b and simplifying leads to

$$\begin{aligned}\frac{\partial^2}{\partial b^2}E_{2,b}(z)=\frac{1}{2}\Bigg[&-\frac{1}{2}\log(z)\left\{-\frac{1}{2}z^{\frac{1}{2}(1-b)}\log(z)\left(e^{\sqrt{z}}+e^{-i\pi(1-b)-\sqrt{z}}\right)+i\pi z^{\frac{1}{2}(1-b)}e^{i\pi(b-1)-\sqrt{z}}\right\}\\&+i\pi\sqrt{z}e^{-\sqrt{z}}z^{-\frac{b}{2}}e^{i\pi(b-1)}\left(i\pi-\frac{1}{2}\log(z)\right)\Bigg]-\frac{\pi\sin(\pi b)}{2}\int_0^\infty\frac{e^{i\left(\sqrt{t}-\frac{\pi}{2}(1-b)\right)}}{t+z}t^{\frac{1}{2}(1-b)}\,dt\\&+\frac{1}{2}\cos(\pi b)\int_0^\infty\frac{e^{i\left(\sqrt{t}-\frac{\pi}{2}(1-b)\right)}}{t+z}t^{\frac{1}{2}(1-b)}(i\pi-\log(t))\,dt\\&+\frac{1}{8\pi}\sin(\pi b)\int_0^\infty\frac{e^{i\left(\sqrt{t}-\frac{\pi}{2}(1-b)\right)}}{t+z}t^{\frac{1}{2}(1-b)}(i\pi-\log(t))^2\,dt,\end{aligned}$$

so that

(4.11)
$$\frac{\partial^2}{\partial b^2}E_{2,b}(z)\bigg|_{b=1}=\frac{1}{4}\log^2(z)\cosh(\sqrt{z})-\frac{i\pi}{2}e^{-\sqrt{z}}\log(z)-\frac{\pi^2}{2}e^{-\sqrt{z}}-\frac{1}{2}\int_0^\infty\frac{e^{i\sqrt{t}}(i\pi-\log(t))}{t+z}\,dt.$$

Now, employing the change of variable $t=u^2$, we have

$$\begin{aligned}\int_0^\infty\frac{e^{i\sqrt{t}}(i\pi-\log(t))}{t+z}\,dt&=2\int_0^\infty\frac{ue^{iu}(i\pi-2\log(u))}{u^2+z}\,du\\&=2\pi i\int_0^\infty\frac{u\cos(u)}{u^2+z}\,du-2\pi\int_0^\infty\frac{u\sin(u)}{u^2+z}\,du-4\int_0^\infty\frac{ue^{iu}\log(u)}{u^2+z}\,du.\end{aligned}\tag{4.12}$$

Next, from Lemma 4.1,

$$\int_0^\infty\frac{u\cos(u)}{u^2+z}\,du=(\sinh\mathrm{Shi}-\cosh\mathrm{Chi})(\sqrt{z}).\tag{4.13}$$

Moreover,

(4.14)
$$\int_0^\infty\frac{ue^{iu}\log(u)}{u^2+z}\,du=\int_0^\infty\frac{u\cos(u)\log(u)}{u^2+z}\,du+i\int_0^\infty\frac{u\sin(u)\log(u)}{u^2+z}\,du,$$

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Now, from [50, p. 537, Formula 2.6.32.8],[8]

(4.15)
$$\int_0^\infty \frac{u\sin(u)\log(u)}{u^2+z}\,du = \frac{\pi}{2}\left\{e^{-\sqrt{z}}\log(\sqrt{z}) - \frac{1}{2}\left(e^{\sqrt{z}}\mathrm{Ei}(-\sqrt{z}) + e^{-\sqrt{z}}\mathrm{Ei}(\sqrt{z})\right)\right\}.$$

Hence, along with Lemma 4.1, (3.30), and (4.15) imply

(4.16)
$$\int_0^\infty \frac{u\sin(u)\log(u)}{u^2+z}\,du = \frac{\pi}{2}\left\{e^{-\sqrt{z}}\log(\sqrt{z}) + (\sinh \mathrm{Shi} - \cosh \mathrm{Chi})(\sqrt{z})\right\}.$$

Therefore, from (4.12), (4.13), (3.36), (4.14), and (4.16), we arrive at

(4.17)
$$\int_0^\infty \frac{e^{i\sqrt{t}}(i\pi - \log(t))}{t+z}\,dt = -\pi^2 e^{-\sqrt{z}} - 2\pi i e^{-\sqrt{z}}\log(\sqrt{z}) - 4\int_0^\infty \frac{u\cos(u)\log(u)}{u^2+z}\,du.$$

Finally, substituting (4.17) in (4.11), we arrive at

$$\frac{\partial^2}{\partial b^2}E_{2,b}(z)\bigg|_{b=1} = \log^2(\sqrt{z})\cosh(\sqrt{z}) + 2\int_0^\infty \frac{u\cos(u)\log(u)}{u^2+z}\,du.$$

Now, let $z = w^2$ to arrive at (4.10). ∎

5 Proof of the main results

This section is devoted to proving the transformation for the Lambert series for logarithm given in Theorem 1.1, the asymptotic expansion of this series in Theorem 1.2, and a mean value theorem in Theorem 1.3.

5.1 Proof of Theorem 1.1

We first prove the result for $y > 0$ and then extend it to $\mathrm{Re}(y) > 0$ by analytic continuation.

