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**RESEARCH ARTICLE**

# Fano-type surfaces with large cyclic automorphisms

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**Abstract**
We give a characterisation of Fano-type surfaces with large cyclic automorphisms. As an application, we give a characterisation of Kawamata log terminal 3-fold singularities with large class groups of rank at least 2.

## 1. Introduction

Fano-type varieties are one of the three building blocks of algebraic varieties. Research into Fano-type surfaces was initiated with the study of smooth Fano surfaces by del Pezzo in the late 19th century. His work culminated with the famous classification of smooth Fano surfaces, nowadays known as *del Pezzo surfaces*. Such surfaces belong to nine families distinguished by their degree. The Borisov–Alexeev–Borisov conjecture predicted that this boundedness behaviour holds for mildly singular Fano-type varieties. This conjecture was proven in dimension 2 by Alexeev [Ale94] . In [Bir04] , Birkar proved the theory of complements for surfaces and used this theory to give a second proof of the boundedness of weak log Fano surfaces. In [Bir19] , Birkar proved the boundedness of log canonical complements for Fano-type varieties of arbitrary dimension. This machinery was used to give a positive answer to the Borisov–Alexeev–Borisov conjecture [Bir21] . A better understanding of the geometry and complements of Fano-type surfaces often guides important developments in higher-dimensional birational geometry. In general, a novel understanding of Fano-type surfaces is usually the cornerstone of theorems about Fano-type varieties of arbitrary dimension.

In this article, we study Fano-type surfaces with large cyclic automorphisms. These appear naturally in the study of Kawamata log terminal (klt) 3-fold singularities with large Cartier index. The main aim of this paper is to understand how the existence of large cyclic automorphisms on Fano-type surfaces reflects in their geometry. Moreover, we study which invariants of the Fano-type surface $X$ will control this concept of *largeness*. For instance, it is easy to see that del Pezzo surfaces with large finite abelian automorphism groups of rank 2 must be toric – that is, they are either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $\mathbb{P}^2$ at up to three points. More generally, we expect the existence of a constant $N(\epsilon)$, depending only on $\epsilon$, satisfying the following: given an $\epsilon$-log canonical Fano surface $X$ with $\mathbb{Z}_m \leq \mathrm{Aut}(X)$ and $m \geq N(\epsilon)$, $X$ can be endowed with a $\mathbb{G}_m$-action. In the two previous examples, we see how the existence of large cyclic automorphisms of $X$ reflects in the geometry of the variety as the existence of a torus action. In these examples, the invariant of the Fano-type surface $X$ that controls the concept of *large* for the automorphism group is the minimal log discrepancy. At the same time, it is known that if a Fano-type surface has a large finite abelian automorphism group of rank $k$, then $k$ is at most 2. This is a consequence

of the Jordan property for the birational automorphism group of Fano-type varieties (see, e.g., [PS14, PS16, PS17] ). Hence, our examples show maximal rank behaviour.

The main result of this article is a characterisation of Fano-type surfaces with large cyclic automorphisms. Surprisingly, the concept of *largeness* is effective and depends only on the dimension. Thus, it is not necessary to bound any invariant of the Fano-type surfaces as in the examples already given. Our first theorem gives a characterisation of Fano-type surfaces with large cyclic automorphism groups:

**Theorem 1.** *There exists a positive integer N satisfying the following: Let X be a Fano-type surface so that $G := \mathbb{Z}_m \leq \mathrm{Aut}(X)$ with $m \geq N$. Then there exist*

1. *a subgroup $A \leq G$ of index at most N,*
2. *a boundary B on X and*
3. *an A-equivariant birational map $X \dashrightarrow X'$*

*satisfying the following conditions:*

1. *The pair $(X, B)$ is log canonical, the divisor B is G-invariant and $N(K_X + B) \sim 0$.*
2. *The log crepant transform $(X', B')$ of $(X, B)$ on $X'$ is a log pair.*
3. *The pair $(X', B')$ admits a $\mathbb{G}_m$-action.*
4. *There are group monomorphisms $A < \mathbb{G}_m \leq \mathrm{Aut}(X', B')$.*

In [Pro15] and [Nak17] some other criteria for the existence of $\mathbb{G}_m$-actions on surfaces are obtained. For the convenience of the reader, we introduce some definitions to restate Theorem 1 less technically:

**Definition 1.** We say that a log canonical projective pair $(X, B)$ admits a *crepant equivalent torus action* if it admits a crepant equivalent model $(X', B')$ endowed with a $\mathbb{G}_m$-action. Note that we require that $(X', B')$ be a log pair. In particular, the birational map $X' \dashrightarrow X$ (resp., $X \dashrightarrow X'$) only extracts divisors with log discrepancies in the interval $[0, 1]$ with respect to $(X, B)$ (resp., $(X', B')$). We say that a projective variety $X$ admits a *log crepant equivalent torus action* if $(X, B)$ admits a crepant equivalent torus action for some boundary $B$ on $X$.

The following is a more natural way to state Theorem 1:

**Theorem A.** *A Fano-type surface X with a large cyclic automorphism admits a log crepant equivalent torus action.*

Here, *large* means that $|G|$ is larger than a universal constant, as in the statement of Theorem 1.

Now we introduce a concept to measure the cardinality of generators of finite subgroups of bounded index of a given finite group. Let $G$ be a finite group. We define the *rank up to index N*, denoted by $r_N(G)$, to be the minimum rank among subgroups of index at most $N$. For $k \leq \mathrm{rank}(G)$, we define the *k-generation* order to be the maximum $N$ such that $|G| \geq N$ and $r_N(G) \geq k$. The *k-generation* of a finite group $G$ is denoted by $g_k(G)$. Our second theorem gives a characterisation of Fano-type surfaces with a finite automorphism group with large 2-generation. In particular, we obtain a characterisation of Fano-type surfaces with large abelian automorphism groups of rank 2.

**Theorem 2.** *There exists a positive integer N satisfying the following: Let X be a Fano-type surface so that $G \leq \mathrm{Aut}(X)$ is a finite group with $g_2(G) \geq N$. Then there exist*

1. *an abelian normal subgroup $A \leq G$ of index at most N*
2. *a boundary B on X and*
3. *an A-equivariant birational map $X \dashrightarrow X'$*

*satisfying the following conditions:*

1. *The pair $(X, B)$ is log canonical, G-invariant and $K_X + B \sim 0$.*
2. *The log crepant transform $(X', B')$ of $(X, B)$ on $X'$ is a log pair.*
3. *The pair $(X', B')$ is a log Calabi–Yau toric surface.*
4. *There are group monomorphisms $A < \mathbb{G}_m^2 \leq \mathrm{Aut}(X', B')$.*

*In particular, $B'$ is the reduced toric boundary.*

Analogously to Theorem 1, we introduce some definitions to state Theorem 2 less technically:

**Definition 2.** We say that a log canonical projective pair $(X, B)$ is *crepant equivalent toric* if $(X, B)$ is crepant equivalent to a projective toric pair. We say that a projective variety $X$ is *log crepant equivalent toric* if $(X, B)$ is crepant equivalent toric for some boundary $B$ on $X$.

The following is a more natural way to state Theorem 2:

**Theorem B.** *A Fano-type surface with a large finite abelian group of automorphisms of rank 2 is crepant equivalent log toric.*

Here, *large* means that $g_2(G)$ is larger than a universal constant, as in the statement of Theorem 2. The main ingredients for the proofs of these theorems are the Jordan property for finite birational automorphism groups of Fano-type varieties, the theory of $G$-invariant complements, characterisation of toric varieties using complexity, the $G$-invariant minimal model program for surfaces and a characterisation of formally toric surface morphisms due to Shokurov [Sho00, Theorem 6.4]. Our theorems can be stated in a more general setting for log pairs $(X, \Delta)$ on a Fano-type surface $X$ so that $(X, \Delta)$ is $\mathbb{Q}$-complemented (see Theorems 3.1 and 3.2). Note that Theorems 1 and 2 are already nontrivial for smooth Fano-type surfaces as those may for unbounded families (see Remark 2.1). Now we give two applications of them.

Our first application is related to the fundamental group of the smooth locus of a Fano-type surface. It is known that such a group is finite [FKL93, GZ94, GZ95, KM99] . Moreover, it can encode much of the geometry of the surface [Xu09] . We prove that if such a group is large enough, then $X$ is log crepant equivalent to a finite quotient of a toric variety. We make this concept precise in the following definition:

**Definition 3.** We say that a log canonical projective pair $(X, B)$ is a *crepant equivalent toric quotient* if $(X, B)$ is crepant equivalent to a projective pair which is the quotient of a log Calabi–Yau toric pair by a finite automorphism group of the pair. We say that $X$ is a *log crepant equivalent toric quotient* if $(X, B)$ is a crepant equivalent toric quotient for some boundary $B$ on $X$.

**Theorem 3.** *There exists a positive integer $N$ satisfying the following: Let $X$ be a klt surface with $-K_X$ ample. If $G \leq \pi_1(X^{\mathrm{sm}})$ satisfies $g_2(G) \geq N$, then $X$ is a log crepant equivalent toric quotient.*

In Section 4 we give a more general version of Theorem 3 which also deals with the case of log pairs. We will also consider the case of fundamental groups of rank 1. Furthermore, we prove that there is a bounded cover of $X$ which makes it a log crepant equivalent toric variety.

