Canad. Math. Bull. Vol. **26** (1), 1983

ON THE PRODUCT OF IDEALS

BY

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ABSTRACT. This article introduces the concept of a condensed domain, that is, an integral domain R for which $IJ = \{ij: i \in I, j \in J\}$ for all ideals I and J of R. This concept is used to characterize Bézout domains (resp., principal ideal domains; resp., valuation domains) in suitably larger classes of integral domains. The main technical results state that a condensed domain has trivial Picard group and, if quasilocal, has depth at most 1. Special attention is paid to the Noetherian case and related examples.

1. **Introduction.** Let I and J be ideals of a commutative ring R. The product IJ is, of course, defined to be the ideal of R generated by the set of products, $P = P(I, J) = \{x \in R$: there exist $i \in I$ and $j \in J$ such that $x = ij\}$. Although it is customary to take P itself as the product of I and J in algebraic contexts where only one binary operation (in this case, multiplication) is available, such a convention would be inappropriate for ring theory, since many examples show that P need not be closed under sums. The present note treats the so-called *condensed domains*, the (commutative integral) domains R in which all sets of the form P actually are ideals; that is, where $P(I, J) = IJ$ for all choices of I and J.

As noted in Corollary 2.2 below, familiar examples of condensed domains include all Bézout domains and, in particular, all principal ideal domains and all valuation domains. Our main purpose here is to show that the "condensed" property serves to characterize each of the above types of domains amongst suitably larger classes of domains. In this regard, see Propositions 2.12 and 2.13 and Corollaries 2.6 and 2.8. The main technical tools are Proposition 2.5 and Theorem 2.7: each condensed domain has trivial Picard group and, if quasi-local, has depth at most 1. Special attention is paid to the impact of the "condensed" property in the Noetherian case (Corollaries 2.8 and 2.9 and Theorem 3.3). Further results and related concepts are treated in the brief final section.

Received by The editors September 4, 1981 and in revised form, December 17, 1981.

* Supported in part by a University of Tennessee Faculty Development Grant.

AMS (MOS) subject classifications (1980). Primary 13F05, Secondary 13A15, 13C15, 13E05, 13G05.

https://doi.org/10.4153/CMB-1983-016-2 Published online by Cambridge University Press

Throughout, R denotes a domain with group of units $U(R)$, integral closure R', and quotient field K. Any unexplained terminology is standard, as in [2] and [3].

2. **Main results.** We begin with some characterizations of condensed domains.

PROPOSITION 2.1. *For a domain R, the following conditions are equivalent:*

(1) *$P(I, J) = IJ$ for all two-generated ideals I and J of R;*

(2) *R is a condensed domain;*

(3) *For all positive integers $n \geq 2$ and all ideals $I_1, I_2, \ldots, I_n$ of R, one has $I_1 I_2 \cdots I_n = \{x \in R$: for each $k = 1, 2, \ldots, n$, there exists $i_k \in I_k$ such that $x = i_1 i_2 \cdots i_n\}$;*

(4) *There exists a positive integer $n \geq 2$ such that for all two-generated ideals $I_1, I_2, \ldots, I_n$ of R, one has $I_1 I_2 \cdots I_n = \{x \in R$: for each $k = 1, 2, \ldots, n$, there exists $i_k \in I_k$ such that $x = i_1 i_2 \cdots i_n\}$.*

Proof. (1) $\Rightarrow$ (2): Let I and J be ideals of R. Consider $r \in IJ$; write $r = i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$, with $i_k \in I$ and $j_k \in J$ for each k. For each $2 \leq k \leq n$, let $r_k = i_1 j_1 + i_2 j_2 + \cdots + i_k j_k$. Then $r_2 \in AB$, where the ideals A and B are given by $A = (i_1, i_2) \subset I$ and $B = (j_1, j_2) \subset J$. An application of (1) yields $a \in A$ and $b \in B$ such that $r_2 = ab \in P(A, B) \subset P(I, J)$. Consequently $r_3 = r_2 + i_3 j_3 \in CD$, where the ideals C and D are given by $C = (a, i_3) \subset I$ and $D = (b, j_3) \subset J$. Another application of (1) yields $r_3 \in P(C, D) \subset P(I, J)$. Continuing repeatedly in this way, we find ultimately that $r = r_n \in P(I, J)$, as desired.