The idea is to differentiate both sides of (1.5) with respect to a and then let $a \to 0$. Define

$$F_1(a,y) := \frac{d}{da}\sum_{n=1}^\infty \sigma_a(n)e^{-ny},$$

$$F_2(a,y) := \frac{d}{da}\left\{\frac{1}{2}\left(\left(\frac{2\pi}{y}\right)^{1+a}\operatorname{cosec}\left(\frac{\pi a}{2}\right)+1\right)\zeta(-a) - \frac{1}{y}\zeta(1-a)\right\},$$

(5.1)
$$F_3(a,y) := \frac{d}{da}\left\{\frac{2\pi}{y\sin\left(\frac{\pi a}{2}\right)}\sum_{n=1}^\infty \sigma_a(n)\left(\frac{(2\pi n)^{-a}}{\Gamma(1-a)}{}_1F_2\left(1; \tfrac{1-a}{2}, 1-\tfrac{a}{2}; \tfrac{4\pi^4 n^2}{y^2}\right) - \left(\tfrac{2\pi}{y}\right)^a \cosh\left(\tfrac{4\pi^2 n}{y}\right)\right)\right\},$$

and let

(5.2)
$$G_1(y) = \lim_{a\to 0}F_1(a,y), \qquad G_2(y) = \lim_{a\to 0}F_2(a,y), \qquad G_3(y) = \lim_{a\to 0}F_3(a,y).$$

[8] The formula given in the book has a typo. We have corrected it here.

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Clearly, from (1.5),

$$G_1(y) + G_2(y) = G_3(y). \tag{5.3}$$

Using (1.3), it is readily seen that

$$G_1(y) = \sum_{n=1}^{\infty} \frac{\log(n)}{e^{ny} - 1}. \tag{5.4}$$

By routine differentiation,

$$\begin{aligned} F_2(a, y) = \frac{1}{y}\zeta'(1-a) - \frac{1}{2}\zeta'(-a) - \frac{1}{2}\left(\frac{2\pi}{y}\right)^{1+a} \operatorname{cosec}\left(\frac{\pi a}{2}\right)\zeta'(-a) \\ + \frac{1}{2}\zeta(-a)\left(\frac{2\pi}{y}\right)^{1+a} \operatorname{cosec}\left(\frac{\pi a}{2}\right)\left(\log\left(\frac{2\pi}{y}\right) - \frac{\pi}{2}\cot\left(\frac{\pi a}{2}\right)\right). \end{aligned} \tag{5.5}$$

Letting $a \to 0$ in (5.5) leads to

$$G_2(y) = \frac{1}{4}\log(2\pi) - \frac{1}{2y}\log^2(y) + \frac{\gamma^2}{2y} - \frac{\pi^2}{12y}, \tag{5.6}$$

which is now proved. Note that

$$\begin{aligned} G_2(y) &= \lim_{a\to 0}\left(-\frac{1}{2}\zeta'(-a)\right) \\ &+ \lim_{a\to 0}\left[\frac{1}{y}\zeta'(1-a) - \frac{(2\pi/y)^{1+a}}{2\sin\left(\frac{\pi a}{2}\right)}\left\{\zeta'(-a) - \zeta(-a)\log\left(\frac{2\pi}{y}\right) + \frac{\pi}{2}\zeta(-a)\cot\left(\frac{\pi a}{2}\right)\right\}\right]. \end{aligned} \tag{5.7}$$

We make use of the following well-known Laurent series expansions around $a = 0$:

$$\begin{aligned} \zeta(-a) &= -\frac{1}{2} + \frac{1}{2}\log(2\pi)a + \left[\frac{\gamma^2}{4} - \frac{\pi^2}{48} - \frac{\log^2(2\pi)}{4} + \frac{\gamma_1}{2}\right]a^2 + O(a^3), \\ \zeta'(1-a) &= -\frac{1}{a^2} - \gamma_1 - \gamma_2 a + O(a^2), \\ \left(\frac{2\pi}{y}\right)^{1+a} &= \frac{2\pi}{y} + \frac{2\pi}{y}\log\left(\frac{2\pi}{y}\right)a + \frac{\pi}{y}\log^2\left(\frac{2\pi}{y}\right)a^2 + O(a^3), \\ \operatorname{cosec}\left(\frac{\pi a}{2}\right) &= \frac{2}{\pi a} + \frac{\pi}{12}a + O(a^3), \\ \cot\left(\frac{\pi a}{2}\right) &= \frac{2}{\pi a} - \frac{\pi}{6}a + O(a^3). \end{aligned} \tag{5.8}$$

Hence, substituting (5.8) in (5.7), we get

$$\begin{aligned} G_2(y) = \tfrac{1}{4}\log(2\pi) + \lim_{a\to 0}\Big[-\tfrac{1}{ya^2} - \tfrac{\gamma_1}{y} + O(a) + \Big\{-\tfrac{2}{ya} - \tfrac{2}{y}\log\left(\tfrac{2\pi}{y}\right) + \left(-\tfrac{\pi^2}{12y} - \tfrac{1}{y}\log^2\left(\tfrac{2\pi}{y}\right)\right)a + O(a^2)\Big\} \\ \times\Big\{-\tfrac{1}{2a} + \tfrac{1}{2}\log\left(\tfrac{2\pi}{y}\right) + \left(-\tfrac{\gamma^2}{4} + \tfrac{\pi^2}{16} - \tfrac{\gamma_1}{2} + \tfrac{1}{4}\log^2(2\pi) - \tfrac{1}{2}\log(2\pi)\log\left(\tfrac{2\pi}{y}\right)\right)a + O(a^2)\Big\}\Big] \\ = \frac{1}{4}\log(2\pi) + \lim_{a\to 0}\left[\left(-\frac{1}{y} + \frac{1}{y}\right)\frac{1}{a^2} + \left(-\frac{1}{y}\log\left(\frac{2\pi}{y}\right) + \frac{1}{y}\log\left(\frac{2\pi}{y}\right)\right)\frac{1}{a}\right. \end{aligned}$$

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$$+\left(-\frac{\gamma_1}{y}+\frac{\gamma^2}{2y}-\frac{\pi^2}{12y}+\frac{\gamma_1}{y}-\frac{1}{2y}\left\{\log^2(2\pi)-2\log(2\pi)\log\left(\frac{2\pi}{y}\right)+\log^2\left(\frac{2\pi}{y}\right)\right\}\right)+O(a)\Bigg]$$
$$=\frac{1}{4}\log(2\pi)+\frac{\gamma^2}{2y}-\frac{\pi^2}{12y}-\frac{\log^2 y}{2y}.$$