**Definition 4.** We say that a singularity $x \in (X, B)$ is a *toric quotient singularity* if it is the quotient of a singularity on a toric pair by a finite automorphism group. Note that toric quotient singularities are a generalisation of quotient singularities. They may not be $\mathbb{Q}$-factorial, as toric singularities themselves may not be $\mathbb{Q}$-factorial. Furthermore, toric quotient singularities are always log canonical. We say that a singularity $x \in X$ is a *log toric quotient singularity* if $x \in (X, B)$ is a toric quotient singularity for some boundary $B$ on $X$. We say that a log canonical singularity $x \in X$ is a *log crepant equivalent toric quotient singularity* if it is the cone over a log crepant equivalent toric quotient projective variety. We have natural inclusions of these classes of singularities:

$$\{\text{Quotient singularities}\} \cup \{\text{Toric singularities}\} \subset \{\text{Toric quotient singularities}\}$$
$$\subset \{\text{Log crepant equivalent toric quotient singularities}\}.$$

All these inclusions are strict. We may use the abbreviation *tq* (resp., *lce-tq*) to denote a toric quotient (resp., log crepant equivalent toric quotient).

In terms of the minimal model program, all these classes of singularities behave similarly. The following theorem shows that lce-tq singularities and their deformations characterise klt 3-fold singularities with a large class group:

**Theorem 4.** *There exists a positive integer N satisfying the following: Let $x \in X$ be a $\mathbb{Q}$-factorial klt 3-fold singularity. If $G \leq \mathrm{Cl}(X; x)$ satisfies $g_3(G) \geq N$, then $x \in X$ degenerates to a lce-tq singularity.*

In Section 5 we give more general versions of Theorem 4, in which we also consider the case of rank 2. The case of rank 1 is outside the scope of this article.

It is expected that all the theorems proved in this article can be generalised to higher dimensions, which will be dealt with in future work. This article aims to be a foundation for the study of Fano-type varieties with large abelian automorphism groups and the corresponding applications to the study of klt singularities.

## 2. Preliminaries

In this section, we recall the singularities of the minimal model program, introduce the theory of $G$-invariant complements and prove some preliminary results. Throughout this article, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0. $\mathbb{G}_m^k$ stands for the $k$-dimensional $\mathbb{K}$-torus, and the rank of a finite group is the minimum number of generators. We will use some classic results of toric geometry over $\mathbb{K}$. See [Ful93, CLS11, Cox95] for some references on toric geometry.

### 2.1. Singularities of the minimal model program

In this subsection, we recall classic definitions on singularities of the minimal model program [KM98, Kol13, HK10] :

**Definition 1.** A *contraction* is a morphism of quasi-projective varieties $\phi \colon X \to Y$ such that $\phi_* \mathcal{O}_X = \mathcal{O}_Y$. Note that $Y$ is normal, provided that $X$ is normal. A *fibration* is a contraction with positive-dimensional general fibre.

**Definition 2.** Let $\mathcal{Q}$ be a finite subset of $[0, 1]$. We define the set of *hyperstandard coefficients* associated to $\mathcal{Q}$, denoted by $\mathcal{H}(\mathcal{Q})$, as

$$\left\{ 1 - \frac{q}{m} \;\middle|\; q \in \mathcal{Q}, m \in \mathbb{N} \right\}.$$

If $\mathcal{Q} = \{0, 1\}$, we say that $\mathcal{H}(\mathcal{Q})$ is the set of *standard coefficients*. We say that a log pair $(X, B)$ has standard coefficients if the coefficients of $B$ belong to the set of standard coefficients.

**Definition 3.** A *log pair* $(X, \Delta)$ consists of a normal quasi-projective variety $X$ and an effective $\mathbb{Q}$-divisor $\Delta$ such that the $\mathbb{Q}$-divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Given a projective birational map $\pi \colon Y \to X$ from a normal quasi-projective variety $Y$ and a prime divisor $E$ on $Y$, we define the *log discrepancy* of the log pair $(X, \Delta)$ at the prime divisor $E$ to be

$$a_E(X, \Delta) := 1 - \mathrm{coeff}_E(K_Y - \pi^*(K_X + \Delta)).$$

We say that a log pair $(X, \Delta)$ is *Kawamata log terminal* (resp., *log canonical*) if every log discrepancy is positive (resp., nonnegative), and we may use the usual abbreviation klt (resp., lc).

**Definition 4.** Let $(X, \Delta)$ be a pair with log canonical singularities. A prime divisor $E$ over $X$ is said to be a *log canonical place* if the corresponding log discrepancy equals 0. The image of a log canonical place in $X$ is said to be a *log canonical centre*.

**Definition 5.** A log pair $(X, \Delta)$ is said to be *divisorially log terminal* (dlt) if there is an open set $U \subset X$ which satisfies all of the following properties:

1. The coefficients of $\Delta$ are at most 1.
2. $U$ is smooth and $\Delta|_U$ has simple normal crossing.

3. Any log canonical centre of $(X, \Delta)$ intersects $U$ nontrivially and is given by intersection of the strata of $\lfloor \Delta \rfloor$.

Given a log canonical pair $(X, \Delta)$, we say that $\pi \colon Y \to X$ is a $\mathbb{Q}$-*factorial dlt modification* if $Y$ is $\mathbb{Q}$-factorial, $\pi$ extracts only log canonical places and $\pi^*(K_X + \Delta) = K_Y + \Delta_Y$ is a dlt pair. The existence of $\mathbb{Q}$-factorial dlt modifications for log canonical pairs is well known [KK10] . We say that a dlt pair $(X, \Delta)$ is *purely log terminal* (plt) if $(X, \Delta)$ is dlt and has at most one log canonical centre.

**Definition 6.** Let $(X, \Delta)$ be a log canonical pair and $D$ be a $\mathbb{Q}$-Cartier effective divisor on $X$. We define the *log canonical threshold* of $D$ on $(X, \Delta)$, denoted by $\mathrm{lct}((X, \Delta); D)$, to be the maximum positive rational number $t$ such that $K_X + \Delta + tD$ is log canonical.

**Definition 7.** Let $(X, \Delta)$ be an lc pair and $x \in X$ a closed point. A *purely log terminal blowup* of $(X, \Delta)$ at $x \in X$ is a birational map $\pi \colon Y \to X$ such that

1. $\pi$ has a unique exceptional divisor $E$,
2. the centre of $E$ on $X$ is $x$,
3. $-E$ is ample over $X$ and
4. $(Y, \Delta_Y + E)$ is purely log terminal near $E$.

Here $\Delta_Y$ is the strict transform of $\Delta$ on $Y$.

**Definition 8.** Let $M$ be a positive integer and $(X, \Delta)$ be a log canonical pair. We say that an effective divisor $B \geq \Delta$ is an *M-complement* of the pair $(X, \Delta)$ if the following conditions hold:

1. $(X, B)$ is a log canonical pair and
2. $M(K_X + B) \sim 0$.

Note that if $(X, \Delta)$ admits an $M$-complement, then $-(K_X + \Delta)$ is a pseudoeffective divisor. In this case, we say that $(X, \Delta)$ is *M-complemented*. We may also say that $-(K_X + \Delta)$ is *M-complemented*.

   We say that an effective divisor $B \geq \Delta$ is a $\mathbb{Q}$-*complement* of the pair $(X, \Delta)$ if the following conditions hold:

1. $(X, B)$ is a log canonical pair
2. $K_X + B \sim_{\mathbb{Q}} 0$.

Note that any $M$-complement is a $\mathbb{Q}$-complement. In this case, we say that $(X, \Delta)$ is $\mathbb{Q}$-complemented. We may also say that $-(K_X + \Delta)$ is $\mathbb{Q}$-complemented.

**Definition 9.** Let $X$ be a quasi-projective variety and $X \to Z$ be a projective contraction. We say that $X$ is of *Fano type* over $Z$ if there exists a boundary $\Delta$ big over $Z$ such that $(X, \Delta)$ is klt and $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$. If $Z = \mathrm{Spec}(\mathbb{K})$, then we just say that $X$ is a Fano-type variety. Recall that relatively Fano-type varieties are relative Mori dream spaces over the base [BCHM10] . In particular, every minimal model program for a divisor $D$ on $X$ over $Z$ terminates with a good minimal model or a Mori fibre space over $Z$.

   Let $X$ be a quasi-projective variety and $X \to Z$ be a projective contraction. We say that $X$ is of *log Calabi–Yau type* over $Z$ if there exists a boundary $\Delta$ on $X$ such that $(X, \Delta)$ is log canonical and $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$. If $Z = \mathrm{Spec}(\mathbb{K})$, then we just say that $(X, \Delta)$ is a log Calabi–Yau pair.

**Definition 10.** Let $X$ be a projective variety, $D$ be a prime divisor on $X$ and $\phi \colon X \to Z$ be a fibration. We say that $D$ is $\phi$-*horizontal* if $D$ dominates $Z$. Conversely, we say that $D$ is $\phi$-*vertical* if $D$ does not dominate $Z$. We may say *vertical over Z* (resp., *horizontal over Z*) instead of '$\phi$-vertical' (resp., '$\phi$-horizontal') when the morphism is not labeled. When $\phi$ and $Z$ are clear from the context, we may just say that $D$ is vertical or horizontal.

**Definition 11.** Let $(X, \Delta)$ be a log pair. We say that $(X, \Delta)$ *admits a* $\mathbb{G}_m$-*action* if $X$ admits a $\mathbb{G}_m$-action such that $\Delta$ is invariant.