(2) $\Rightarrow$ (3): This follows easily by induction, in view of the observation that $I_1 I_2 \cdots I_n = (I_1 I_2 \cdots I_{n-1}) I_n$.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Assume (4). Given I and J as in (1), select any nonzero element $s \in R$, and set $I_j = (s)$, a principal (hence, two-generated) ideal of R, for each $3 \leq j \leq n$. If $r \in IJ$ then $rs^{n-2} \in IJ I_3 \cdots I_n$, and so an application of (4) supplies $i \in I$, $j \in J$ and $r_3, \ldots, r_n \in R$ such that $rs^{n-2} = ij(r_3 s) \cdots (r_n s)$. Cancellation in the domain R leads to $r = i(jr_3 \cdots r_n) \in P(I, J)$, completing the proof.

COROLLARY 2.2. *If R is a Bézout domain, then R is condensed. In particular, all principal ideal domains and all valuation domains are condensed.*

Proof. By Proposition 2.1, it is enough to show that $P(I, J) = IJ$ for all finitely generated ideals I and J of R. Since R is a Bézout domain, such I and J are principal ideals. However, a cancellation argument (cf. the above proof that (4) $\Rightarrow$ (1)) yields $P(I, J) = IJ$ whenever I is principal and J is *any* ideal. This completes the proof.

We next give an example of a one-dimensional local condensed domain which is not integrally closed.

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EXAMPLE 2.3. Let F be a field. Set $R = F[[X^2, X^3]]$, the ring of those formal power series over F whose coefficient of X is 0. Then R is a condensed domain.

For a proof, first recall from [1, Exercise 1(a), p. 545] that all nonprincipal ideals of R assume the form (X^k, X^{k+1}), $k \geq 2$. If $I = (X^m, X^{m+1})$ and $J = (X^n, X^{n+1})$ for some $2 \leq m \leq n$, the fact that $X^{n+2} \in J$ readily leads to $IJ = (X^m)J$, so that $IJ = P((X^m), J) \subset P(I, J)$. In view of the earlier comments, this guarantees that R is condensed.

The class of condensed domains is stable under various constructions, such as formation of factor-domains and localization. Our next result generalizes the latter fact.

PROPOSITION 2.4. *Each overring of a condensed domain is itself condensed.*

Proof. Let S be an overring of a condensed domain R. If I and J are ideals of S and if $s \in IJ$, write $s = i_1 j_1 + \cdots + i_n j_n$, with $i_k \in I$ and $j_k \in J$ for each k. By multiplying denominators appropriately, one finds a nonzero element $r \in R$ such that $ri_k, rj_k \in R$ for all k. Consider the ideals $A = (ri_1, \ldots, ri_n)$ and $B = (rj_1, \ldots, rj_n)$ of R. Since $r^2 s \in AB$ and R is condensed, $r^2 s = ab$ for some $a \in A$ and $b \in B$. Note that $a_1 = ar^{-1} \in \sum Ri_k \subset I$ and, similarly, $b_1 = br^{-1} \in J$. As $r^2 s = r^2 a_1 b_1$, cancellation gives $s = a_1 b_1 \in P(I, J)$, as desired.

The import of the next result is that $\mathrm{Pic}(R) = 0$ for each condensed domain R.

PROPOSITION 2.5. *If I is an invertible ideal of a condensed domain R, then I is principal.*

Proof. As usual, set $I^{-1} = \{x \in K : xI \subset R\}$. Invertibility of I guarantees that I^{-1} is a finitely generated R-module (cf. proof of [3, Theorem 58]), thus producing a nonzero element $r \in R$ such that $A = rI^{-1} \subset R$. Since $r = r1 \in r(II^{-1}) = IA$ and R is condensed, there exist $i \in I$ and $a \in A$ such that $r = ia$. Then $b = ar^{-1} \in I^{-1}$ satisfies $r = ibr$, whence cancellation gives $ib = 1$. Therefore, for each $y \in I$, we have $y = y(ib) = (yb)i \in Ri$, so that $I = Ri$, completing the proof.

COROLLARY 2.6 (cf. Zafrullah [7, Corollary 8]). *For a domain R, the following conditions are equivalent:*

(1) *R is a condensed Prüfer domain;*

(2) *R is a Bézout domain.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Our task is to show that if I is a finitely generated ideal of R, then I is principal. Without loss of generality, $I \neq 0$. Since R is a Prüfer domain, I is therefore invertible. Now an applicaton of Proposition 2.5 suffices.