Next, we apply the definitions of $\sigma_a(n)$, ${}_1F_2$ and simplify to obtain

$$F_3(a,y)=\frac{d}{da}\left[\frac{2\pi}{y\sin\left(\frac{\pi a}{2}\right)}\sum_{m,n=1}^{\infty}\left\{(2\pi n)^{-a}\sum_{k=0}^{\infty}\frac{\left(4\pi^2mn/y\right)^{2k}}{\Gamma(1-a+2k)}-\left(\frac{2\pi m}{y}\right)^a\cosh\left(\frac{4\pi^2mn}{y}\right)\right\}\right]$$
$$=\frac{2\pi}{y}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{4\pi^2mn}{y}\right)^{2k}\frac{1}{\sin^2\left(\frac{\pi a}{2}\right)}\left[\frac{(2\pi n)^{-a}}{\Gamma(1-a+2k)}\left\{(\psi(1-a+2k)-\log(2\pi n))\sin\left(\frac{\pi a}{2}\right)\right.\right.$$
$$\left.\left.-\frac{\pi}{2}\cos\left(\frac{\pi a}{2}\right)\right\}-\frac{1}{\Gamma(2k+1)}\left(\frac{2\pi m}{y}\right)^a\left(\log\left(\frac{2\pi m}{y}\right)\sin\left(\frac{\pi a}{2}\right)-\frac{\pi}{2}\cos\left(\frac{\pi a}{2}\right)\right)\right]$$
$$=\frac{2\pi}{y}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{4\pi^2mn}{y}\right)^{2k}\frac{A_1(a)-A_2(a)}{\sin^2\left(\frac{\pi a}{2}\right)},$$

where in the first step, we have interchanged the order of summation and differentiation using the fact [23, p. 35] that the series in the definition of (5.1) converges uniformly as long as $\mathrm{Re}(a)>-1$, and where

$$A_1(a)=A_1(m,n,k;a):=\frac{(2\pi n)^{-a}}{\Gamma(1-a+2k)}\left\{(\psi(1-a+2k)-\log(2\pi n))\sin\left(\frac{\pi a}{2}\right)-\frac{\pi}{2}\cos\left(\frac{\pi a}{2}\right)\right\},$$
$$A_2(a)=A_2(m,n,k;a):=\frac{1}{\Gamma(2k+1)}\left(\frac{2\pi m}{y}\right)^a\left\{\log\left(\frac{2\pi m}{y}\right)\sin\left(\frac{\pi a}{2}\right)-\frac{\pi}{2}\cos\left(\frac{\pi a}{2}\right)\right\}.$$

We now show that

$$\lim_{a\to0}\frac{A_1(a)-A_2(a)}{\sin^2\left(\frac{\pi a}{2}\right)}=\frac{1}{\pi\Gamma(2k+1)}\left\{\log\left(\frac{4\pi^2mn}{y}\right)\log\left(\frac{ny}{m}\right)-2\log(2n\pi)\psi(2k+1)+\psi^2(2k+1)\right.$$
$$\left.-\psi'(2k+1)\right\}. \tag{5.9}$$

Let L denote the limit in the equation given above. The expression inside the limit is of the form $0/0$. By routine calculation, we find that

$$A_1'(a)=\frac{(2\pi n)^{-a}}{\Gamma(1-a+2k)}\left\{-\psi'(1-a+2k)+\frac{\pi^2}{4}+(\psi(1-a+2k)-\log(2\pi n))^2\right\}\sin\left(\frac{\pi a}{2}\right), \tag{5.10}$$

$$A_2'(a)=\frac{(2\pi m/y)^a}{\Gamma(2k+1)}\left\{\log\left(\frac{2\pi m}{y}\right)\left(\log\left(\frac{2\pi m}{y}\right)\sin\left(\frac{\pi a}{2}\right)-\frac{\pi}{2}\cos\left(\frac{\pi a}{2}\right)\right)\right.$$
$$\left.+\frac{\pi}{2}\log\left(\frac{2\pi m}{y}\right)\cos\left(\frac{\pi a}{2}\right)+\frac{\pi^2}{4}\sin\left(\frac{\pi a}{2}\right)\right\}, \tag{5.11}$$

where $'$ denotes differentiation with respect to a. Since $A_1'(0)=A_2'(0)=0$, using L'Hopital's rule again, it is seen that

$$L=\frac{A_1''(0)-A_2''(0)}{\pi^2/2},$$

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Differentiating (5.10) and (5.11) with respect to a, we get

$$A_1''(0) = \frac{\pi}{2\Gamma(2k+1)}\left\{-\psi'(2k+1) + \frac{\pi^2}{4} + (\psi(2k+1) - \log(2\pi n))^2\right\},$$
$$A_2''(0) = \frac{1}{\Gamma(2k+1)}\left\{\frac{\pi}{2}\log^2\left(\frac{2\pi m}{y}\right) + \frac{\pi^3}{8}\right\},$$

whence we obtain (5.9) upon simplification. Therefore, from (5.2) and (5.9), we deduce that

$$\begin{aligned}G_3(y) &= \frac{2}{y}\sum_{m,n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(4\pi^2mn/y)^{2k}}{(2k)!}\left\{\log\left(\frac{4\pi^2mn}{y}\right)\log\left(\frac{ny}{m}\right) - 2\log(2n\pi)\psi(2k+1)\right.\\ &\qquad\left. + \psi^2(2k+1) - \psi'(2k+1)\right\}\\ &= \frac{2}{y}\sum_{m,n=1}^{\infty}\left\{\log\left(\frac{4\pi^2mn}{y}\right)\log\left(\frac{ny}{m}\right)\cosh\left(\frac{4\pi^2mn}{y}\right) - 2\log(2n\pi)\sum_{k=0}^{\infty}\frac{\psi(2k+1)}{\Gamma(2k+1)}\left(\frac{4\pi^2mn}{y}\right)^{2k}\right.\\ &\qquad\left. + \sum_{k=0}^{\infty}\frac{\psi^2(2k+1) - \psi'(2k+1)}{\Gamma(2k+1)}\left(\frac{4\pi^2mn}{y}\right)^{2k}\right\}.\end{aligned}$$