**Definition 12.** Let $(X, \Delta)$ be a log canonical pair. We say that $(Y, \Delta_Y)$ is a *crepant equivalent model* (or *crepant equivalent transformation*) of $(X, \Delta)$ if there is a birational morphism $\pi \colon Y \to X$ which extracts only divisors of with log discrepancy at most 1 with respect to the log pair $(X, \Delta)$, and $\pi^*(K_X + \Delta) = K_Y + \Delta_Y$. This is equivalent to requiring that $(Y, \Delta_Y)$ be a log pair.

**Definition 13.** Let $\pi \colon X \dashrightarrow X'$ be a birational map and $(X, B)$ be a log pair on $X$. The *log crepant transform* of $(X, B)$ on $X'$ is the unique pair $(X', B')$ such that $K_{X'} + B' = p_* q^*(K_X + B)$, where $p \colon Y \to X'$ and $q \colon Y \to X$ give a resolution of the morphism.

Let $X \dashrightarrow X'$ be a birational map. We say that $(X, B)$ and $(X', B')$ are *log crepant equivalent* if the log crepant transform of $(X, B)$ on $X'$ equals $(X', B')$.

**Definition 14.** Let $X$ be a scheme with nodal singularities at codimension 1 points. Denote by $D$ the conductor of $X$ and let $B$ be an effective divisor on $X$ whose support does not contain a prime component of the conductor. Let $\pi \colon X^\eta \to X$ be the normalisation morphism of $X$ and let $B^\eta$ be the divisorial part of $\pi^{-1}(B)$. We say that $(X, B)$ is *semi-log canonical* (slc) if $K_X + B$ is $\mathbb{Q}$-Cartier and $K_{X^\eta} + B^\eta + D^\eta$ is log canonical.

### 2.2. Theory of G-invariant complements

In this subsection, we introduce $G$-invariant complements and prove the existence of bounded $G$-invariant complements. In what follows, we may use the $G$-equivariant version of some statements of the minimal model program. See [CS11a, CS11b, CS12] for some results related to $G$-equivariant singularities of the minimal model program.

**Remark 2.1.** All toric projective varieties are Fano-type varieties. Therefore, smooth projective varieties of any fixed dimension at least 2 form unbounded families.

**Definition 15.** Let $X$ be a projective algebraic variety and $G \leq \mathrm{Aut}(X)$ be a finite automorphism group. We say that a log pair $(X, \Delta)$ is *G-invariant* if $g^*(\Delta) = \Delta$ for every element $g \in G$. Note that this is equivalent to $\mathrm{coeff}_P(\Delta) = \mathrm{coeff}_{g^*P}(\Delta)$ for every prime divisor $P$ on $X$ and for every element $g \in G$. Hence, we can define the coefficients of $\Delta$ over the prime $G$-orbits of $X$.

**Definition 16.** Let $M$ be a positive integer, $X$ be a projective algebraic variety and $G \leq \mathrm{Aut}(X)$ be a finite automorphism group. Let $(X, \Delta)$ be a $G$-invariant log canonical pair. We say that an effective divisor $B \geq \Delta$ is a *G-invariant M-complement* of the pair $(X, \Delta)$ if the following conditions hold:

1. $(X, B)$ is $G$-invariant.
2. $(X, B)$ is a log canonical pair.
3. $M(K_X + B) \sim 0$.

Note that if $(X, \Delta)$ admits a $G$-invariant $M$-complement, then $-(K_X + \Delta)$ is a pseudoeffective divisor.

We say that an effective divisor $B \geq \Delta$ is a *G-invariant $\mathbb{Q}$-complement* ($G\mathbb{Q}$-complement) of the pair $(X, \Delta)$ if the following conditions hold:

1. $(X, B)$ is $G$-invariant.
2. $(X, B)$ is a log canonical pair.
3. $K_X + B \sim_{\mathbb{Q}} 0$.

Note that any $G$-invariant $M$-complement is a $G$-invariant $\mathbb{Q}$-complement.

**Definition 17.** Let $(X, \Delta)$ be a $G$-equivariant klt pair. A *G-equivariant Mori fibre space* $X \to Z$ is a $G$-equivariant contraction such that $\rho^G(X/Z) = 1$ and $-K_X$ is ample over $Z$.

**Proposition 2.2.** *Let $X$ be an algebraic variety and $G \leq \mathrm{Aut}(X)$ be a finite automorphism group. Let $(X, \Delta)$ be a G-invariant log canonical pair and $\pi \colon X \to Y$ be the quotient by $G$. Then there exists a log canonical pair $(Y, \Delta_Y)$ such that $\pi^*(K_Y + \Delta_Y) = K_X + \Delta$. Furthermore, the coefficients of $\Delta_Y$ have the form $\frac{1-m+\lambda}{m}$, where $m$ is a positive integer and $\lambda$ is a coefficient of $\Delta$.*

*Proof.* First we prove the statement for $K_X$. By the Hurwitz formula, we can write $K_X = \pi^*(K_Y) + R$, where $R$ is the ramification divisor. Let $P$ be a prime divisor of $X$ contained in the support of $R$. Then for every element $g \in G$, we have $\operatorname{coeff}_{g^*P}(X) = \operatorname{mult}_P(\pi) = \operatorname{ram}_P(\pi) + 1$. This means that all the prime divisors on the $G$-orbit of $P$ appear with the same coefficient on $R$. Let $P_1, \ldots, P_k$ be the image of all the $G$-orbits of prime components of $R$ and let $r_i = \operatorname{ram}_{P_i}(\pi)$ be the multiplicity of $\pi$ at such prime $G$-orbits. Then we have $\pi^*\left(\sum_{i=1}^k \frac{r_i-1}{r_i}P_i\right) = R$. We conclude that

$$K_X = \pi^*\left(K_Y + \sum_{i=1}^k \frac{r_i-1}{r_i}P_i\right).$$

This pullback formula also follows from [Sho92, Proposition 2.1]. Now we consider the $G$-invariant boundary $\Delta$. For every $G$-orbit of a prime divisor $P$ on $X$, we define $b_i := \operatorname{coeff}_P(\Delta)$ and $n_i := \operatorname{ram}_P(\pi)$. Let $Q_1, \ldots, Q_s$ be the images on $Y$ of all the $G$-orbits contained in the support of $\Delta$. Then we have $\Delta = \pi^*\left(\sum_{i=1}^s \frac{b_i}{n_i}Q_i\right)$. Thus we have an equality

$$K_X + B = \pi^*\left(K_Y + \sum_{i=1}^k \frac{r_i-1}{r_i}P_i + \sum_{i=1}^s \frac{b_i}{n_i}Q_i\right). \tag{2.1}$$

We define

$$\Delta_Y := \sum_{i=1}^k \frac{r_i-1}{r_i}P_i + \sum_{i=1}^s \frac{b_i}{n_i}Q_i.$$

The fact that $(Y, \Delta_Y)$ is a log canonical pair follows from [Sho92, Proposition 2.2]. Note that equation (2.1) implies that the coefficients of $\Delta_Y$ have the form $\frac{m-1+\lambda}{m}$, where $m$ is a positive integer and $\lambda$ is a coefficient of $\Delta$. Indeed, if $n_i > 1$ for some $i \in \{1, \ldots, s\}$, then $P_j = Q_i$ for some $j$ and $r_i = n_j$.     $\square$

**Remark 2.3.** Let $X \to Y$ be a Galois quotient by the group $G$ and let $(X, \Delta)$ be a $G$-invariant log canonical pair. The log canonical pair $(Y, \Delta_Y)$ constructed in Proposition 2.2 will be called the *quotient* of the log canonical pair $(X, \Delta)$ by $G$. Note that the pair $(Y, \Delta_Y)$ is unique.

**Proposition 2.4.** *The quotient of a Fano-type variety by a finite automorphism group is of Fano type.*

*Proof.* Let $\phi \colon X \to Y$ be the quotient by the finite group $G$ acting on $X$ and let $\Delta$ be a big boundary on $X$ so that $(X, \Delta)$ is klt and $K_X + \Delta \sim_{\mathbb{Q}} 0$. We consider the big boundary $\Delta^G = \sum_{g \in G} g^*\Delta/|G|$ which satisfies the conditions that $K_X + \Delta^G \sim_{\mathbb{Q}} 0$ and $(X, \Delta^G)$ is klt. By Proposition 2.2, there exists a boundary $\Delta_Y$ on $Y$ such that $K_X + \Delta^G = \pi^*(K_Y + \Delta_Y)$. Thus $(Y, \Delta_Y)$ is a klt pair. By the Riemann–Hurwitz formula, we have $\pi^*(-K_Y) = -K_X + E$, where $E$ is an effective divisor. Hence, $\pi^*(-K_Y)$ is a big divisor, and thus we can write $\pi^*(-K_Y) \sim_{\mathbb{Q}} A + F$, where $A$ is a $G$-invariant ample divisor and $F$ is a $G$-invariant effective divisor. Therefore, the push-forward of $\pi^*(-K_Y)$ with respect to $\pi$ is big as well. Thus, we have that $-K_Y$ is big, so $\Delta_Y$ is big. We conclude that $Y$ is a Fano-type variety.     $\square$

**Proposition 2.5.** *Let $X$ be a Fano-type variety and $G \leq \operatorname{Aut}(X)$ be a finite subgroup. Let $(X, \Delta)$ be a log canonical $G$-invariant pair. Assume that $(X, \Delta)$ admits a $\mathbb{Q}$-complement. Then $(X, \Delta)$ admits a $G$-invariant $\mathbb{Q}$-complement.*

*Proof.* Let $B \geq \Delta$ be a $\mathbb{Q}$-complement for $(X, \Delta)$. Consider the divisor $\Gamma := \frac{1}{|G|}\sum_{g \in G} g^*B$. Note that $\Gamma \geq \Delta$; indeed, for every element $g \in G$ we have $g^*B \geq g^*\Delta = \Delta$, so $\sum_{g \in G} g^*B \geq |G|\Delta$. The pair $(X, \Gamma)$ is log canonical and $G$-invariant. Furthermore, by construction, we have that $K_X + \Gamma \sim_{\mathbb{Q}} 0$. We conclude that $\Gamma \geq \Delta$ is a $G$-invariant $\mathbb{Q}$-complement of $(X, \Delta)$.     $\square$

The following theorem is the boundedness of $G$-invariant complements for Fano-type varieties. It is a consequence of the boundedness of complements proved in [Bir19].

