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THE HAUSDORFF COMPLETION OF THE SPACE OF CLOSED SUBSETS OF A MODULE

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ABSTRACT. In this paper, we show that the lattice of closed subsets of the completion, in the Jacobson radical topology, of a finitely generated module M is isomorphic to the completion, under the Hausdorff topology, of the lattice of closed subsets of M. This extends submodule-theoretic results for complete modules to modules satisfying Chevalley's Theorem. We show that the lattice of submodules of every finitely generated module over a semi-local ring R is complete in the Hausdorff topology if and only if R is complete in the Jacobson radical topology.

Throughout, all rings are commutative with identity 1, and all modules are unitary. M denotes a finitely generated module over a semi-local ring R, and $J = J(R)$ denotes the Jacobson radical of R.

Recall that the J-adic metric d is defined on M by $d(x,y) = 1/2^{\delta(x,y)}$, where $\delta(x,y) = \sup\{n \in \mathbf{Z} \mid x - y \in J^n M\}$. Recall also that the Hausdorff metric h is defined on the space of nonempty subsets of M which are closed under the J-adic metric by

$$h(A,B) = \max\Big\{\sup_{a\in A}\big\{\inf_{b\in B}\{d(a,b)\}\big\}, \sup_{b\in B}\big\{\inf_{a\in A}\{d(a,b)\}\big\}\Big\}.$$

For any R-module M, we denote by $\hat{M}$ the completion of M in the $J = J(R)$-adic topology. We denote the Jacobson radical of $\hat{R}$ by $\hat{J}$.

We denote by $K_R(M)$ the collection of nonempty subsets of M closed under the J-adic topology, and by $\widehat{K_R(M)}$ the completion of $K_R(M)$ in the Hausdorff topology. We denote by $K_r(M)^*$ the order completion $K_R(M) \cup \{\emptyset\}$ of $K_R(M)$. We denote by $\widehat{K_R(M)}^*$ the set $\widehat{K_R(M)} \cup \{\emptyset\}$, which is the order completion of $\widehat{K_R(M)}$ with respect to the order induced on it by $\subseteq$. Finally, we denote by $L_R(M)$ the lattice of R-submodules of M. All submodules of M are closed under the J-adic topology, so $L_R(M) \subseteq K_R(M) \subseteq \widehat{K_R(M)} \subset \widehat{K_R(M)}^*$.

We note that $K_R(M)^*$ is a complete, modular lattice with respect to the order $\subseteq$. We also note that $A + B$ is empty if either A or B is empty.

In this paper, we obtain a representation of $\widehat{K_R(M)}^*$ as $K_{\hat{R}}(\hat{M})^*$ (Theorem 2). This map can be used to extend submodule-theoretic results for complete modules to the larger class of modules satisfying $L_R(M) = \widehat{L_R(M)}$. We show that $L_R(M) = \widehat{L_R(M)}$ for all finitely generated R-modules M if and only if R is complete under the J-adic metric d (Theorem 3).

It is useful to collect the following well-known results into a lemma for ease of reference. They can all be found in [3, 4 or 5], for example.

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LEMMA 1. *Let M be a finitely generated module over R.*

(a) If $S \subseteq M$, then the closure of S in M is given by $\bar{S} = \bigcap_{i=1}^{\infty}(S + J^iM)$.

(b) $\hat{R}$ is a semi-local ring with Jacobson radical $\hat{J} = \hat{R}J$.

(c) $\hat{M}$ is a finitely generated module over $\hat{R}$ with $\hat{M} = \hat{R}M$.

(d) $J^i\hat{M} \cap M = J^iM$.

(e) $\hat{J}M = J\hat{M}$.

COROLLARY 1. *For any $S \subseteq M$ and for each $i \geq 0$, $S + J^iM$ is closed in M.*

Given Cauchy sequences $C = \{C_i\}_{i=1}^{\infty}$ and $D = \{D_i\}_{i=1}^{\infty}$ in $K_R(M)$, we write $C \sim_H D$ if $\lim_{i\to\infty} h(C_i, D_i) = 0$. We write $C \leq_H D$ if, given any positive integer n, $C_i \subseteq D_i + J^nM$ for large i.