Therefore, invoking Lemma 4.2, using (4.2) and Theorem 4.4 with $w = 4\pi^2mn/y$, we are led to

$$\begin{aligned}G_3(y) &= \frac{2}{y}\sum_{m,n=1}^{\infty}\left\{\log\left(\frac{4\pi^2mn}{y}\right)\log\left(\frac{ny}{m}\right)\cosh\left(\frac{4\pi^2mn}{y}\right) - 2\log(2n\pi)\left[(\sinh\operatorname{Shi} - \cosh\operatorname{Chi})\left(\frac{4\pi^2mn}{y}\right)\right.\right.\\ &\quad\left.\left. + \log\left(\frac{4\pi^2mn}{y}\right)\cosh\left(\frac{4\pi^2mn}{y}\right)\right] + \log^2\left(\frac{4\pi^2mn}{y}\right)\cosh\left(\frac{4\pi^2mn}{y}\right) + 2\int_0^\infty\frac{u\cos(u)\log(u)\,du}{u^2 + (4\pi^2mn/y)^2}\right\}\\ &= \frac{4}{y}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^\infty\frac{u\cos(u)\log(u/(2\pi n))}{u^2 + (4\pi^2mn/y)^2}\,du,\end{aligned}$$

where in the last step, we used Lemma 4.1 with $w = 4\pi^2mn/y$.

Next, an application of Theorem 3.4 with $w = 2\pi n/y$ and then of (2.2) with $w = 4\pi^2n/y$ in the second step further yields

$$\begin{aligned}G_3(y) &= \frac{4}{y}\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\int_0^\infty\frac{u\cos(u)\log(uy/(2\pi n))}{u^2 + (4\pi^2mn/y)^2}\,du - \log(y)\sum_{m=1}^{\infty}\int_0^\infty\frac{u\cos(u)}{u^2 + (4\pi^2mn/y)^2}\,du\right\}\\ &= \frac{1}{y}\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi in}{y}\right) - \frac{1}{2}\log^2\left(\frac{2\pi in}{y}\right) + \psi_1\left(-\frac{2\pi in}{y}\right) - \frac{1}{2}\log^2\left(-\frac{2\pi in}{y}\right) + \frac{y}{4n}\right.\\ &\quad\left. + (\gamma + \log(y))\left(\psi\left(\frac{2\pi in}{y}\right) + \psi\left(-\frac{2\pi in}{y}\right) - 2\log\left(\frac{2\pi n}{y}\right)\right)\right\}\\ &= \frac{1}{y}\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi in}{y}\right) - \frac{1}{2}\log^2\left(\frac{2\pi in}{y}\right) + \psi_1\left(-\frac{2\pi in}{y}\right) - \frac{1}{2}\log^2\left(-\frac{2\pi in}{y}\right) + \frac{y}{4n}\right\}\\ &\quad + \frac{(\gamma + \log(y))}{y}\sum_{n=1}^{\infty}\left\{\left(\psi\left(\frac{2\pi in}{y}\right) + \psi\left(-\frac{2\pi in}{y}\right) - 2\log\left(\frac{2\pi n}{y}\right)\right)\right\},\end{aligned} \tag{5.12}$$

where the validity of the last step results from the fact that invoking Theorem 3.2, once with $z = 2\pi i n/y$, and again with $z = -2\pi i n/y$, and adding the two expansions yields

$$\psi_1\left(\frac{2\pi i n}{y}\right) - \frac{1}{2}\log^2\left(\frac{2\pi i n}{y}\right) + \psi_1\left(-\frac{2\pi i n}{y}\right) - \frac{1}{2}\log^2\left(-\frac{2\pi i n}{y}\right) + \frac{y}{4n} = O_y\left(\frac{\log(n)}{n^2}\right). \tag{5.13}$$

Therefore, from (5.3), (5.4), (5.6), and (5.12), we arrive at

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1} + \frac{1}{4}\log(2\pi) - \frac{1}{2y}\log^2(y) + \frac{y^2}{2y} - \frac{\pi^2}{12y} \\
&= \frac{1}{y}\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi i n}{y}\right) - \frac{1}{2}\log^2\left(\frac{2\pi i n}{y}\right) + \psi_1\left(-\frac{2\pi i n}{y}\right) - \frac{1}{2}\log^2\left(-\frac{2\pi i n}{y}\right) + \frac{y}{4n}\right\} \\
&\quad + \frac{(\gamma + \log(y))}{y}\sum_{n=1}^{\infty}\left\{\left(\psi\left(\frac{2\pi i n}{y}\right) + \psi\left(-\frac{2\pi i n}{y}\right) - 2\log\left(\frac{2\pi n}{y}\right)\right)\right\},
\end{aligned}$$

which, upon rearrangement, gives (1.13).

Now, (1.14) can be immediately derived from (1.13) by invoking (1.6) to transform the first series on the right-hand side of (1.13), multiplying both sides of the resulting identity by y followed by simplification and rearrangement. This completes the proof for $y > 0$. Using Weierstrass' theorem on analytic functions, it is not difficult to see that all of the infinite series are analytic in $\mathrm{Re}(y) > 0$. Since the other expressions in (1.14) are analytic in this region too, the result holds in $\mathrm{Re}(y) > 0$ by analytic continuation.

The theorem just proved allows us to obtain the complete asymptotic expansion of $\sum_{n=1}^{\infty} d(n)\log(n)e^{-ny}$ as $y \to 0$. This is derived next.