**Theorem 2.6.** *Let n be a positive integer and let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $M := M(\Lambda, n)$, depending only on $\Lambda$ and n, satisfying the following: Let X be an n-dimensional Fano-type variety and $\Delta$ be a boundary on X such that the following conditions hold:*

1. *$G \leq \mathrm{Aut}(X)$ is a finite subgroup,*
2. *$(X, \Delta)$ is log canonical and G-invariant,*
3. *the coefficients of $\Delta$ belong to $\Lambda$ and*
4. *$-(K_X + \Delta)$ is $\mathbb{Q}$-complemented.*

*Then there exists a boundary $B \geq \Delta$ on X such that the following hold:*

1. *$(X, B)$ is G-invariant,*
2. *$(X, B)$ is log canonical and*
3. *$M(K_X + B) \sim 0$.*

*Proof.* Let $(Y, \Delta_Y)$ (the pair constructed in Proposition 2.2) be the quotient of $(X, \Delta)$ by $G$. In particular, we have $K_X + \Delta = \pi^*(K_Y + \Delta_Y)$. By Proposition 2.4, we know that $Y$ is a Fano-type variety. By Proposition 2.5, we know that $(X, \Delta)$ admits a $G\mathbb{Q}$-complement. Hence $(Y, \Delta_Y)$ admits a $\mathbb{Q}$-complement $\Gamma_Y \geq \Delta_Y$. In particular, it is a Mori dream space. Note that $-(K_Y + \Delta_Y)$ is a pseudoeffective divisor and the log pair $(Y, \Delta_Y)$ is log canonical. We claim that the coefficients of $\Delta_Y$ belong to a set $\mathcal{H}(\Lambda) \subset \mathbb{Q}$ which depends only on $\Lambda$, satisfies the descending chain condition and has rational accumulation points. By Proposition 2.2, the coefficient of $\Delta_Y$ at a prime divisor $P$ has the form

$$\mathrm{coeff}_P(\Delta_Y) := 1 - \frac{1}{m} + \frac{\lambda}{m},$$

were $\lambda \in \Lambda$ and $m$ is the multiplicity index of $X \to Y$ at a prime component of $Y$. Note that it suffices to find an $M$-complement for $(Y, \Delta_Y)$ and pull it back to $X$. Moreover, we may run a minimal model program for $-(K_Y + \Delta_Y)$. This minimal model program is denoted by $Y \dashrightarrow Z$. It terminates given that $Y$ is a Mori dream space. Since the divisor $-(K_Y + \Delta_Y)$ is pseudoeffective, the divisor $-(K_Z + \Delta_Z)$ is semiample. Note that $(Z, \Delta_Z)$ remains log canonical and $\mathbb{Q}$-complemented. Indeed, all the steps of this minimal model program are $\mathbb{Q}$-trivial for the log canonical pair $(Y, \Gamma_Y)$ and $\Gamma_Z \geq \Delta_Z$. By [Bir19, Proposition 6.1.(3)], it suffices to produce an $M$-complement for $(Z, \Delta_Z)$. The existence of this $M$-complement is proved in [FM20, Theorem 1.2]. Furthermore, $M$ depends only on $\mathcal{H}(\Lambda)$ and $n$, and thus only on $\Lambda$ and $n$.    □

**Proposition 2.7.** *Let M be a positive integer. There exists a positive $M' := M'(M)$, depending only on M, satisfying the following: Let X be a projective surface and $G \leq \mathrm{Aut}(X)$ be a finite automorphism group. Let $(X, B)$ be a G-invariant log canonical pair such that $M(K_X + B) \sim 0$. Let $\phi\colon X \to C$ be a G-equivariant fibration to a curve C. Then we can write*

$$K_X + B \sim \phi^*(M'(K_C + B_C)),$$

*where $M'(K_C + B_C) \sim 0$ and $(C, B_C)$ is invariant with respect to the action induced on the base.*

*Proof.* We have an exact sequence

$$1 \to G_f \to G \to G_b \to 1,$$

where $G_b$ is the quotient group acting on $C$ and $G_f$ is the subgroup of $G$ acting fibre-wise. We obtain

the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\phi \downarrow & & \downarrow \phi_Y \\
C & \xrightarrow{\pi_C} & C_Y .
\end{array}
$$

Here $Y$ is the quotient of $X$ by $G$ and $C_Y$ is the quotient of $C$ by $G_b$. Observe that the coefficients of the log pair $(Y, B_Y)$ belong to the hyperstandard set $\mathcal{H}\left(\mathbb{N}\left[\frac{1}{M}\right] \cap [0,1]\right)$. Note that this set depends only on $M$. Hence, we may apply the effective canonical bundle formula (see, e.g., [PS01] ). We obtain a log pair $\left(C_Y, B_{C_Y}\right)$ with

$$
K_Y + B_Y \sim \phi_Y^* \left(M'\left(K_{C_Y} + B_{C_Y}\right)\right),
$$

where $M'\left(K_{C_Y} + B_{C_Y}\right) \sim 0$. Here $M'$ depends only on $\mathcal{H}\left(\mathbb{N}\left[\frac{1}{M}\right] \cap [0,1]\right)$, which in turn depends only on $M$. Hence the pair obtained by pulling back $K_{C_Y} + B_{C_Y}$ to $C$ satisfies the conditions of the statement. $\qquad\square$

### 2.3. *k-Generation order*

In this subsection, we collect some lemmas about the $k$-generation order and recall the Jordan property for finite birational automorphism groups of Fano-type varieties.

**Definition 18.** We define the *rank* of a finite group $G$, denoted by $\mathrm{rank}(G)$, to be the minimum number of generators of $G$. We define the *rank up to index $N$*, denoted by $r_N(G)$, to be the minimum rank among subgroups of $G$ of index at most $N$. If $r_N(G) = k$, then any subgroup generated by at most $k-1$ elements has index larger than $N$. In particular, if $r_N(G) \geq k$, then the minimum rank of $G$ is at least $k$. Note that $r_1(G) = \mathrm{rank}(G)$ and $r_N(G) = 1$ for every $N \geq |G|$. In general, $r_N(G) \in \{1, \dots, \mathrm{rank}(G)\}$ for every positive integer $N$.

**Definition 19.** Let $G$ be a finite group. For $k \leq \mathrm{rank}(G)$, we define the *k-generation order*, denoted by $g_k(G)$, to be

$$
\max\{N \mid |G| \geq N \text{ and } r_N(G) \geq k\}.
$$

Note that any finite group satisfies $g_1(G) = |G|$. So having large 1-generation is the same as having large cardinality.

**Lemma 2.8.** *Let $G$ be a finite group with $g_k(G) \geq N$ and let $A \leq G$ be a subgroup of index at most $J$. Then $g_k(A) \geq N/J$.*

*Proof.* Note that $|A| \geq N/J$. Assume that $r_{N/J}(A) < k$. Then there exists a subgroup $H \leq A$ of index $\leq N/J$ which is generated by strictly fewer than $k$ elements. Note that $H \leq G$ is a subgroup of index $\leq N$ which is generated by strictly fewer than $k$ elements. This contradicts the fact that $r_N(G) \geq k$. We conclude that $|A| \geq N/J$ and $r_{N/J}(A) \geq k$. Thus, we have $g_k(A) \geq N/J$. $\qquad\square$

**Lemma 2.9.** *Let $G$ be a finite group with $g_2(G) \geq N$. Assume that we have an exact sequence*

$$
1 \to \mathbb{Z}_n \to G \to \mathbb{Z}_m \to 1.
$$

*Then $\min\{m, n\} \geq N$.*

*Proof.* If $m < N$, then $\mathbb{Z}_n$ is a cyclic subgroup of $G$ of index bounded by $N$, contradicting the fact that $r_N(G) \geq 2$. On the other hand, if $n < N$, we can consider an element $h$ which maps to $1_m \in \mathbb{Z}_m$. Let

$H \leq G$ be the cyclic subgroup generated by $h$. Note that the order of $h$ in $G$ is at least $m$; hence the index of $H$ in $G$ is at most $n$. We conclude that $H$ is a cyclic subgroup of $G$ with index bounded by $N$. This contradicts the fact that $r_N(G) \geq 2$. □