(2) $\Rightarrow$ (1): As any Bézout domain is a Prüfer domain, the assertion follows from Corollary 2.2.

We next present our main technical result. As usual, we shall take $\operatorname{depth}(R) \leq 1$ to mean that R contains no R-sequence of length greater than 1.

THEOREM 2.7. *If R is a quasilocal condensed domain, then* $\operatorname{depth}(R) \leq 1$.

Proof. If not, then the maximal ideal M of R contains an R-sequence x, y. Consider the ideals $I = (x, y^2)$ and $J = (x^2, y)$ of R. As $x^3 + y^3 \in IJ$ and R is condensed, there exist $a, b, c, d \in R$ such that

$$x^3 + y^3 = (ax + by^2)(cx^2 + dy).$$

Expanding and regrouping reveal that $(1 - ac)x^3 \in (y)$. Since y, x is an R-sequence, $1 - ac \in (y) \subset M$, so that $ac \in U(R)$. Thus both a and c are units of R; similarly, so are b and d. Consequently $ad + bcxy \in U(R)$. However, the above equation yields $(ad + bcxy)xy \in (x^3, y^3)$, whence $xy = rx^3 + sy^3$ for suitable $r, s \in R$. As $sy^3 \in (x)$ and x, y is an R-sequence, $s \in (x)$; write $s = ex$ for some $e \in R$. Similarly, $r = fy$ for some $f \in R$. Since $xy = (fx^2 + ey^2)xy$, cancellation gives $1 = fx^2 + ey^2 \in M$, the desired contradiction, completing the proof.

COROLLARY 2.8. *For a domain R, the following conditions are equivalent:*
(1) *R is Noetherian,* $\operatorname{gl\,dim}(R) < \infty$ *and R is condensed;*
(2) *R is Noetherian, integrally closed and condensed;*
(3) *R is a principal ideal domain.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Since R is Noetherian, $\sup\{\operatorname{gl\,dim}(R_M): M$ is a maximal ideal of $R\} = \operatorname{gl\,dim}(R) < \infty$. For each M, R_M is therefore a regular local ring, hence integrally closed; thus, R is integrally closed as well.

(2) $\Rightarrow$ (3): Assume (2). If the Krull dimension of R exceeds 1, the generalized principal ideal theorem (cf. [3, Theorem 152]) provides a height 2 prime P of R. Then R_P is a Macaulay ring [3, Exercise 25, p. 104], and so has an R_P-sequence of length 2. However, Proposition 2.4 guarantees that R_P is condensed, whence Theorem 2.7 gives $\operatorname{depth}(R_P) \leq 1$, a contradiction. Hence, $\dim(R) \leq 1$. Accordingly, R satisfies Noether's conditions for a Dedekind domain. As each nonzero ideal of R is therefore invertible, an application of Proposition 2.5 yields (3).

(3) $\Rightarrow$ (1): Trivial.

By Corollaries 2.2 and 2.8, any Dedekind domain which is not a principal ideal domain is locally condensed (in the obvious sense) but not condensed.

COROLLARY 2.9. *Let R be a Noetherian condensed domain. Then* $\dim(R) \leq 1$ *and R' is a principal ideal domain.*

Proof. If $\dim(R) > 1$, one argues as above that R has a height 2 prime P, so that $S = R_p$ is a two-dimensional Noetherian condensed domain. By the Mori–Nagata theorem (cf. [4, Theorem 33.12]), S' is also Noetherian. Since Proposition 2.4 assures that S' is condensed, Corollary 2.8 yields that S' is a principal ideal domain, whence $2 = \dim(S) = \dim(S') \leq 1$, a contradiction. Therefore $\dim(R) \leq 1$. As R' is Noetherian (by the Krull–Akizuki theorem [3, Theorem 93]) and condensed (by Proposition 2.4), Corollary 2.8 gives that R' is a principal ideal domain, as asserted.

We are grateful to the referee for the observations in the following two paragraphs.