5.2 Proof of Theorem 1.2

From (3.13), as $y \to 0$,

$$\psi\left(\pm\frac{2\pi i n}{y}\right) \sim \log\left(\pm\frac{2\pi i n}{y}\right) \mp \frac{y}{4\pi i n} - \sum_{k=1}^{\infty}\frac{B_{2k}}{2k}\left(\pm\frac{2\pi i n}{y}\right)^{-2k}$$

Inserting the above asymptotics into the first sum of the right-hand side of (1.13), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\left\{\log\left(\frac{2\pi n}{y}\right) - \frac{1}{2}\left(\psi\left(\frac{2\pi i n}{y}\right) + \psi\left(-\frac{2\pi i n}{y}\right)\right)\right\} &\sim \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}}{2k}\left(\frac{2\pi n}{y}\right)^{-2k} \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}\zeta(2k)}{2k}\left(\frac{2\pi}{y}\right)^{-2k} \\
&= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{2k}^2 y^{2k}}{(2k)(2k)!},
\end{aligned} \tag{5.14}$$

where in the last step, we used Euler's formula

$$\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}B_{2m}}{2(2m)!} \qquad (m \geq 0). \tag{5.15}$$

Next, we need to find the asymptotic expansion of the second sum on the right-hand side of (1.13). Invoking Theorem 3.2 twice, once with $z = 2\pi in/y$, and again, with $z = -2\pi in/y$, and inserting them into the second sum of the right-hand side of (1.13), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\psi_1\left(\frac{2\pi in}{y}\right) + \psi_1\left(-\frac{2\pi in}{y}\right) - \frac{1}{2}\left(\log^2\left(\frac{2\pi in}{y}\right) + \log^2\left(-\frac{2\pi in}{y}\right)\right) + \frac{y}{4n}\right\}\\
&\sim \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}}{k}\left(\frac{y}{2\pi n}\right)^{2k}\left[\frac{1}{2k-1} + \sum_{j=1}^{2k-2}\frac{1}{j} - \log\left(\frac{2\pi n}{y}\right)\right]\\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}}{k}\left(\frac{y}{2\pi n}\right)^{2k}\left[\frac{1}{2k-1} + \left(\sum_{j=1}^{2k-2}\frac{1}{j} - \log\left(\frac{2\pi}{y}\right)\right) - \log n\right]\\
&= \sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}}{k}\left(\frac{y}{2\pi}\right)^{2k}\left[\zeta(2k)\left\{\frac{1}{2k-1} + \left(\sum_{j=1}^{2k-2}\frac{1}{j} - \log\left(\frac{2\pi}{y}\right)\right)\right\} + \zeta'(2k)\right]
\end{aligned}$$

(5.16)

$$= -\sum_{k=1}^{\infty}\left[\frac{B_{2k}^2 y^{2k}}{(2k-1)(2k)(2k)!} + \frac{B_{2k}^2 y^{2k}}{2k(2k)!}\left(\sum_{j=1}^{2k-2}\frac{1}{j} - \log\left(\frac{2\pi}{y}\right)\right) + \frac{(-1)^{k+1}B_{2k}\zeta'(2k)}{k}\left(\frac{y}{2\pi}\right)^{2k}\right],$$

where in the penultimate step, we used $\zeta'(s) = -\sum_{n=1}^{\infty}\log(n)n^{-s}$ for $\mathrm{Re}(s) > 1$, and in the last step, we used (5.15). Substituting (5.14) and (5.16) in (1.13), we see that as $y \to 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1}\\
&\sim -\frac{1}{4}\log(2\pi) + \frac{1}{2y}\log^2(y) - \frac{y^2}{2y} + \frac{\pi^2}{12y} - \frac{2}{y}(\gamma + \log(y))\left\{-\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{2k}^2 y^{2k}}{(2k)(2k)!}\right\}\\
&\quad - \frac{1}{y}\sum_{k=1}^{\infty}\left[\frac{B_{2k}^2 y^{2k}}{(2k-1)(2k)(2k)!} + \frac{B_{2k}^2 y^{2k}}{2k(2k)!}\left(\sum_{j=1}^{2k-2}\frac{1}{j} - \log\left(\frac{2\pi}{y}\right)\right) + \frac{(-1)^{k+1}B_{2k}\zeta'(2k)}{k}\left(\frac{y}{2\pi}\right)^{2k}\right]\\
&= -\frac{1}{4}\log(2\pi) + \frac{1}{2y}\log^2(y) - \frac{y^2}{2y} + \frac{\pi^2}{12y} + \sum_{k=1}^{\infty}\frac{B_{2k}y^{2k-1}}{k}\left\{\frac{B_{2k}}{2(2k)!}\left(\gamma - \sum_{j=1}^{2k-1}\frac{1}{j} + \log(2\pi)\right) + \frac{(-1)^k\zeta'(2k)}{(2\pi)^{2k}}\right\}\\
&= -\frac{1}{4}\log(2\pi) + \frac{1}{2y}\log^2(y) - \frac{y^2}{2y} + \frac{\pi^2}{12y} + \frac{y}{6}\left\{\frac{1}{24}(\gamma - 1 + \log(2\pi)) - \frac{1}{4\pi^2}\left[\frac{\pi^2}{6}(\gamma + \log(2\pi)\right.\right.\\
&\quad \left.\left.- 12\log(A))\right]\right\} + \sum_{k=2}^{\infty}\frac{B_{2k}y^{2k-1}}{k}\left\{\frac{B_{2k}}{2(2k)!}\left(\gamma - \sum_{j=1}^{2k-1}\frac{1}{j} + \log(2\pi)\right) + \frac{(-1)^k\zeta'(2k)}{(2\pi)^{2k}}\right\}\\
&= \frac{1}{2y}\log^2(y) + \frac{1}{y}\left(\frac{\pi^2}{12} - \frac{y^2}{2}\right) - \frac{1}{4}\log(2\pi) + \frac{y}{12}\left(\log(A) - \frac{1}{12}\right)\\
&\quad + \sum_{k=2}^{\infty}\frac{B_{2k}y^{2k-1}}{k}\left\{\frac{B_{2k}}{2(2k)!}\left(\gamma - \sum_{j=1}^{2k-1}\frac{1}{j} + \log(2\pi)\right) + \frac{(-1)^k\zeta'(2k)}{(2\pi)^{2k}}\right\}.
\end{aligned}$$

This completes the proof.

An application of the result proved above in the theory of the moments of the Riemann zeta function is given next.

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5.3 Proof of Theorem 1.3

Our goal is to link $\int_0^\infty \zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}\,dt$ with $\sum_{n=1}^\infty \frac{\log(n)}{\exp(2\pi i n e^{-i\delta})-1}$ and then invoke Theorem 1.2. Our treatment is similar to that of Atkinson [3].