**Lemma 2.10.** *Let $G$ be a finite group with $g_k(G) \geq N$ and $k \geq 2$. Consider the exact sequence*

$$1 \to C \to G \to G' \to 1,$$

*where $C$ is a cyclic group. Then, $g_{k-1}(G') \geq N$.*

*Proof.* Assume that $|G'| \leq N$. Then $C$ is a cyclic group of index bounded by $N$ on $G$. This contradicts the fact that $r_N(G) \geq 2$, and hence we have $|G'| > N$. On the other hand, assume that $r_N(G') < k-1$. This means that we can find a subgroup $H'$ of $G'$ generated by strictly fewer than $k-1$ elements with index at most $N$. Let $H_0$ be a subgroup of $G$ surjecting onto $H'$. We may assume that $H_0$ is generated by strictly fewer than $k-1$ elements as well. Let $H$ be the subgroup of $G$ generated by the image of the generator of $C$ and $H_0$; then $H$ is generated by strictly fewer than $k$ elements and furthermore has index at most $N$ on $G$, leading to a contradiction of the fact that $r_N(G) \geq k$. We conclude that $r_N(G') \geq k-1$, and hence $g_{k-1}(G') \geq N$. □

**Lemma 2.11.** *Let $G$ be a finite group and $H$ be a subgroup. If $g_2(H) \geq N$, then $g_2(G) \geq N$.*

*Proof.* Note that $g_2(H) \geq N$ implies that $|H| \geq N$, so $|G| \geq N$. We need to check that $r_N(G) \geq 2$. Assume the opposite holds – that is, there is a cyclic subgroup $\mathbb{Z}_m \leq G$ of index at most $N$. Then $\mathbb{Z}_m \cap H$ is a cyclic subgroup of $H$ of index at most $N$, contradicting the fact that $r_N(H) \geq 2$. Hence we have $r_N(G) \geq 2$, and thus $g_2(G) \geq 2$. □

**Lemma 2.12.** *There exists a constant $l := l(r, k, J)$, depending only on $r$, $k$ and $J$, satisfying the following: Let $G < \operatorname{Aut}(\mathbb{G}_m^k)$ be a finite subgroup. Suppose that there exists a normal abelian subgroup of $G$ of rank $r$ with index at most $J$. Then there exists an abelian characteristic subgroup $A \leq G$ of index at most $l$ such that $A < \mathbb{G}_m^k$.*

*Proof.* We have an exact sequence

$$1 \to \mathbb{G}_m^k \to \operatorname{Aut}\left(\mathbb{G}_m^k\right) \to \operatorname{GL}_k(\mathbb{Z}) \to 1.$$

There exists a constant $h := h(k)$, depending only on $k$, such that every finite order element of $\operatorname{Gl}_k(\mathbb{Z})$ has order at most $h$. Set $l := h!J$. We have that $G^l = \{g^l \mid g \in G\}$ is a characteristic subgroup of $G$. By construction, we have $G^l < \mathbb{G}_m^k$. Furthermore, we have $G^l \leq A_0$. In particular, the group $G^l$ is an abelian characteristic subgroup of $G$. We need to check that it has bounded index on $G$. We have subgroups

$$A_0^l \leq G^l \leq A_0 \leq G.$$

Note that the index of $A_0^l$ in $A_0$ is at most $l^r$. We conclude that the index of $A := G^l$ in $G$ is bounded by $l^r J$. □

**Definition 20.** Let $\mathcal{G}$ be a class of finite groups. We say that $\mathcal{G}$ is *uniformly Jordan* if there is a constant $J(\mathcal{G})$ such that for any group $G \in \mathcal{G}$ there is a normal abelian subgroup $A \leq G$ of index at most $J(\mathcal{G})$.

**Theorem 2.13.** (Compare [PS16, Theorem 1.8]). *The class of finite birational automorphism groups of Fano-type varieties of dimension $n$ is uniformly Jordan.*

### 2.4. Del Pezzo surfaces with large automorphisms

In this subsection, we prove some propositions about del Pezzo surfaces with large automorphism groups.

**Lemma 2.14.** *Let M and n be positive integers. There exists a positive integer $N := N(M, n)$, depending only on M and n, satisfying the following: Let $A < \mathrm{Aut}(X_n)$ be a finite abelian subgroup, where $X_n$ is either $\mathbb{P}^n$ or $(\mathbb{P}^1)^n$, and let B be an A-invariant M-complement of $\mathbb{P}^n$. If $g_r(A) \geq N(M, n)$, then B is a $\mathbb{G}_m^r$-invariant boundary for some $\mathbb{G}_m^r < \mathrm{Aut}(X_n)$. Moreover, there is a subgroup $A' < A$ of index at most $N(M, n)$ such that $A' < \mathbb{G}_m^r < \mathrm{Aut}(X_n)$.*

*Furthermore, if $r = n$, then B is the torus invariant boundary, $A \simeq \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_n}$ acts as multiplication by roots of unity on the torus and $\min\{k_1, \ldots, k_n\} \geq N(M, n)$.*

*Proof.* Note that $\mathrm{Aut}(X_n)$ is either $\mathrm{PGL}_{n+1}(\mathbb{K})$ or $S_n \rtimes \mathrm{PGL}_2(\mathbb{K})$. By the Jordan property of $\mathrm{Aut}(X_n)$, there exists a subgroup $A' < A$ of index at most $c(n)$ which is contained in a maximal torus $\mathbb{G}_m^n$ of $\mathrm{Aut}(X_n)$. Hence the group $A'$ has rank at most $n$, so we can write $A' \simeq \mathbb{Z}_{l_1} \oplus \cdots \oplus \mathbb{Z}_{l_n}$ with $l_1 \geq \cdots \geq l_n$. By Lemma 2.8, we have $g_r(A') \geq g_r(A)/c(n)$. Applying Lemma 2.10, we see that $\min\{l_1, \ldots, l_r\} \geq g_r(A)/c(n)$. We may replace $A'$ with $\mathbb{Z}_{l_1} \oplus \cdots \oplus \mathbb{Z}_{l_r}$ which is contained in $\mathbb{G}_m^r < \mathbb{G}_m^n$. Let $P \subset \mathbb{P}^n$ be a prime divisor which appears on $B$. If $P$ is $A'$-invariant but not $\mathbb{G}_m^r$-invariant, then it has degree at least $g_n(A)/c(n)$. If $P$ is in the support of $B$, then it appears on $B$ with coefficient at least $1/M$. Thus we conclude that for $g_n(A)/c(n) \geq 4M$, the support of $B$ is $\mathbb{G}_m^r$-invariant. Hence if $g_n(A) \geq 4c(n)M$, then $B$ is $\mathbb{G}_m^r$-invariant.

From now on, we assume that $r = n$. In this case, $B$ must be the unique torus invariant boundary. Since $A$ fixes the boundary, we conclude that $A < \mathrm{Aut}(\mathbb{G}_m^n)$. Since $A$ commutes with $A'$, we conclude that $A$ is actually a subgroup of the torus, provided that $g_n(A)/c(n) > 1$. Applying Lemma 2.10 again, we see that $A \simeq \mathbb{Z}_{k_1} \oplus \cdots \oplus_{k_n}$ with $\min\{k_1, \ldots, k_n\} \geq N(m, N)$. Hence it suffices to take $N(M, n) = 4c(n)M$. This concludes the proof. $\qquad\square$

**Proposition 2.15.** *Let M be a positive integer. There exists a positive integer $N := N(M)$, depending only on M, satisfying the following: Let X be a del Pezzo surface and $G \leq \mathrm{Aut}(X)$ be a finite subgroup with $g_r(G) \geq N$, $r \in \{1, 2\}$. Let B be an effective divisor on X satisfying the following conditions:*

1. *$(X, B)$ is G-invariant,*
2. *$(X, B)$ is log canonical and*
3. *$M(K_X + B) \sim 0$.*

*Then $(X, B)$ admits a $\mathbb{G}_m^r$-action. Furthermore, there exists an abelian subgroup $A \leq G$ with index bounded by N, such that $A < \mathbb{G}_m^r \leq \mathrm{Aut}(X, B)$.*

*Proof.* If $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then the claim follows from Lemma 2.14. Hence, we may assume it is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

By the Jordan property, we may find an abelian normal subgroup $A_0 \leq G$ of index at most $J(2)$. Here $J := J(2)$ is a constant. By Lemma 2.8, we know that $g_2(A_0) \geq g_2(G)/J$. Passing to a subgroup $A_1$ of $A_0$ of bounded index $I$, we may assume that $A_1$ acts trivially on $\mathrm{Pic}(X)$, and hence we obtain an $A_1$-equivariant contraction $X \to \mathbb{P}^2$. The $A_1$-action on $\mathbb{P}^2$ fixes $9 - K_X^2$ points. By Lemma 2.8, we know that $g_2(A_1) \geq g_2(G)/JI$. Let $B_{\mathbb{P}^2}$ be the push-forward of $B$ to $\mathbb{P}^2$ – note that $(\mathbb{P}^2, B_{\mathbb{P}^2})$ is $A_1$-invariant and $M(K_{\mathbb{P}^2} + B_{\mathbb{P}^2}) \sim 0$. Assume that $g_2(G) \geq JIN(M, 2)$, where $N(M, 2)$ is as in Lemma 2.14. By Lemma 2.14, we can find an abelian subgroup $A_2 < A_1$ of index at most $N(M, 2)$ such that $A_2 < \mathbb{G}_m^r < \mathrm{Aut}(\mathbb{P}^2)$ and $B_{\mathbb{P}^2}$ is $\mathbb{G}_m^r$-invariant. Note that $A_2$ and $\mathbb{G}_m^r$ have the same fixed points, and hence the exceptional curves of $X \to \mathbb{P}^2$ map to $A_2$-invariant points and thus to $\mathbb{G}_m^r$-invariant points. Thus $(X, B)$ admits a $\mathbb{G}_m^r$-action such that $A_2 < \mathbb{G}_m^r \leq \mathrm{Aut}(X, B)$. We conclude that it is enough to consider $N := JIN(M, 2)$. $\qquad\square$

### 2.5. Formally toric surface morphisms

In this subsection, we recall an invariant called complexity. We also recall a characterisation of toric morphisms using complexity. First we will introduce the complexity of a log pair with respect to a contraction. In [BMSZ18], the authors introduce a more general version of complexity, but for our purposes the following definition suffices:

**Definition 21.** A *decomposition* $\Sigma$ of an effective divisor $B$ is an expression of the form $B := \sum_{i=1}^{k} b_i B_i$, where the $B_i$s are effective Weil divisors and the $b_i$s are nonnegative real numbers. We denote by $|\Sigma| := \sum_{i=1}^{k} b_i$ the *norm* of the decomposition. Each effective divisor $B$ on an algebraic variety $X$ has a natural decomposition given by the prime components of the support of $B$.