If A is a (Noetherian) local domain of dimension at least 2, then there exist nonzero nonunits x and y of A such that $(x):y = (x):y^2$. (For instance, let M be the maximal ideal of A, choose $0 \neq x \in M$, use the prime avoidance lemma to find z in M but outside each of the minimal primes of (x), use the Noetherian property to find n such that $(x):z^n = (x):z^i$ for all $i > n$, and set $y = z^n$.) Hence, by arguing as in Theorem 2.7, no such A can be condensed. (Otherwise, argue as in Theorem 2.7 that $(1-ac)x^3 \in (y)$. If $1-ac$ is a unit, then $x^3 \in (y)$, so that x is in any minimal prime of y, contradicting the fact that height $(x, y) = 2$. Thus a and c are units, and similarly so are b and d. As before, $xy = rx^3 + sy^3$, whence $sy \in (x):y^2 = (x):y$, and $s \in (x):y^2 = (x):y$, producing $t \in R$ such that $sy = tx$. This leads to $y = rx^2 + ty^2$ and, since $1-ty$ is a unit, to $y \in (x^2)$, contradicting height $(x, y) = 2$.)

The assertions of the preceding paragraph apply in particular to $(A =)$ S, the ring in the proof of Corolary 2.9. Accordingly, those assertions may replace the second and third sentences of that proof. Thus the first assertion of Corollary 2.9 may be obtained without appeal to the Mori–Nagata theorem.

Corollary 2.10. *For a domain R, the following conditions are equivalent:*
(1) *$R[X]$ is condensed;*
(2) *$R[[X]]$ is condensed;*
(3) *R is a field.*

Proof. If R is a field, then both $R[X]$ and $R[[X]]$ are principal ideal domains. Consequently (3) implies both (1) and (2). We shall next prove that (2) $\Rightarrow$ (3), leaving the similar details for (1) $\Rightarrow$ (3) for the reader. If (2) $\Rightarrow$ (3) fails, let $r \in R$ be a nonzero nonunit, choose a maximal ideal M of R containing r, and set $S = R[[X]]_{(M,X)}$. As $R[[X]]$ is supposed condensed, Proposition 2.4 assures that S is condensed, so that Theorem 2.7 yields depth$(S) \leq 1$. However r, X is evidently an $R[[X]]$-sequence and, hence, also an S-sequence (cf. [3, Theorem 133]), the desired contradiction. This completes the proof.

We next collect some examples which illuminate some of the preceding material. To motivate parts (b) and (c) of Examples 2.11, note that any local (Noetherian) one-dimensional integrally closed domain is a DVR and, hence, condensed.

EXAMPLES 2.11. (a) The converse of Corollary 2.9 is false. Indeed, there exists a Noetherian one-dimensional domain R such that R' is a principal ideal domain and R is not condensed. One such example is given by $R = F[X^2, X^3]$, the ring of those polynomials over a field F whose coefficients of X is 0. It is well-known that $\mathrm{Pic}(R) \neq 0$ (indeed, $\mathrm{Pic}(R) \cong F$ *qua* abelian groups), and so the fact that R is not condensed is a consequence of Proposition 2.5. The other assertions are immediate.

(b) There exists a local (Noetherian) one-dimensional domain R such that $\mathrm{Pic}(S) = 0$ for each overring S of R and R is not condensed. (Necessarily, R is not integrally closed. Indeed, the example given below is not even seminormal.)

For the construction, let $R = \mathbb{R} + \mathbb{R}X^3 + \mathbb{R}X^5 + \mathbb{R}X^6 + \mathbb{R}X^8 + \ldots$, the ring of those formal power series over $\mathbb{R}$ whose coefficients of X, X^2, X^4 and X^7 are 0. Since $R' = \mathbb{R}[[X]]$ is a valuation domain, each overring S of R is (quasi-)local (cf. [5, Proposition 2.34]), whence $\mathrm{Pic}(S) = 0$ by [1, Proposition 5, p. 113]. As $R' = R[X]$ is Noetherian, it follows by Eakin's theorem (cf. [3, Exercise 15, p. 54]) that R is also Noetherian; one-dimensionality follows by integrality. Moreover, the unique maximal ideal of R is given by $M = X\mathbb{R}[[X]] \cap R = (X^3, X^5)$. We claim that $X^6 + X^{10}$, which is evidently in M^2, is not in $P(M, M)$. Otheriwse, $X^6 + X^{10} = (X^3 + aX^5 + \cdots)(X^3 + bX^5 + \cdots)$ for suitable real coefficients $a, b, \ldots$. Equating corresponding coefficients of X^8 (resp., X^{10}) yields $a + b = 0$ (resp., $ab = 1$), whence $a^2 = -1$, the desired contradiction, proving the claim. Therefore R is not condensed. Finally, R is not seminormal since $y = X^4 \in R' \setminus R$ satisfies $y^2, y^3 \in R$.