First note that, for $\mathrm{Re}(y)>0$,

$$\sum_{n=1}^\infty d(n)\log(n)e^{-ny}=\sum_{n=1}^\infty d(n)\log(n)\frac{1}{2\pi i}\int_{(c)}\Gamma(s)(ny)^{-s}ds, \tag{5.17}$$

where $c=\mathrm{Re}(s)>1$. Using $(1*\log)(n)=\sum_{d|n}\log d=\frac{1}{2}d(n)\log(n)$ and interchanging the order of summation and integration on the right-hand side of (5.17), we obtain

$$\begin{aligned}\sum_{n=1}^\infty d(n)\log(n)e^{-ny}&=\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\left(\sum_{n=1}^\infty\frac{d(n)\log(n)}{n^s}\right)y^{-s}ds\\&=\frac{2}{2\pi i}\int_{(c)}\Gamma(s)\sum_{n=1}^\infty\frac{(1*\log)(n)}{n^s}y^{-s}ds\\&=-\frac{2}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta'(s)y^{-s}ds.\end{aligned}\tag{5.18}$$

Letting $y=2\pi i e^{-i\delta}$ so that $0<\mathrm{Re}(\delta)<\pi$, we deduce that

$$\sum_{n=1}^\infty d(n)\log(n)\exp\left(-2\pi i n e^{-i\delta}\right)=-\frac{2}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta'(s)(2\pi i e^{-i\delta})^{-s}ds. \tag{5.19}$$

We would like to shift the line of integration from $\mathrm{Re}(s)=c$ to $\mathrm{Re}(s)=1/2$. The integrand in (5.19) has a third-order pole at $s=1$. Therefore, using Cauchy's residue theorem on the rectangular contour with sides $[c-iT,c+iT],\left[c+iT,\frac{1}{2}+iT\right],\left[\frac{1}{2}+iT,\frac{1}{2}-iT\right]$, and $\left[\frac{1}{2}-iT,c-iT\right]$, where $T>0$, noting that (2.1) implies that the integrals along the horizontal line segments tend to zero as $T\to\infty$, we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta'(s)(2\pi i e^{-i\delta})^{-s}ds\\&=\frac{1}{2\pi i}\int_{(\frac{1}{2})}\Gamma(s)\zeta(s)\zeta'(s)(2\pi i e^{-i\delta})^{-s}ds+\mathrm{Res}_{s=1}\Gamma(s)\zeta(s)\zeta'(s)(2\pi i e^{-i\delta})^{-s}\\&=\frac{1}{4\pi i}\int_{(\frac{1}{2})}\frac{\zeta(1-s)\zeta'(s)(ie^{-i\delta})^{-s}}{\cos\left(\frac{\pi s}{2}\right)}ds+\frac{1}{2\pi i e^{-i\delta}}\left(\frac{\gamma^2}{2}-\frac{\pi^2}{12}-\frac{1}{2}\log^2(2\pi i e^{-i\delta})\right),\end{aligned}\tag{5.20}$$

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where, in the last step, we used the functional equation (3.25) with s replaced by $1-s$. Hence, from (5.19) and (5.20), we arrive at

$$\frac{e^{-\frac{i\delta}{2}}}{2i}\int_{(\frac{1}{2})}\frac{\zeta(1-s)\zeta'(s)e^{-is(\frac{\pi}{2}-\delta)}}{\cos\left(\frac{\pi s}{2}\right)}ds=-\pi e^{-\frac{i\delta}{2}}\sum_{n=1}^{\infty}d(n)\log(n)\exp\left(-2\pi ine^{-i\delta}\right)$$
$$+\frac{ie^{\frac{i\delta}{2}}}{2}\left(\gamma^2-\frac{\pi^2}{6}-\log^2(2\pi ie^{-i\delta})\right). \tag{5.21}$$

We next consider the difference of the integrals on the left-hand sides of (1.16) and (5.21) and employ the change of variable $s=\frac{1}{2}+it$ in the second integral in the first step below to see that

$$\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}dt-\frac{e^{-\frac{i\delta}{2}}}{i}\int_{(\frac{1}{2})}\frac{\zeta(1-s)\zeta'(s)}{2\cos\left(\frac{\pi s}{2}\right)}e^{-is(\frac{\pi}{2}-\delta)}ds$$
$$=\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}dt-e^{-\frac{i\delta}{2}}\int_{-\infty}^{\infty}\frac{\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)}{\left(e^{\frac{i\pi}{2}(\frac{1}{2}+it)}+e^{-\frac{i\pi}{2}(\frac{1}{2}+it)}\right)}e^{-i(\frac{1}{2}+it)(\frac{\pi}{2}-\delta)}dt$$
$$=\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}dt-\int_0^{\infty}\frac{\zeta\left(\frac{1}{2}+it\right)\zeta'\left(\frac{1}{2}-it\right)}{\left(e^{\frac{i\pi}{4}+\frac{\pi t}{2}}+e^{-\frac{i\pi}{4}-\frac{\pi t}{2}}\right)}e^{-\frac{i\pi}{4}-\frac{\pi t}{2}+\delta t}dt$$
$$-\int_0^{\infty}\frac{\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)}{\left(e^{\frac{i\pi}{4}-\frac{\pi t}{2}}+e^{-\frac{i\pi}{4}+\frac{\pi t}{2}}\right)}e^{-\frac{i\pi}{4}+\frac{\pi t}{2}-\delta t}dt$$
$$=\int_0^{\infty}\frac{\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)}{\left(e^{\frac{i\pi}{4}-\frac{\pi t}{2}}+e^{-\frac{i\pi}{4}+\frac{\pi t}{2}}\right)}e^{\frac{i\pi}{4}-\frac{\pi t}{2}-\delta t}dt-\int_0^{\infty}\frac{\zeta\left(\frac{1}{2}+it\right)\zeta'\left(\frac{1}{2}-it\right)}{\left(e^{\frac{i\pi}{4}+\frac{\pi t}{2}}+e^{-\frac{i\pi}{4}-\frac{\pi t}{2}}\right)}e^{-\frac{i\pi}{4}-\frac{\pi t}{2}+\delta t}dt. \tag{5.22}$$