Let $(X, B)$ be a log canonical pair and $X \to C$ be a contraction. Let $\Sigma$ be a decomposition of $B$ and let $c \in C$ be a closed point. We assume that every irreducible component of $B$ intersects the fibre over $c$. The *complexity* of $(X, B)$ over $c$, denoted by $\gamma_c(X, B)$, is defined as

$$\gamma_c(X, B) := \inf\{\dim(X) + \rho_c(X/C) - |\Sigma| \mid \Sigma \text{ is a decomposition of } B\},$$

were $\rho_c(X/C)$ is the relative Picard rank of $X$ over $c \in C$. The *local complexity* of a log pair $(X, B)$ at a closed point $x \in X$ is just the complexity over $x$ with respect to the identity morphism.

We recall some results about the complexity of log Calabi–Yau pairs:

**Theorem 2.16.** (Compare [Kol92, Theorem 18.22]). *Let $(X, B)$ be a log canonical pair and $x \in X$ be a closed point. Then $\gamma_x(X, B) \geq 0$. Furthermore, if $\gamma_x(X, B) = 0$, then $(X, \lfloor B \rfloor)$ is formally toric.*

**Theorem 2.17.** (Compare [Sho00, Theorem 6.4]). *Let $X$ be a surface, $(X, B)$ be a log Calabi–Yau pair and $X \to C$ be a fibration to a curve. Let $c \in C$ be a closed point. Then $\gamma_c(X, B) \geq 0$. Furthermore, if $\gamma_c(X, B) = 0$, then $(X, \lfloor B \rfloor)$ is formally toric over $c \in C$.*

**Theorem 2.18.** (Compare [BMSZ18, Theorem 1.2]). *Let $(X, B)$ be a log Calabi–Yau projective pair. Then $\gamma(X, B) \geq 0$. Furthermore, if $\gamma(X, B) = 0$, then $(X, \lfloor B \rfloor)$ is isomorphic to a toric log pair.*

**Proposition 2.19.** *Let $(X, B)$ be a log Calabi–Yau surface and let $X \to C$ be a fibration to a curve. Assume that $\lfloor B \rfloor_{\mathrm{hor}}$ has two irreducible components over $c \in C$. Then one of the following statement holds:*

1. *If $\mathrm{lct}_c((X, B); \pi^*(c)) > 0$, then $(X, B)$ is formally toric over $c \in C$.*
2. *If $\mathrm{lct}_c((X, B); \pi^*(c)) = 0$, then there is a crepant equivalent birational transformation of $(X, B)$ over $c$, which is an isomorphism outside $\pi^{-1}(c)$, and is formally toric over $c \in C$.*

Proposition 2.19 is a particular case of Proposition 2.22.

## 2.6. G-equivariant formally toric surface morphisms

In this subsection, we introduce a $G$-equivariant version of complexity. We prove the $G$-equivariant versions of some of the results from the previous subsection.

**Definition 22.** Let $X$ be a normal quasi-projective variety and $G \leq \mathrm{Aut}(X)$ be a finite subgroup. We say that an effective Weil divisor $D$ on $X$ is *G-prime* or *G-irreducible* if it $G$-invariant and is not the sum of two proper $G$-invariant effective Weil divisors. For instance, the $G$-orbit of a prime reduced divisor on $X$ is $G$-prime.

**Definition 23.** Let $X$ be a quasi-projective normal variety and $G \leq \mathrm{Aut}(X)$ be a finite automorphism group. Let $\phi \colon X \to Z$ be a $G$-equivariant fibration. We have an exact sequence $1 \to G_{\phi, f} \to G \to G_{\phi, b} \to 1$, where $G_{\phi, b}$ is the finite automorphism group induced on the base and $G_{\phi, f}$ is the finite automorphism group acting fibre-wise. Analogously, $G_{\phi, f}$ is the kernel of the natural surjective homomorphism $G \to G_{\phi, b}$. We call $G_{\phi, f}$ (resp., $G_{\phi, b}$) the *fibre-wise automorphism group* (resp., *base automorphism group*). If $\phi$ is clear from the context, we will omit it from the notation.

**Definition 24.** Let $X$ be an algebraic variety and let $G \leq \mathrm{Aut}(X)$ be a finite subgroup. A *G-invariant decomposition* $\Sigma$ of an effective $G$-invariant divisor $B$ is an expression of the form $B := \sum_{i=1}^{k} b_i B_i$, where the $B_i$s are effective $G$-invariant Weil divisors and the $b_i$s are nonnegative real numbers. We denote by $|\Sigma|^G := \sum_{i=1}^{k} b_i$ the *G-invariant norm* of the decomposition. Each $B$ on $X$ has a natural decomposition given by the $G$-orbits of the prime decomposition of $B$.

Let $(X, B)$ be a $G$-invariant log canonical pair and $X \to C$ be a $G$-equivariant contraction. Let $\Sigma$ be a $G$-invariant decomposition of $B$ and $c \in C$ be a closed point. Assume that every $G$-prime component of $B$ intersects the $G$-invariant fibre of $c$. The *$G$-complexity* or *$G$-invariant complexity* of $(X, B)$ over $c$, denoted by $\gamma_c^G(X, B)$, is defined as

$$\gamma_c^G(X, B) := \inf \left\{ \dim(X) + \rho_c^G(X/C) - |\Sigma|^G \;\middle|\; \Sigma \text{ is a } G\text{-invariant decomposition of } B \right\},$$

where $\rho_c^G(X/C)$ is the $G$-invariant relative Picard rank of $X$ over $c \in C$.

**Theorem 2.20.** *Let $X$ be a surface and $G \leq \operatorname{Aut}(X)$ be a finite subgroup. Let $(X, B)$ be a $G$-invariant log Calabi–Yau pair and $X \to C$ be a $G$-equivariant fibration to a curve. Let $c \in C$ be a closed point. Then $\gamma_c^G(X, B) \geq 0$. Furthermore, if $\gamma_c^G(X, B) = 0$, then $(X, \lfloor B \rfloor)$ is $G$-equivariantly formally toric over the $G_b$-orbit of $c$ in $C$.*

*Proof.* Quotienting $X$ by $G$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
(X, B) & \xrightarrow{\;/G\;} & (Y, B_Y) \\
\pi \downarrow & & \downarrow \pi_Y \\
C & \xrightarrow{\;/G_b\;} & C_Y.
\end{array}
$$

Let $c_0$ be the image of the $G_b$-orbit of $c$ in $C$. We claim that we have the inequalities

$$\gamma_c^G(X, B) \geq \gamma_{c_0}(Y, B_Y) \geq 0.$$

Indeed, $\dim(X) = \dim(Y)$, $\rho_c^G(C/X) = \rho_{c_0}(Y/C_Y)$ and

$$|\Sigma|^G = \sum_{i=1}^k b_i \leq \sum_{i=1}^k \left( 1 - \frac{1}{m_i} + \frac{b_i}{m_i} \right) = |\Sigma|, \tag{2.2}$$

where $m$ is the multiplicity index of $X \to Y$ at a $G$-prime component $B_i$ of coefficient $b_i$. Note that the equality of equation (2.2) holds if and only if $b_i = 1$ or $m_i = 1$. Hence, the inequality $\gamma_c^G(X, B) \geq 0$ holds. Furthermore, the equality $\gamma_c^G(X, B) = \gamma_{c_0}(Y, B_Y)$ holds if and only if $X \to Y$ ramifies only at divisors of $\lfloor B \rfloor$. Assume that $\gamma_c^G(X, B) = 0$; then we have $\gamma_{c_0}(Y, B_Y) = 0$. By Theorem 2.17, we conclude that $(Y, \lfloor B_Y \rfloor)$ is formally toric over $c_0$. Furthermore, $X \to Y$ ramifies only at divisors of $\lfloor B_Y \rfloor$. Let $D_Y$ be the toric boundary of $Y$ over $c_0$; in particular, we have $\lfloor B_Y \rfloor \subset D_Y$. Note that the complement $U_Y$ of $D_Y$ on $Y$ has Picard rank 0. Let $(X, D_X)$ be the log pullback of $(Y, D_Y)$ to $X$. Shrinking around the $G_b$-orbit of $c$, we may assume that the prime components of $D_X$ generate the relative Picard group of $X$ over $C$. Let $k$ be the number of prime components of $D_X$. We have two $\mathbb{Q}$-linearly independent equivalence relations among the irreducible components of $D_X$. These two relations come from pulling back the corresponding relations on $Y$ over $C$. Thus, $\rho_c(X/C) \leq k - 2$. We conclude that

$$\gamma_c(X, D_X) = \dim(X) + \rho_c(X/C) - |D_X| \leq 2 + (k-2) - k \leq 0.$$

Thus the Picard rank of $X$ over the $G_b$-orbit of $c$ is $k - 2$, and $(X, D_X)$ is locally a log Calabi–Yau toric pair over $c$. Since $\lfloor B \rfloor \subset D_X$, we conclude that $(X, \lfloor B \rfloor)$ is formally toric over $c$. $\qquad\square$