The following clarifies the relationship between $\leq_H$ and $\sim_H$ and shows that $\leq_H$ induces an order on the completion of $K_R(M)$ in the Hausdorff topology.

LEMMA 2. *Let M be a finitely generated module over a semi-local ring M with Jacobson radical J. Then, for A and B in $K_R(M)$, $h(A,B) \leq 1/2^n$ if, and only if, $A + J^nM = B + J^nM$.*

PROOF. Assume $A + J^nM = B + J^nM$. Then for each $a \in A$ there exists $b_a \in B$ with $a - b_a \in J^nM$. It follows that, for $a \in A$, $\inf_{b\in B}\{d(a,b)\} \leq 1/2^n$, and hence that $\sup_{a\in A}\{\inf_{b\in B}\{d(a,b)\}\} \leq 1/2^n$. By symmetry, $h(A,B) \leq 1/2^n$.

Now, assume $A + J^nM \neq B + J^nM$, say $A \not\subseteq B + J^nM$. Choose $a \in A$ so $a \notin B + J^nM$. Then, for any $b \in B$, $d(a,b) \geq 1/2^{n-1}$. It follows that $\sup_{a\in A}\{\inf_{b\in B}\{d(a,b)\}\} \geq 1/2^{n-1}$.

The following simplifies calculations with Cauchy sequences in $K_R(M)$.

COROLLARY 2. *Any Cauchy sequence $C = \{C_i\}_{i=1}^{\infty}$ in $K_R(M)$ is equivalent to a decreasing Cauchy sequence C' in $K_R(M)$. If C lies in $L_R(M)$ then C' can be chosen from $L_R(M)$ as well.*

PROOF. Let $C = \{C_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $K_R(M)$. By extracting a subsequence if necessary, we can assume $h(C_i, C_j) \leq 1/2^n$ for all i and $j \geq n$. Then $C_i + J^nM = C_j + J^nM$ for all i and $j \geq n$. It follows that the sequence $\{C_i + J^iM\}_{i=1}^{\infty}$ is Cauchy, decreasing and equivalent to C. ∎

We require some information on extensions and contractions.

LEMMA 3. *If $S \subseteq M$, then $(S + J^i\hat{M}) \cap M = S + J^iM$.*

PROOF. If $S = \emptyset$, the result is clear, so assume $S \neq \emptyset$.

($\subseteq$). Assume $s + \hat{\jmath} = m \in M$; $s \in S$, $\hat{\jmath} \in J^i\hat{M}$. Then $\hat{\jmath} = m - s \in J^i\hat{M} \cap M = J^iM$ (Lemma 1), so $m \in S + J^iM$.

($\supseteq$). This is clear. ∎

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LEMMA 4. *If $T \subseteq \hat{M}$, then $\big((T+J^i\hat{M})\cap M\big)+J^i\hat{M} = T+J^i\hat{M}$.*

PROOF. If T is empty, both sides of the equation evaluate to $\emptyset$. Assume $T \neq \emptyset$.

($\subseteq$). This is clear.

($\supseteq$). Assume $t \in T$. Then either $t \in M$, in which case $t \in \big((T+J^i\hat{M})\cap M\big)+J^i\hat{M}$, or there exists $m \in M$ with $d(m,t) \leq 1/2^i$. In the latter case, $t-m \in J^i\hat{M}$ and $m \in (T+J^i\hat{M})\cap M$, so $t = m+(t-m) \in \big((T+J^i\hat{M})\cap M\big)+J^i\hat{M}$. ∎

LEMMA 5. *Assume $\{C_i\}_{i=1}^\infty \subseteq K_R(M)$. Then $\{C_i\}_{i=1}^\infty$ is Cauchy in $K_R(M)$ if and only if $\{C_i+J^i\hat{M}\}_{i=1}^\infty$ is Cauchy in $K_{\hat{R}}(\hat{M})$.*

PROOF. Assume $\{C_i\}_{i=1}^\infty$ is a Cauchy sequence in $K_R(M)$. Then, given n, $C_i+J^nM = C_j+J^nM$ for large i and j, so $C_i+J^n\hat{M} = C_j+J^n\hat{M}$ for large i and j. Hence (Lemma 2), $\{C_i+J^i\hat{M}\}_{i=1}^\infty$ is Cauchy in $K_{\hat{R}}(\hat{M})$.

