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THE EVOLUTION OF COMET ORBITS AS PERTURBED BY URANUS AND NEPTUNE

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When the perturbing planets are Uranus and Neptune, the perturbations on comets are so much weaker than with Jupiter and Saturn that a study of the comets' orbital evolution, using exact numerical integration, would require 200 times more revolutions. This is hardly practical with present computers. Here we describe results with a simulation approach, the "Monte Carlo (random walk) method." The proper distribution shape for the perturbations in energy are found from a few thousand numerical integrations, then this distribution of perturbations is applied to millions of simulated orbit-revolutions. This method reproduces earlier Jupiter results in 1/500 the former computation time. We find that Neptune can capture near-parabolic comets with perihelia in the range of 30 to 34 AU, increasing their 1/a-values and decreasing their perihelia until they reach a region where Uranus can interact. Uranus in turn passes some of these on to Saturn, who passes some to Jupiter. Ultimately a few reach the orbits of the visible short-period comets. The process requires about 200,000 comet orbit-revolutions, 4 x 10^8 years, and the efficiency is one in 6000. The rest of the comets are ejected on hyperbolic orbits.

When comets have their perihelia at 22 to 34 AU, in the outermost parts of the planetary region, the pertinent planetary perturbations are the weak ones by Neptune. It is well known that Neptune is 107 times less effective than Jupiter would be in a comparable single interaction, but it is not always noted that the net effect of a number n of random perturbations in sequence depends on the square of this number. Neptune must interact 107 x 107n times (that is, for 11000n comet orbital revolutions) to cause the same average net change in the comet's 1/a-value that would be caused by n revolutions interacting with Jupiter.

The number of computations required is not as large as the 11,000 to one ratio implies, because only a rather small change in the comet's energy is required of Neptune. To capture a comet it is only necessary for Neptune to increase the comet's 1/a-value, and simultaneously reduce its perihelion, so that the comet can reach the control of Uranus. Nonetheless, enormous numbers of perturbations are necessary. This study of the effects of Neptune and Uranus has required simulation of some 3 x 10^8 orbital revolutions, each requiring 5 x 10^{-6} seconds on a CDC 7600 computer. The random walk method has made the computation feasible. In contrast, the Monte Carlo exact numerical integration method would have required 500 times as much computation time.

A random walk method for simulation natural stochastic processes has been applied in many fields, including perturbations of comets. See Lyttleton and

Hammersley (1963), for example. Indeed, the papers by Weissman (1976) and by Rickman and Vaghi (1976) in this volume apply Monte Carlo methods, specifically random walk methods, to their problems.

Let U be a dimensionless measure of the perturbation in 1/a caused by a single planet p. The actual perturbation $u = \Delta(1/a)$ in AU^{-1} (reciprocal astronomical units) is found from U using $u = (m_p / a_p)U$, where m_p is the mass of the planet in units of the sun's mass, and a_p is its orbital radius. Our study (Everhart 1968) finds the distribution function f(U), which is symmetrical about U = 0, to be fit well by the empirical expression

$$f(\pm U) = G \exp(-(D|U| - 1)^2) + F / (U^2 + B^2)^{3/2}. \qquad (1)$$

The parameters D, F and B depend on inclination i, dimensionless perihelion distance q/a_p, and a dimensionless measure a_p/a of the present energy of the orbit, but the orbits are random in the other two angular elements and in the mean anomaly of the planet when the comet arrives at perihelion. Here G is adjusted to give the distribution an area of unity for positive values of U. The paper cited above tabulates G, D, F, and B for near-parabolic orbits, and shows that any function describing the distribution must vary as $1/|U|^3$ for large perturbations. In fact the inverse U-cubed portion is far more important than the Gaussian part of the distribution.