(c) There exists a quasi-local one-dimensional integrally closed domain R such that R satisfies accp (ascending chain condition on principal ideals) and R is not condensed.

For the construction, let k be a field of characteristic 0, let Y and Z be algebraically independent commuting indeterminates over k, set $F = k(Y, Z)$, and consider the valuation domain $V = F[[X]]$. Note $V = F + M$, where $M = XV$. We claim that $R = k + M$ has the asserted properties.

Indeed, it is well known that R is quasi-local, one-dimensional and integrally closed (cf. [3, Exercise 5, p. 52]). Next, observe that

$$k = k(Y^2 + Y, Z) \cap k(Y^2, Z) \cap k(Y, Z^2) \cap k(Y, Z + Z^2).$$

If L_i $(1 \leq i \leq 4)$ denote the fields intersected above, then $D_i = L_i + M$ is Noetherian for each i (cf. [2, Exercise 8(3), p. 271]). Then, as a finite intersection of domains each having accp, $R = \bigcap D_i$ necessarily also satisfies accp.

Finally, to show that R is not condensed, let I be the ideal of R generated by $X(Y, Z)k[Y, Z]$. We claim that $X^2Y^2 + X^2Z^3$, which is evidently in I^2, is not in $P(I, I)$. Otherwise, a degree argument (and cancellation of X^2) would lead to an expression of $Y^2 + Z^3$ as a product of two elements in $(Y, Z)k[Y, Z]$,

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contradicting the fact that Y^2+Z^3 is an irreducible element of $k[Y, Z]$. This establishes the claim and completes the proof.

PROPOSITION 2.12. *For a domain R, the following conditions are equivalent:*
(1) *R is a condensed* GCD-*domain;*
(2) *R is a Bézout domain.*

Proof. (2) $\Rightarrow$ (1): If R is a Bézout domain, then R is condensed by Corollary 2.2, and it is well-known that R is a GCD-domain.

(1) $\Rightarrow$ (2): Assume (1). By Corollary 2.6, it is enough to show that R is a Prüfer domain. As the conditions in (1) are preserved by localizaton (cf. Proposition 2.4), we may assume that R is quasilocal, say with maximal ideal M, and our task is then to prove that R is a valuation domain. Note that R, being a GCD-domain, is necessarily a so-called finite-conductor domain, in the sense that $Ra \cap Rb$ is finitely generated for all $a, b \in R$. Accordingly, by the *proof* of [6, Lemma 3.9], it suffices to establish that M is R-flat. Now if r and s are in M then Theorem 2.7 guarantees that r, s is not an R-sequence; thus by [3, Exercise 5, p. 102], a greatest common divisor d of r and s must lie in M. In particular, $(r, s) \subset (d) \subset M$. Consequently M is a direct limit of (flat) principal ideals of R, and so M is R-flat [1, Proposition 2(ii), p. 14], completing the proof.

By passing to the quasilocal case of Proposition 2.12, we see that valuation domains may be characterized as the quasilocal condensed GCD-domains.

A compansion result to Proposition 2.12 characterizes principal ideal domains as the condensed unique factorization domains. One way to see this is via the following generalization.

PROPOSITION 2.13. *For a domain R, the following conditions are equivalent:*
(1) *R is a condensed Krull domain;*
(2) *R is a principal ideal domain.*

Proof. We need only tend to the proof that (1) $\Rightarrow$ (2). Assume (1). If $\dim(R) \leq 1$ then R is Noetherian (cf. [3, Exercise 8, p. 83]) and integrally closed, so that (2) follows from Corollary 2.8. Without loss of generality, we may therefore assume $n = \dim(R) > 1$. There is no harm in further assuming that R is quasilocal, with maximal ideal M. By Theorem 2.7, $\text{depth}(R) \leq 1$, whence [3, Exercise 4(b), p. 83] implies that h, the height of M in R, is at most 1. (Note that the result quoted from [3] may be applied because R is a Krull domain.) But $h = n > 1$, the desired contradiction, completing the proof.

Since the integral closure of any Noetherian domain must be a Krull domain [4, Theorem 33.10(1)], Proposition 2.13 readily leads to another proof of Corollary 2.9.

3. **Further results.** This section begins with a characterization of valuation domains which, quite apart from any possible intrinsic interest, yields another proof that all valuation domains are condensed.