Conversely, assume that $\{C_i\}_{i=1}^\infty$ is a sequence in $K_R(M)$ and $\{C_i+J^i\hat{M}\}_{i=1}^\infty$ is Cauchy in $K_{\hat{R}}(\hat{M})$. Then, (Lemma 2) given n, $C_i+J^n\hat{M} = C_j+J^n\hat{M}$ for large i and j. It follows that $(C_i+J^n\hat{M})\cap M = (C_j+J^n\hat{M})\cap M$, and hence (Lemma 3) that $C_i+J^nM = C_j+J^nM$, for large i and j. By Lemma 2, $\{C_i\}_{i=1}^\infty$ is Cauchy in $K_R(M)$. ∎

Elements of $\widehat{K_R(M)}$ can be thought of as points or as equivalence classes of Cauchy sequences of elements of $K_R(M)$. For any Cauchy sequence $C = \{C_i\}_{i=1}^\infty$ in $K_R(M)$, we denote both by $[C]_H$. We note that $\widehat{K_R(M)}$ is naturally ordered with $[C]_H = [\{C_i\}_{i=1}^\infty]_H \leq [\{D_i\}_{i=1}^\infty]_H = [D]_H$ if, given n, $C_i \subseteq D_i+J^nM$ for large i. We define $\emptyset \leq D$ for all $D \in \widehat{K_R(M)}$.

THEOREM 1. *Let $\{C_i\}_{i=1}^\infty$ be a decreasing sequence in $K_R(M)$ with $\bigcap_{i=1}^\infty C_i = C_0$. The the following are equivalent.*

1. *$\lim_{i\to\infty} C_i = C_0$.*
2. *$\{C_i\}_{i=1}^\infty$ converges in $K_R(M)$.*
3. *$C_i \subseteq C_0+J^nM$ for fixed n and large i.*

PROOF. Clearly (1) implies (2). Assume (2) with $\lim_{i\to\infty} C_i = L$. Then for fixed n and large i, $h(C_i,L) \leq 1/2^n$, and so $L+J^nM = C_i+J^nM$, for large i. Fix k. Then $L \subseteq C_k+J^nM$ for all n, so $L \subseteq C_k$. By the choice of k, it follows that $L \subseteq \bigcap_{k=1}^\infty C_k = C_0$. Then $C_i \subseteq C_i+J^nM = L+J^nM \subseteq C_0+J^nM$, for large i. Hence, (2) implies (3). Now, assume $C_i \subseteq C_0+J^nM$ for fixed n and large i. Then $C_i+J^nM = C_0+J^nM$ for large i, so (Lemma 2) $h(C_i,C_0) \leq 1/2^n$. Hence also (3) implies (1). ∎

A famous theorem of Chevalley, when stated for modules, says that in a complete module over a semi-local ring, any decreasing sequence $\{C_i\}_{i=1}^\infty$ of submodules satisfies $C_i \subseteq (\bigcap_{j=1}^\infty C_j)+J^nM$ for fixed n and large i [5, Theorem 13, p. 270]. We say that a subspace S of $K_R(M)$ satisfies Chevalley's Theorem if every decreasing sequence $\{C_i\}_{i=1}^\infty$ in S with nonempty intersection satisfies $C_i \subseteq (\bigcap_{j=1}^\infty C_j)+J^nM$ for fixed n and large i. The following relates Chevalley's Theorem to the completeness of $K_R(M)$ and $L_R(M)$ under the Hausdorff metric h.

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COROLLARY 3. *Let M be a finitely generated R-module.*

1. $\widehat{K_R(M)} = K_R(M)$ *if and only if* $K_R(M)$ *satisfies Chevalley's Theorem.*
2. $\widehat{L_R(M)} = L_R(M)$ *if and only if* $L_R(M)$ *satisfies Chevalley's Theorem.*

The following gives a precise description of $\widehat{K_R(M)}$ and $\widehat{L_R(M)}$ in general.