Before tabulating the parameters used here, we next examine the evolutionary track specified by the several Tisserand quantities C_p, with index-values p = 1, 2, 3, 4 referring to Jupiter, Saturn, Uranus, Neptune. These arise from the Jacobi integral of the 3-body problem, but when applied to the current 6-body problem (Sun, comet, 4 planets) none of the values C_p are constant. Numerical experiments show that the value of C_p is not changed much by perturbations caused by the planet p, though it is changed by the effects of other planets. Thus we take C_4 (for Neptune) to remain constant until an object evolves to a perihelion where Uranus is in control. Then it has a new temporary constant C_3 until the perihelion decreases to where Saturn is in control, and so on. Here

$$\begin{aligned} C_p &= a_p/a + 2((q/a_p)\ (2\ q/a))^{1/2} \cos i, \quad \text{for } 1/a \neq 0 \\ &= (8q_0/a_p)^{1/2} \cos i \quad\quad\quad , \text{ for } 1/a = 0, \end{aligned} \qquad (2)$$

where q and q_0 are perihelion values, the subscript 0 referring to a parabolic orbit. There is a well-known barrier at $C_p = 3$, or $q_0 = 1.125\ a_p$, but for $C_p < 3$ one can fix C_p, i, and vary a_p/a in Eq. (2), solving for q/a_p. One finds that q/a_p decreases as a_p/a increases, this determining the evolutionary track. This eliminates q as an independent quantity and reduces the problem to a one-dimensional random walk. Use of Eq. (2) in this way requires an arbitrary decision about inclination i. Here we make the approximation that i does not change for a given comet, though it can have different values for different comets.

Numerical work has established the values in Table I. For this the inclination values i follow a sinusoidal population distribution extending from 0^o to 9^o where it is cut off. The median value is 6^o. For 1/2 = 0 these values are for single comet-revolutions, but they are for 5 comet-revolutions when 1/a > 0. Of course this speeds up the calculation 5-fold, but it also should be more accurate, because it makes a fair allowance for the correlation between perturbations in consecutive revolutions. The value of D we use in Eq. (1) is 0.5 in all cases.

In carrying out the Monte Carlo random walk, one choses random values of ±U in such a way that they have the population distribution of Eq. (1) with parameters according to the region in Table I. These are transformed to u-values, and the cumulative sum of these u-values is the 1/a-value in AU^{-1}. Figure 1

TABLE I

F- AND B-VALUES TO USE WITH EQ. (1) FOR INCLINATIONS WITH A MEDIAN VALUE OF 6^{o}. (FOR OTHER SMALL VALUES OF i WE FIND THAT (APPROXIMATELY) F VARIES AS i^{-1} AND B AS $i^{-1/2}$)

C_p (average)		2.36	2.61	2.75	2.9
1/a = 0	q_0/a_p	0.60 - 0.80	0.80 - 0.90	0.90 - 1.00	1.00 - 1.125
	F	75	148	220	160
	B	14	22	25	20
$1/a = 0.3/a_p$	q/a_p	0.48 - 0.69	0.69 - 0.81	0.81 - 0.93	0.93 - 1.06
	F	374	526	900	768
	B	25	30	50	30
$1/a = 0.6/a_p$	q/a_p	0.35 - 0.56	0.65 - 0.68	0.68 - 0.82	0.82 - 1.00
	F	366	564	1066	2740
	B	25	40	40	60
$1/a = 0.9/a_p$	q/a_p	0.23 - 0.40	0.40 - 0.51	0.51 - 0.66	0.66 - 0.88
	F	354	574	814	3250
	B	20	30	30	90
$1/a = 1.1/a_p$	q/a_p	0.16 - 0.30	0.30 - 0.40	0.40 - 0.52	0.52 - 0.71
	F	334	560	734	3280
	B	20	30	25	70

shows the average 1/a-value attained by those comets still in elliptical orbits after n revolutions, starting from 1/a = 0. The factor of 11,000 revolutions, mentioned already, between Jupiter's effects and those of Neptune is evident. Since the distribution in U is symmetrical about zero, many comets (and eventually all of them) arrive at negative 1/a-values and are lost to infinity. This attrition is the same on all plots, following the $n^{-1/2}$ law illustrated in Fig. 1 on page 499 in Everhart (1974). Figure 1 and the $n^{-1/2}$ law are concerned with averages, and they do not tell us the number of comets that actually reach a given 1/a-value regardless of the number of revolutions required. A somewhat different plan of experiment is described next.