From [29, p. 127, Equation (20)], we have

$$\zeta\left(\frac{1}{2}-it\right)\ll|t|^{\frac{1}{4}+\frac{\varepsilon}{2}}\quad\text{and}\quad\zeta'\left(\frac{1}{2}+it\right)\ll|t|^{\frac{1}{4}+\frac{\varepsilon}{2}},$$

whence, by reverse triangle inequality, we have

$$\left|\frac{\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)}{\left(e^{\frac{i\pi}{4}-\frac{\pi t}{2}}+e^{-\frac{i\pi}{4}+\frac{\pi t}{2}}\right)}e^{\frac{i\pi}{4}-\frac{\pi t}{2}-\delta t}\right|\ll\frac{|t|^{\frac{1}{2}+\varepsilon}e^{-\mathrm{Re}(\delta)t-\frac{\pi t}{2}}}{e^{\frac{\pi t}{2}}-e^{-\frac{\pi t}{2}}}=\frac{|t|^{\frac{1}{2}+\varepsilon}e^{(-\mathrm{Re}(\delta)-\pi)t}}{1-e^{-\pi t}}.$$

This implies that the first integral is analytic for $\mathrm{Re}(\delta)>-\pi$. Similarly, the second integral is analytic for $\mathrm{Re}(\delta)<\pi$. Consequently, both integrals of (5.22) are analytic in δ in $|\delta|<\pi$.

From (5.21) and above discussion, we can say that the expression

$$\phi(\delta):=\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}dt+\pi e^{-\frac{i\delta}{2}}\sum_{n=1}^{\infty}d(n)\log(n)\exp\left(-2\pi ine^{-i\delta}\right)$$
$$-ie^{\frac{i\delta}{2}}\left(\frac{\gamma^2}{2}-\frac{\pi^2}{12}-\frac{\log^2(2\pi ie^{-i\delta})}{2}\right) \tag{5.23}$$

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is an analytic function of δ in $|\delta| < \pi/2$. Hence, $\phi(\delta) + ie^{\frac{i\delta}{2}}\left(\frac{\gamma^2}{2} - \frac{\pi^2}{12} - \frac{\log^2(2\pi i e^{-i\delta})}{2}\right)$ can be expressed as a power series in δ for $|\delta| < \pi/2$.

Letting $y = 2\pi i(e^{-i\delta} - 1)$ in the asymptotic expansion of $\sum_{n=1}^{\infty} d(n)\log(n)e^{-ny}$ in (1.15), we find that as $\delta \to 0$ (within the region $0 < \mathrm{Re}(\delta) < \pi$),

$$\begin{aligned}
&\pi e^{-\frac{i\delta}{2}} \sum_{n=1}^{\infty} d(n)\log(n)\exp\left(-2\pi i n e^{-i\delta}\right)\\
&= \pi e^{-\frac{i\delta}{2}} \sum_{n=1}^{\infty} d(n)\log(n)\exp\left(-2\pi i n(e^{-i\delta}-1)\right)\\
&= \pi e^{-\frac{i\delta}{2}}\left[\frac{1}{2\pi i(e^{-i\delta}-1)}\left(\log^2(2\pi i(e^{-i\delta}-1)) + \frac{\pi^2}{6} - \gamma^2\right) - \frac{1}{2}\log(2\pi)\right.\\
&\quad + \sum_{k=2}^{m-1} \frac{B_{2k}(2\pi i(e^{-i\delta}-1))^{2k-1}}{k}\left\{\frac{B_{2k}}{(2k)!}\left(\gamma - \sum_{j=1}^{2k-1}\frac{1}{j} + \log(2\pi)\right) + \frac{2(-1)^k\zeta'(2k)}{(2\pi)^{2k}}\right\}\\
&\quad \left. + \frac{2\pi i(e^{-i\delta}-1)}{6}\left(\log A - \frac{1}{12}\right) + O\left((e^{-i\delta}-1)^{2m-1}\right)\right].
\end{aligned} \tag{5.24}$$

Now, observe that upon expanding $e^{-i\delta}$ as a power series in δ and simplifying, we have

$$\begin{aligned}
\log^2(2\pi i(e^{-i\delta}-1)) &= \log^2(2\pi\delta) + 2\log(2\pi\delta)\log\left(1 + \frac{i^3\delta}{2!} - \frac{i^4\delta^2}{3!} + \dots\right)\\
&\quad + \log^2\left(1 + \frac{i^3\delta}{2!} - \frac{i^4\delta^2}{3!} + \dots\right).
\end{aligned} \tag{5.25}$$

As $\delta \to 0$, the second expression on the above right-hand side simplifies to

$$\begin{aligned}
2\log(2\pi\delta)\log\left(1 + \frac{i^3\delta}{2!} - \frac{i^4\delta^2}{3!} + \dots\right) &= 2\log(2\pi\delta)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{i^3\delta}{2!} - \frac{i^4\delta^2}{3!} + \dots\right)^k\\
&= -2\log(2\pi\delta)\sum_{k=1}^{2m-1} a_k\delta^k + O\left(\delta^{2m}\log(\delta)\right),
\end{aligned} \tag{5.26}$$

and, using the power series expansion of $\log^2(1+x)$, we see that the third expression becomes

$$\log^2\left(1 + \frac{i^3\delta}{2!} - \frac{i^4\delta^2}{3!} + \dots\right) = \sum_{k=2}^{2m-1} b_k\delta^k + O\left(\delta^{2m}\right), \tag{5.27}$$

where a_k and b_k are effectively computable constants. Inserting (5.26) and (5.27) in (5.25), we find that

$$\log^2(2\pi i(e^{-i\delta} - 1) = \log^2(2\pi\delta) - 2\log(2\pi\delta)\sum_{k=1}^{2m-1} a_k\delta^k + \sum_{k=2}^{2m-1} b_k\delta^k + O\left(\delta^{2m}\log(\delta)\right). \tag{5.28}$$