PROPOSITION 3.1. *For a domain R, the following conditions are equivalent:*
(1) *R is a valuation domain;*
(2) *If an indexed subset $\{i_k\}$ of R generates the ideal I of R, then $I = \bigcup (i_k)$;*
(3) *If indexed subsets $\{i_m\}$ and $\{j_n\}$ of R generate ideals I and J, respectively, then $IJ = \bigcup (i_m j_n)$.*

Proof. $(1) \Rightarrow (2)$: Assume (1). Given $\{i_k\}$ as in (2) and $r \in I$, write $r = r_1 i_{k_1} + \cdots + r_n i_{k_n}$ for some indexes $k_1, \ldots, k_n$ and some coefficients $r_1, \ldots, r_n \in R$. By (1), we may relabel so that $i_{k_m} \in (i_{k_1})$ for all $2 \le m \le n$, so that $r \in (i_{k_1})$, as desired.

$(2) \Rightarrow (3)$: Given I and J as in (3), note that $\{i_m j_n\}$ generates IJ. Apply (2).

$(3) \Rightarrow (1)$: Let $r, s \in R$. Consider the ideals I and J of R generated by the sets $\{r, s\}$ and $\{1\}$, respectively. Since $r+s \in I = IR = IJ$, (3) assures that $r+s \in (r) \cup (s)$. Without loss of generality, $r+s \in (r)$. Then $s = (r+s) - r \in (r)$, establishing (1), to complete the proof.

It is clear that condition (3) of Proposition 3.1 implies that R is condensed. Note also that the list of equivalent conditions in Proposition 3.1 may be augmented by three conditions analogous to (1), (3) and (4) in the statement of Proposition 2.1.

In view of the arguments given for parts (b) and (c) of Examples 2.11, it seems convenient to say that a domain R is *semicondensed* in case $I^2 = P(I, I)$ for each ideal I of R. This definition permits us to state a sharpening of the first assertion in Corollary 2.9. First, we shall need the next result.

LEMMA 3.2. *Each overring of a semicondensed domain is itself semicondensed.*

Proof. (Sketch) Modify the proof of Proposition 2.4 in the natural way, by applying the semicondensed property to the ideal $(ri_1, \ldots, ri_n, rj_1, \ldots, rj_n)$.

THEOREM 3.3. *If R is a Noetherian semicondensed domain, then* $\dim(R) \le 1$.

Proof. We claim first that $\operatorname{depth}(R_p) \le 1$ for each prime P of R. If the claim fails, then Lemma 3.2 permits us to suppose that R is quasilocal, say with maximal ideal M, containing an R-sequence x, y. By the proof of [3, Theorem 125], we may change notation so as to assume $S = \{x, y, z_1, \ldots, z_n\}$ is a minimal generating set for M as an ideal of R. (Possibly $n = 0$, in which case $M = (x, y)$.) Consider the ideal $I = (x, y)$ of R. As $x^2 + y^3 \in I^2$ and R is semicondensed, there exist $a, b, c, d \in R$ such that

$$x^2 + y^3 = (ax + by)(cx + dy).$$

As $(1-ac)x^2 \in (y)$ and y, x is an R-sequence, $1-ac = ry$ for some $r \in R$; consequently, both a and c are in $U(R)$. Moreover, the equation displayed above yields $(bd-y)y = (rx-bc-ad)x$ after cancellation of y. The R-sequence property then guarantees that $bd-y \in (x)$ and $rx-bc-ad \in (y)$. From the first of these, we see that either b or d is in M; from the second, that $bc+ad \in M$. It follows from the earlier information about a and c that *both* b and d are in M. Thus, writing $bd-y = sx$ for some $s \in R$, we have $M = (x, sx+y, z_1, \ldots, z_n) = (x, bd, z_1, \ldots, z_n) = (x, z_1, \ldots, z_n)$ since $bd \in M^2$. This contradicts minimality of S and establishes the claim.

Next, if $\dim(R) > 1$, one argues as in Corollaries 2.8 and 2.9 that R has a height 2 prime P, so that $T = R_p$ is two-dimensional, Noetherian and (thanks to Lemma 3.2) semicondensed. Combining the Mori–Nagata theorem with Lemma 3.2 reveals that T' is a two-dimensional, Noetherian, integrally closed, semicondensed domain. By the result of the preceding paragraph, each localization of T' at a prime has depth at most 1. Accordingly, one may now argue as in the proof that $(2) \Rightarrow (3)$ in Corollary 2.8, concluding that $\dim(T') \leq 1$, the desired contradiction, to complete the proof.

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