Let us start many hypothetical comets at $1/a = 20 \times 10^{-6}$ AU^{-1}, as from Oort's cloud, with median $i = 6^{o}$, and with perihelia in the range of 30 to 34 AU. This is at the limit of Neptune's control. Here C_4 is about 2.9, and the evolution will follow the last column of Table I. Initially F and B are 160 and 20, and the perturbations of each revolution are simulated one at a time. Whenever a comet attains a negative 1/a-value it is discarded, being lost on a hyperbolic orbit. When 1/a reaches $0.15/a_p$, or 0.005 AU^{-1}, which is halfway to the next region in Table I, the F- and B-values are changed to 768 and 30, and the revolutions are simulated 5 at a time, and so on, proceeding in this random walk down this column in Table I.

Starting 12,230 such comets we find that 18 actually reached 1/2 = 0.0395 AU^{-1}. The median value was 197,000 orbit revolutions and the median time was 4.4×10^8 years, found by summing $a^{3/2}$ for all revolutions. Thus the efficiency was 18/12,230. All the rest were ejected to infinity on hyperbolic orbits.

The value of 0.0395 AU^{-1} was precalculated, for then the q-value under Neptune's control (calculated from C_4 and 1/a) has decreased to the range of 19.2 to 21.6 AU, and C_3 for Uranus is now 2.9. In other words, Uranus' barrier has been crossed and Uranus can take over. (In a more exact model there would

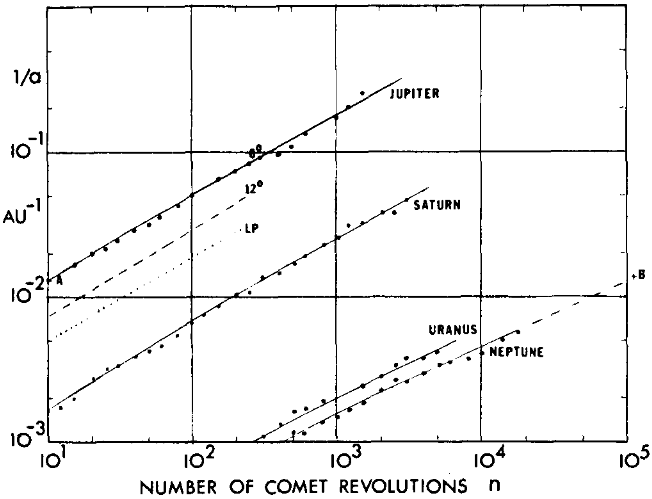

Figure 1. Average 1/a vs revolution number n for comets starting in parabolic orbits with 1/a = 0. The initial perihelia q_o of the comets is 1.0 to 1.125 times the orbital radius of the planet acting. There is a factor of 11,000 between the point A on the left end of the Jupiter line, and the point B at the same 1/a-value on the (extrapolated) right end of the Neptune line. All lines are for a median inclination of 6^o, except the dashed 12^o line for Jupiter. The dotted line marked LP is the action of Jupiter on 6^o comets with initial perihelia typical of the visible long period comets. Only about 2% of those starting survive for 1000 revolutions on any of these lines, and only 1% survive for 4000 revolutions. The rest are lost to hyperbolic orbits.

be a transition region where both Neptune and Uranus would act on the comet. Perturbations by Neptune leave C_4 constant, but they change C_3. In like manner perturbations by Uranus do not change C_3, but C_4 would then vary. This refinement is not included in this work, but an analogous study of C_2 and C_1 for Saturn and Jupiter appears in Sec. III of Everhart (1973) which shows the validity of this discussion. The random walk is no longer one-dimensional, since each object does not follow the same evolutionary track on the (q, 1/a) plane).

A second experiment started 69 comets at 1/2 = 0.0395 AU^{-1} under the control of Uranus, and these must reach 0.0686 AU^{-1} and perihelia less than 19.7 AU if Saturn is to take control with C_2 = 2.9. Out of the 69 so started, 28 were ejected to infinity by Uranus, one case was in limbo (undecided after 10^6 revolutions), and 40 reached the control of Saturn. The efficiency of this transfer was 40/69. A third experiment started 500 comets at 0.0686 AU^{-1} under the influence of Saturn, and 229 of these reached 0.113 AU^{-1} and perihelia less than 5.8 AU, where control passes to Jupiter. A fourth experiment started these same 229 comets at 0.113 AU^{-1} under Jupiter's influence, and of these, 92 reached perihelia where they would be easily visible as short-period comets.