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Substitute the above estimate for $\log^2(2\pi i(e^{-i\delta}-1)$ in (5.24) to find upon simplification that as $\delta \to 0$,

$$\begin{aligned}
&\pi e^{-\frac{i\delta}{2}} \sum_{n=1}^{\infty} d(n)\log(n)\exp\left(-2\pi i n e^{-i\delta}\right)\\
&= \frac{1}{4\sin\left(\frac{\delta}{2}\right)}\left(\log^2(2\pi\delta) - 2\log(2\pi\delta)\sum_{k=1}^{2m-1} a_k\delta^k + \sum_{k=2}^{2m-1} b_k\delta^k\right.\\
&\qquad\left. + O\left(\delta^{2m}\log(\delta)\right) + \frac{\pi^2}{6} - \gamma^2\right) - \frac{\pi}{2}e^{-\frac{i\delta}{2}}\log(2\pi)\\
&\quad + \sum_{k=1}^{m-1} \pi c_k' e^{-\frac{i\delta}{2}}(2\pi i(e^{-i\delta}-1))^{2k-1} + O\left((e^{-i\delta}-1)^{2m-1}\right)\\
&= \frac{1}{4\sin\left(\frac{\delta}{2}\right)}\left(\log^2(2\pi\delta) - 2\log(2\pi\delta)\sum_{k=1}^{2m-1} a_k\delta^k + \sum_{k=2}^{2m-1} b_k\delta^k + \frac{\pi^2}{6} - \gamma^2\right)\\
&\quad + \sum_{k=0}^{2m-2} c_k\delta^k + O\left(\delta^{2m-1}\log(\delta)\right),
\end{aligned} \tag{5.29}$$

where $c_0 = -(\pi/2)\log(2\pi)$. In the last step, we used the fact that $1/\sin\left(\frac{\delta}{2}\right) = O(1/\delta)$.

Consequently, from (5.23) and (5.29), as $\delta \to 0$ in $|\arg(\delta)| < \pi/2$, we arrive at (1.16) upon simplification which involves the use of the estimate

$$\frac{1}{\sin(\frac{\delta}{2})} = \frac{2}{\delta} + \sum_{n=1}^{m-1} r_n\delta^{2n-1} + O\left(\delta^{2m-1}\right)$$

as $\delta \to 0$, where r_n are constants. This completes the proof.

6 Concluding remarks

The main highlight of this paper was to derive (1.13), that is, an exact transformation for the series $\sum_{n=1}^{\infty}\frac{\log(n)}{e^{ny}-1}$, or equivalently, for $\sum_{n=1}^{\infty} d(n)\log(n)e^{-ny}$, where $\mathrm{Re}(y) > 0$. Such a transformation was missing from the literature up to now. One of the reasons for this could be the lack of availability of the transformation in (1.5) until [23] appeared, which, in turn, resulted from evaluating an integral with a combination of Bessel functions as its associated kernel. This underscores the importance of the applicability of integral transforms in number theory.

In addition, it is to be emphasized that in order to derive (1.13), several new intermediate results, interesting in themselves, had to be derived, for example, the ones given in Theorems 3.3, 3.4, and 4.4. This also shows how important $\psi_1(z)$, and, in general, $\psi_k(z)$, are in number theory, and further corroborates Ishibashi's quote given in the introduction.

The transformation in (1.13) has a nice application in the study of moments of $\zeta(s)$, namely, to obtain the asymptotic expansion of $\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta'\left(\frac{1}{2}+it\right)e^{-\delta t}\,dt$ as $\delta \to 0$. However, considering the fact that (1.13) holds for *any* y with $\mathrm{Re}(y) > 0$, we expect further applications of it in the future, in particular, in the theory of modular forms.

A generalization of (1.13), equivalently, of (1.14), may be obtained for the series $\sum_{n=1}^{\infty} d(n)\log^k(n)e^{-ny}$. This would require differentiating (1.5) k times with respect to a and then letting $a \to 0$. This would involve Ishibashi's higher analogs of Deninger's $R(z)$ defined in (3.4). Ishibashi [36, Theorem 1] gave a Plana-type formula for $R_k(x)$, $x > 0$, which is easily seen to hold for complex x in the right-half plane $\mathrm{Re}(x) > 0$. This formula involves a polynomial in $\log(t)$ defined by $S_k(t) := \sum_{j=0}^{k-1} a_{k,j}\log^j(t)$, where $a_{k,j}$ are recursively defined by $a_{k,j} = -\sum_{r=0}^{k-2}\binom{k-1}{r}\Gamma^{(k-r-1)}(1)a_{r+1,j}$, $0 \le j \le k-2$, with $a_{1,0} = 1$ and $a_{k,k-1} = 1$.

Observe that $S_1(t) = 1$ and $S_2(t) = \gamma + \log(t)$, and so the numerators of the summands of the series on the left-hand sides of (1.6) and (1.14) are $S_1(ny)$ and $S_2(ny)$ respectively. In view of this, we speculate the left-hand side of (1.14) to generalize to $\sum_{n=1}^{\infty}\dfrac{S_{k+1}(ny)}{e^{ny}-1}$ and the right-hand side to involve $\psi_k(z)$.

Since (5.18) can be generalized to

$$\sum_{n=1}^{\infty}\frac{\log^k(n)}{e^{ny}-1} = -\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta^{(k)}(s)y^{-s}\,ds \qquad (c = \mathrm{Re}(s) > 1, \mathrm{Re}(y) > 0),$$

on account of the fact that $\zeta^{(k)}(s) = (-1)^k\sum_{n=1}^{\infty}\log^k(n)n^{-s}$, once a generalization of (1.14) is obtained, it would possibly give us the complete asymptotic expansion of $\int_0^{\infty}\zeta\left(\frac{1}{2}-it\right)\zeta^{(k)}\left(\frac{1}{2}+it\right)e^{-\delta t}\,dt$ as $\delta \to 0$. Details will appear elsewhere.

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