THEOREM 2. *Let M be a finitely generated module over the semi-local ring R. Then the map* $\psi: K_{\hat{R}}(\hat{M})^* \longrightarrow \widehat{K_R(M)}^*$ *defined by* $\psi(S) = [\{(S+J^i\hat{M})\cap M\}_{i=1}^{\infty}]_H$ *for* $S \neq \emptyset$, *and* $\psi(\emptyset) = \emptyset$ *is a lattice isomorphism of* $K_{\hat{R}}(\hat{M})^*$ *onto* $\widehat{K(M)}^*$. *The isomorphism carries* $L_{\hat{R}}(\hat{M})$ *onto* $\widehat{L_R(M)}$.

PROOF. Define ψ is as in the statement of the theorem. Let S be any element of $K_{\hat{R}}(\hat{M})^*$. By Lemma 5, the sequence $\{(S+J^i\hat{M})\cap M\}_{i=1}^{\infty}$ is Cauchy in $K_R(M)$. Hence, ψ is a map from $K_{\hat{R}}(\hat{M})^*$ to $\widehat{K_R(M)}^*$. It is clear that ψ is order preserving. Assume S and T are elements of $K_{\hat{R}}(\hat{M})$ and $\psi(S) \leq \psi(T)$. Then, given n, $(S+J^i\hat{M})\cap M+J^nM \subseteq (T+J^j\hat{M})\cap M+J^nM$ for large i and j. Then $(S+J^i\hat{M})\cap M+J^i\hat{M}+J^n\hat{M} \subseteq (T+J^j\hat{M})\cap M+J^j\hat{M}+J^n\hat{M}$ for large i and j.

By Lemma 4, this gives $S+J^n\hat{M} \subseteq T+J^n\hat{M}$. As n is arbitrary and S and T are closed, it follows that $S \subseteq T$. Hence, $\psi(S) \leq \psi(T)$ if and only if $S \subseteq T$.

Now, let $\{C_i\}_{i=1}^{\infty}$ be any decreasing Cauchy sequence in $K_R(M)$. Then $\{C_i+J^i\hat{M}\}_{i=1}^{\infty}$ is a decreasing Cauchy sequence in $K_{\hat{R}}(\hat{M})$. Let $\hat{C}_0 = \bigcap_{i=1}^{\infty}(C_i + J^i\hat{M})$. We show $\lim_{i\to\infty}(C_i+J^i\hat{M}) = \hat{C}_0$.

By extracting a subsequence of $\{C_i\}_{i=1}^{\infty}$ if necessary, we can assume that $h(C_i, C_j) \leq 1/2^n$ for all $n \geq 1$ and all $i, j \geq n$. Hence, $C_n + J^nM = C_j + J^nM$ for $j \geq n$. Fix n and $c_n \in C_n$. Then $c_n = c_{n+1} + m_n$, for some $c_{n+1} \in C_{n+1}$ and $m_n \in J^nM$. Continue to get $\{c_j\}_{j=n}^{\infty}$ and $\{m_j\}_{j=n}^{\infty}$ with $c_j = c_{j+1} + m_j$ for all $j \geq i$. Set $\sigma_j = \sum_{r=n}^{j} m_r$. Then $c_n = c_{r+1} + \sigma_r$ for all $r \geq n$. Also, as $m_r \in J^rM$, necessarily $\lim_{r\to\infty} m_r = 0$, and hence $\{\sigma_j\}_{j=i}^{\infty}$ converges in $\hat{M}$, say to σ_0. It follows that $\{c_j\}_{j=1}^{\infty}$ also converges, say to c_0. As $\sigma_n \in J^nM$ necessarily $\sigma_0 \in J^nM$ as well. Likewise, $c_j \in C_j$ for all $j \geq n$, so $c_0 \in C_0$. As $c_n = c_0 + \sigma_0$, it follows by the choice of c_n that $C_n \subseteq C_0 + J^nM$. Hence $\lim_{i\to\infty} C_i = C_0$. As $\{C_i\}_{i=1}^{\infty} \sim_H \{C_i + J^iM\}_{i=1}^{\infty}$, it follows that $\psi(\hat{C}_0) = [\{C_i\}_{i=1}^{\infty}]_H$.