**Theorem 2.21.** *Let $X$ be a normal projective variety of dimension $n$ and $G \leq \operatorname{Aut}(X)$ be a finite automorphism group. Let $(X, B)$ be a $G$-invariant log Calabi–Yau pair. Then $\gamma^G(X, B) \geq 0$. Furthermore, if $\gamma^G(X, B) = 0$, then $(X, \lfloor B \rfloor)$ is isomorphic to a toric pair.*

*Proof.* Let $(Y, B_Y)$ be the quotient of $(X, B)$ by $G$. We know that $\gamma^G(X, B) \geq \gamma(Y, B_Y) \geq 0$. This proves the first statement. Moreover, equality holds if and only if $X \to Y$ ramifies only at $\lfloor B \rfloor$. Assume

that $\gamma^G(X, B) = 0$; then $(Y, \lfloor B_Y \rfloor)$ is a toric pair. Let $(Y, D_Y)$ be the toric log Calabi–Yau structure. Denote by $(X, D_X)$ the log pullback of $(Y, D_Y)$ to $X$. Note that $X \to Y$ is unramified over $\mathbb{G}_m^n$. Hence, we have $X \setminus \operatorname{supp}(D_X) \simeq \mathbb{G}_m^n$. We conclude that the Picard group of $X$ is generated by the irreducible components of $D_X$. Let $k$ be the number of such irreducible components; then $\rho(X) \leq k$. We claim that there are at least $n\,\mathbb{Q}$-linearly independent linear relations among the irreducible components of $D_X$. Indeed, this statement holds among the prime components of $D_Y$ in $Y$. Pulling back these linear relations, we obtain the corresponding relations on $X$. Thus $\rho(X) \leq k - n$, where $n$ is the dimension of $X$. Now we can proceed to compute the complexity of $(X, D_X)$. Note that

$$\gamma(X, D_X) = \rho(X) + \dim(X) - |D_X| \leq k - n + n - k = 0.$$

By Theorem 2.18, we conclude that $\rho(X) = k - n$ and $(X, D_X)$ is a toric log Calabi–Yau pair. Since $\lfloor B \rfloor \subset D_X$, we conclude that $(X, \lfloor B \rfloor)$ is a toric pair as well. □

**Proposition 2.22.** *Let $X$ be a surface and $G \leq \operatorname{Aut}(X)$ be a finite automorphism group. Let $(X, B)$ be a $G$-invariant log Calabi–Yau pair and $X \to C$ be a $G$-equivariant fibration to a curve. Assume that $\lfloor B \rfloor_{\mathrm{hor}}$ has two irreducible components over $c \in C$. Then one of the following statements holds:*

1. *If $\operatorname{lct}((X, B); \pi^*(c)) > 0$, then $(X, B)$ is $G$-equivariantly formally toric over the $G_b$-orbit of $c$ in $C$.*
2. *If $\operatorname{lct}((X, B); \pi^*(c)) = 0$, then there is a crepant equivalent $G$-equivariant birational transformation of $(X, B)$ over the $G_b$-orbit of $c$ in $C$ which is an isomorphism outside the $G$-orbit of $\pi^{-1}(c)$ and is $G$-equivariantly formally toric over the $G_b$-orbit of $c$ in $C$.*

*Proof.* Assume that $\operatorname{lct}_c((X, B); \pi^*(c)) = t > 0$. Denote by $G_b c$ the $G_b$-orbit of $c$ in $C$. Since this log canonical threshold is positive, the two irreducible components of $\lfloor B \rfloor_{\mathrm{hor}}$ are disjoint over $G_b c$. Furthermore, the log pair $(X, B + t\pi^*(G_b c))$ is a $G$-equivariant strictly log canonical pair. This pair is log Calabi–Yau (up to shrinking around $G_b c$). Let $(Y, B_Y)$ be a $G$-equivariant dlt modification of $(Y, B + t\pi^*(G_b c))$ over $C$; then $B_Y$ contains a $G$-prime vertical curve with coefficient 1 mapping onto $G_b c$. We call such a curve $E$. Note that $E$ may be reducible. Furthermore, $\operatorname{coeff}_P(B_Y) > 0$ for every prime divisor $P$ mapping to $c$. We run a $G$-equivariant minimal model program for $(Y, B_Y - \epsilon E)$ over $C$, where $\epsilon$ is a positive small rational number. This minimal model program terminates with a $G$-equivariant model $(Z, B_Z - \epsilon E_Z)$ over $C$, where $B_Z$ is the push-forward of $B_Y$ to $Z$ and $E_Z$ is the push-forward of $E$ to $Z$. Note that $(Z, B_Z)$ is a $\mathbb{Q}$-factorial $G$-equivariant log Calabi–Yau pair with a $G$-equivariant fibration to $C$. Furthermore, by construction we have $\rho_c^G(Z/C) = 1$, and $\gamma_c(Z, B_Z) = 0$. By Theorem 2.20, we conclude that $(Z, B_Z)$ is $G$-equivariantly formally toric over $G_b c$. Note that the morphism $Y \to Z$ is $G$-equivariant and extracts only prime divisors with log discrepancy in the interval $[0, 1)$ with respect to $(Z, B_Z)$. We conclude that $(Y, B_Y)$ is $G$-equivariantly formally toric over $G_b c$ in $C$. Finally, since $Y \to X$ is a $G$-equivariant contraction over $G_b c$ in $C$, we conclude that $(X, B)$ is $G$-equivariantly formally toric over the $G_b$-orbit of $c$ in $C$. Indeed, $Y \to X$ is the morphism over $C$ defined by a semiample $G$-invariant toric divisor on $Y$.

Assume that $\operatorname{lct}_c((X, B); \pi^*(c)) = 0$. We can apply the connectedness of log canonical centres [HH19, Theorem 1.2]. We conclude that either the components of $\lfloor B \rfloor_{\mathrm{hor}}$ are connected by components of $\lfloor B \rfloor_{\mathrm{ver}}$ or two components of $\lfloor B \rfloor_{\mathrm{hor}}$ intersect over $G_b c$. In the former case, we can take a $G$-prime effective divisor $E \subset \lfloor B \rfloor_{\mathrm{vert}}$ and run a $G$-equivariant minimal model program for $(X, B - \epsilon E)$ over $C$. It terminates with a model $(Y, B_Y - \epsilon E_Y)$ for which $\gamma_c(Y, B_Y) = 0$. By Theorem 2.20, we have that $(Y, B_Y)$ is $G$-equivariantly formally toric over $c$. On the other hand, in the latter case, we take a $G$-equivariant $\mathbb{Q}$-factorial dlt modification $(Y, B_Y)$ of $(X, B)$ over $c \in C$. Let $E$ be an effective $G$-prime divisor contained in $\lfloor B_Y \rfloor$ which maps to $c$. Note that we can always find such a divisor $E$ by possibly replacing $(Y, B_Y)$ with a higher $G$-equivariant $\mathbb{Q}$-factorial dlt modification. Run a $G$-equivariant minimal model program for $(Y, B_Y - \epsilon E)$ over $C$ which terminates with a $G$-equivariant minimal model $(Z, B_Z - \epsilon E_Z)$. In this $G$-equivariant minimal model, the $G$-invariant divisor $B_Z$ is reduced. Note that $\gamma_c(Z, B_Z) = 0$. Hence, we conclude that $(Z, B_Z)$ is formally toric over $c \in C$. Observe that in any case, the $G$-equivariant birational map $X \dashrightarrow Z$ over $C$ is an isomorphism over $C \setminus G_b c$. □

**Corollary 2.23.** *Let $X$ be a surface and $G \leq \mathrm{Aut}(X)$ be a finite automorphism group. Let $(X, B)$ be a $G$-equivariant log Calabi–Yau pair and $X \to C$ be a $G$-equivariant fibration to a curve. Assume that $\lfloor B \rfloor_{\mathrm{hor}}$ has two irreducible components over $C$. Then there is a crepant equivalent $G$-equivariant birational map $X \dashrightarrow Y$ over $C$ such that $(Y, B_Y)$ is $G$-equivariantly formally toric over every point of $C$.*

*Proof.* Let $c \in C$ be a point such that $\mathrm{lct}((X, B); \pi^*(c)) > 0$; then $(X, B)$ is $G$-equivariantly formally toric over the $G_b$-orbit of $c$ in $C$, by Proposition 2.22. We conclude that $(X, B)$ is $G$-equivariantly formally toric over the open set $U$ of $C$ on which this log canonical threshold is positive. Assume that there is a unique closed point $c_0$ in $C$ over whose orbit this log canonical threshold is 0.