The overall efficiency of the capture process for near-parabolic comets with perihelia between 30 and 34 AU and median i = 6^o is thus (18/12,230) (40/69) (229/500) (92/229) = 0.00016. Thus one in 6000 of these are captured to the orbit of a visible short-period comet, the rest being lost on hyperbolic orbits.

For other initial perihelion distances the 1/a-values at which control passes to Uranus are different. An experiment with the near-parabolic perihelia in the range of 27 to 30 AU, again at median i = 6^o, found one in 3300 of these captured to the orbit of a visible short-period comet. With initial perihelia in the range of 24 to 27 AU the ratio was one in 7000.

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Results and Comments:

1. In a time span well under that of the age of the solar system, comets with initial perihelia up to 34 AU and inclinations of 6° can be captured ultimately to the orbits of visible short-period comets through the cooperation of Neptune, Uranus, Saturn, and then Jupiter. The overall efficiency depends on the perihelia range, but is one in several thousand.

2. Starting the hypothetical comets as before but with perihelia in the range of 15 to 21 AU so that initial capture is by Uranus, we find that one in 1700 to 4600 of these ultimately appear as short-period comets. When the initial perihelia are in the range of 7.6 to 10.7 AU, initial capture by Saturn results in an efficiency of about one in 700. When the initial perihelia are in the range of 5.2 to 5.9 AU such capture by Jupiter alone has an efficiency of about one in 128. Using Monte Carlo exact numerical integrations, we have earlier found a ratio of one in 135 for roughly these same conditions (Everhart 1972).

3. It is interesting to compare the rapidity of initial capture by the 4 planets. In each case we start with near-parabolic comets ($i = 6^{\circ}$ median) whose perihelia are in the range of 1.0 to 1.125 the planet's orbital radius and require that they be brought to the control of the next planet inside (or to short periods in the case of Jupiter). The number of comet-revolutions required and the times are median values.

Neptune	197,000 rev	4.4×10^{8} yrs
Uranus	270,000 rev	3.5×10^{8} yrs
Saturn	4,500 rev	4.0×10^{6} yrs
Jupiter	1,100 rev	2.0×10^{5} yrs

4. The comets' inclinations were changed in a series of experiments involving near-parabolic comets whose initial perihelia lay in the range of 9.5 to 10.7 AU. Here Saturn began to capture, passing some on to Jupiter. (The F- and B-values vary with inclination as noted in the caption to Table I). When the median inclination was 3° then one in 470 of these ultimately showed as a short-period comet, and the evolution required a median of 3.5×10^{6} years, measured from first perihelion passage. For a median i of 6° the efficiency was one in 650 and the median time was 4.5×10^{6} years. For 12° it was one in 990 and 8.5×10^{6} years, and for 24° the efficiency was one in 1100, requiring a median time of 1.5×10^{7} years.

5. There is validity to the concept of "comet families" associated with each planet, provided that these are characterized by ranges in perihelia. Thus comets with perihelia between 22 and 34 AU belong to Neptune's family. The old practise of grouping comet families according to their periods or aphelia is not correct, because these quantities do not specify the planet causing the largest perturbations and controlling the evolution. All comets whose current perihelia are less than 5.8 AU belong to Jupiter's family regardless of their period.

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DISCUSSION

VAGHI: Do you think that resonance effects, which are necessarily neglected in random walk simulations of orbital evolution, could possibly play an important role in capture process?

EVERHART: The resonance effects, the librations, the Trojans, the horseshoe orbits, which add such variety to the numerical integration results, are lost in the random walk simulation, but we have been able to show agreement with the capture process result calculated by either method.

WHIPPLE: It would be valuable if you could calculate the energy loss to Neptune and Uranus in ejecting the average comet.

EVERHART: This depends on the masses of the comets, but the energy loss is very small because most of those comets which leave a hyperbolic orbit have very little excess energy. Such data can be calculated with programs now in hand.

SINGER: Your results indicate a preferred selection of low inclination comets. Assuming an isotropic source, what is the distribution in inclination of observed comets?

EVERHART: There will be a preferred selection of those comets which have low inclinations. However, I have not fully studied the random walk in inclination and its correlation with the random walk in energy and perihelia.