It is clear that ψ carries $L_{\hat{R}}(\hat{M})$ onto $\widehat{L_R(M)}$. ■

$L_R(M)$ is an "$L_R(R)$-module." The natural extension of the scalar multiplication makes $\widehat{L_R(M)}$ an $\widehat{L_R(R)}$-module. If ρ is the isomorphism of $L_{\hat{R}}(\hat{R})$ onto $\widehat{L_R(R)}$ corresponding to the map ψ in the proof of Theorem 2, then $\psi(AN) = \rho(A)\psi(N)$, for $A \in \widehat{L_R(R)}$ and $N \in \widehat{L_R(M)}$.

It is possible for $L_R(M)$ to be complete without M complete. For example, a one-dimensional regular local ring need not be complete. On the other hand, if R is complete, then every finitely generated R-module M is complete. It is natural to ask if $L_R(R)$ complete implies the lattice $L_R(M)$ of every finitely generated R-module is complete. The following gives a definitive answer.

THEOREM 3. *Let R be a semi-local ring. Then the following are equivalent.*

1. *For every finitely generated R-module M, $K_R(M)$ is complete in the Hausdorff topology.*
2. *For every finitely generated R-module M, $L_R(M)$ is complete in the Hausdorff topology.*
3. *$L_R(R \oplus R)$ is complete in the Hausdorff topology.*
4. *R is complete in the J-adic topology.*

PROOF. (1) implies (2): This is clear. (2) implies (3): Clear. (3) implies (4): Let $m_1, \ldots m_n$ be the maximal ideals of R, and let M be a finitely generated R-module with $L_R(M)$ complete under the metric h. Then $\hat{M}$ is a finitely generated module over $\hat{R} = \widehat{R_{m_1}} \oplus \cdots \oplus \widehat{R_{m_n}}$. This induces a decomposition of $\hat{M}$ which induces a decomposition of M by contraction. Hence, we can assume that R is local with maximal ideal m. Fix $C \in L_{\hat{R}}(\hat{M})$. Under the map ψ of Theorem 2, let $C_0 = \psi(C) = \lim_{i\to\infty}(C + J^i\hat{M}) \cap M = \bigcap_{i=1}^{\infty}(C + J^i\hat{M}) \cap M$.

Also, $\bigcap_{i=1}^{\infty}(\hat{R}C_0 + J^i\hat{M}) \cap M = \bigcap_{i=1}^{\infty} \hat{R}(C_0 + J^iM) \cap M = \bigcap_{i=1}^{\infty}(C_0 + J^iM) = C_0 = \psi(\hat{R}C_0)$. It follows that $C = \hat{R}C_0$. As C is arbitrary, it follows that every submodule C of $\hat{M}$ is extended. When applied to the cyclic submodules, it follows that every element $\hat{c} \in \hat{M}$ is of the form $\hat{u}c$ for some $c \in M$ and unit $\hat{u} \in \hat{R}$. When applied to $M = R \oplus R$, it follows that every element of $\hat{R} \oplus \hat{R}$ is a unit multiple of an element of $R \oplus R$. In particular, $(1, \hat{c})$ is a unit multiple of an element $(r, s) \in R \oplus R$, and likewise (r, s) is a unit multiple of $(1, \hat{c})$. In the latter case, the unit is clearly r. But then $\hat{c} = rs \in R$, so $\hat{R} \subseteq R$. (4) implies (1): Assume R is complete. Let M be a finitely generated R-module. By Theorem 3, $L_{\hat{R}}(\hat{M}) = L_R(M)$ is complete in the Hausdorff topology. ∎

We note that modules with submodule lattices satisfying Chevalley's Theorem have been called quasi-complete. See, for example, [1, 2].

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