We have $\mathrm{lct}((X, B); \pi^*(c_0)) = 0$. By Proposition 2.22, there is a crepant equivalent $G$-equivariant birational transformation $(Y, B_Y)$ of $(X, B)$ over the $G_b$-orbit of $c_0$ in $C$, which is an isomorphism outside the $G$-orbit of $\pi^{-1}(c)$, and is $G$-equivariantly formally toric over the $G_b$-orbit of $c$ in $C$. Note that after this birational transformation of $(X, B)$, the log pair $(Y, B_Y)$ remains $G$-equivariantly formally toric over $U$. Hence $Y \to C$ is $G$-equivariantly formally toric over $C$ for the log pair $(Y, B_Y)$. When $C$ has several closed points over which the log canonical threshold is 0, we proceed inductively. $\square$

## 2.7. Surface fibrations with large fibre-wise automorphisms

In this subsection, we study $G$-equivariant fibrations of Fano-type surfaces such that the induced automorphism on a general fibre is large.

**Proposition 2.24.** *Let $M$ be a positive integer. There exists a positive integer $N := N(M)$, depending only on $M$, satisfying the following: Assume that the following conditions hold:*

1. *$X$ is a Fano-type surface and $G \leq \mathrm{Aut}(X)$ is a finite subgroup,*
2. *$(X, B)$ is a $G$-invariant log Calabi–Yau pair such that $M(K_X + B) \sim 0$,*
3. *$X \to C$ is a $G$-equivariant fibration to a curve and*
4. *$G_f$ is abelian with $|G_f| \geq N$.*

*Then, up to passing to a $G$-equivariant crepant equivalent birational model of $(X, B)$, we may assume that $\lfloor B \rfloor_{\mathrm{hor}}$ has two prime components.*

*Proof.* First we reduce to the case in which $B_{\mathrm{hor}}$ is a reduced divisor. We denote by $(F, B_F)$ the restriction of $(X, B)$ to a general fibre $F$. Note that $M(K_F + B_F) \sim 0$. By Lemma 2.14, we may assume that $G_f$ is a cyclic group acting as multiplication by a root of unity on $F \simeq \mathbb{P}^1$. Note that the coefficients of $B_F$ are larger than $\frac{1}{2M}$. Hence, we conclude that for $N \geq 4M$, the pair $(F, B_F)$ is isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$. Thus all horizontal components of $B$ are reduced, and furthermore, $B$ has either one or two horizontal components.

We reduce to the case in which $C_0 := B_{\mathrm{hor}}$ is normal. Assume that it is not a normal curve. Since it must be a semi-log canonical curve, it has a node $c \in C_0$. Note that $c_0 \in X$ is a log canonical centre of $(X, B)$ by inversion of adjunction. Let $c$ be the image of $c_0$ in $C$. We can extract a $G$-prime log canonical $E$-centre over the $G$-orbit of $c_0$ in $X$. We call $Y \to X$ a $G$-equivariant birational map. We run a $G$-equivariant minimal model program for $(Y, B_Y - E)$ over $C$ and get a model $(Z, B_Z)$. Note that all the hypotheses on $(X, B)$ are satisfied by $(Z, B_Z)$. Furthermore, the curve $B_{Z, \mathrm{hor}}$ is normal over $C \setminus c$ and is normal on a neighbourhood of the fibre over $c$. Hence, $B_{Z, \mathrm{hor}}$ is a normal curve. Replacing the pair $(X, B)$ with $(Z, B_Z)$, we may assume that $C_0 := \lfloor B \rfloor_{\mathrm{hor}}$ is a normal curve.

Note that we have a finite morphism $C_0 \to C$. We prove that it must be unramified. Assume that $C_0 \to C$ ramifies over a point $c \in C$. Let $c_0$ be the preimage of $c$ in $C_0$. The multiplicity index of $C_0 \to C$ at $c$ equals 2; otherwise the boundary $B_F$ has three or more points with coefficient 1, leading to a contradiction. Let $Y \to X$ be the normalisation of the base change induced by $C_0 \to C$. Hence, over the $G_b$-orbit of $c$ in $C$, the finite morphism $Y \to X$ is the quotient by an involution $\tau$ on $Y$. Let $(Y, B_Y)$ be the log pullback of $(X, B)$ to $Y$. By abuse of notation, we will also denote by $\tau$ the induced involution on $C_0$.

We claim that $G_f = \langle g_f \rangle$ acts as a group of automorphisms on $(Y, B_Y)$ such that $g_f \tau g_f^{-1} \in \langle \tau \rangle$. We name $\pi \colon X \to C$. Indeed, we have the following commutative diagram:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ \pi\ } & C & \xleftarrow{\ /\tau\ } & C_Y \\
{\scriptstyle g_f}\downarrow & & {\scriptstyle \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{C_Y}} \\
X & \xrightarrow{\ \pi\ } & C & \xleftarrow{\ /\tau\ } & C_Y.
\end{array}
$$

Hence, we have an induced automorphism $g_f \in \mathrm{Aut}(Y)$ acting as multiplication by a root of unity on a general fibre. Furthermore, we have the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ g_f\ } & Y \\
{\scriptstyle /\tau}\downarrow & & \downarrow{\scriptstyle /\tau} \\
X & \xrightarrow{\ g_f\ } & X.
\end{array}
$$

By abuse of notation, we are calling both automorphisms $g_f$. Thus, we conclude that the following diagram commutes:

$$
\begin{array}{ccccccc}
Y & \xrightarrow{\ g_f^{-1}\ } & Y & \xrightarrow{\ \tau\ } & Y & \xrightarrow{\ g_f\ } & Y \\
{\scriptstyle /\tau}\downarrow & & {\scriptstyle /\tau}\downarrow & & {\scriptstyle /\tau}\downarrow & & \downarrow{\scriptstyle /\tau} \\
X & \xrightarrow{\ g_f^{-1}\ } & X & \xrightarrow{\ \mathrm{id}_X\ } & X & \xrightarrow{\ g_f\ } & X.
\end{array}
$$

Since the automorphism on $X$ obtained by composing the bottom horizontal arrows is $\mathrm{id}_X$, we conclude that the automorphism on $Y$ obtained by composing the upper horizontal arrows is contained in $\langle \tau \rangle$. This proves the claim.

We denote by $G_Y := \langle g_f, \tau \rangle$. Note that $B_Y$ has two horizontal irreducible components. By Proposition 2.22, we know that $(Y, B_Y)$ is $G_f$-equivariantly formally toric over $c_0$. Observe that the toric boundary of $Y$ is $G_Y$-invariant; hence $G_Y$ acts as an automorphism group of the complement of the toric boundary. The complement of the toric boundary, locally over $c_0$, is isomorphic to $D_0 \times \mathbb{G}_m$ (where $D_0$ is the punctured disk). In the coordinates $(t_1, t_2)$ of $D_0 \times \mathbb{G}_m$, the automorphism $\tau$ acts as $\left(-t_1, \lambda t_1^a t_2^{-1}\right)$, where $\lambda \neq 0$ and $a \in \mathbb{Z}$. On the other hand, $g_f$ acts as $\left(t_1, \mu_{|G_f|} t_2\right)$, where $\mu_{|G_f|}$ is a $|G_f|$-root of unity. Thus, the relation $g_f \tau g_f^{-1} \in \langle \tau \rangle$ holds if and only if $|g_f| = |G_f| \leq 4$. We deduce that for $|G_f| > 4$, the finite map $C_0 \to C$ must be unramified. Since $C$ is isomorphic to $\mathbb{P}^1$, we conclude that $C_0$ must have two connected components.

We conclude that the statement of the proposition holds for $N := \max\{5, 4M\}$. $\qquad\square$

### 2.8. Surface fibrations with large base automorphisms

In this subsection, we study $G$-equivariant fibrations of Fano-type surfaces such that the induced automorphism on the base is large.

**Proposition 2.25.** *Let $M$ be a positive integer. There exists a positive integer $N := N(M)$, depending only on $N$, satisfying the following: Assume that the following conditions hold:*

1. *$X$ is a Fano-type surface, $G \leq \mathrm{Aut}(X)$ is a finite automorphism group and $\rho^G(X) = 2$.*
2. *$(X, B)$ is a $G$-invariant log Calabi–Yau pair such that $M(K_X + B) \sim 0$.*
3. *$X \to C$ is a $G$-equivariant fibration to a curve.*
4. *$G_b$ is an abelian group with $|G_b| \geq N$.*

5. *The fibres over $\{0\}$ and $\{\infty\}$ are G-prime components of $\lfloor B \rfloor$.*
6. *The second extremal ray of the G-equivariant cone of curves defines a birational contraction.*

*Then $(X, B)$ satisfies one of the following:*

1. *It is a G-equivariant log Calabi–Yau toric pair or*
2. *it $(X, B)$ admits a $\mathbb{G}_m$-action with $H < \mathbb{G}_m \leq \mathrm{Aut}(X, B)$, and $H \leq G$ has index bounded by $N$.*

*If we assume that $G_f$ is abelian with $|G_f| \geq N$, the first case holds.*

*Proof.* Let $F_0$ and $F_\infty$ be the fibres at zero and infinity for the morphism $X \to C$. By assumption, they are $G$-prime components of $\lfloor B \rfloor$. Let $X \to X_0$ be the $G$-equivariant morphism defined by the second extremal ray of the $G$-equivariant cone of curves. We are assuming it is a birational map. Note that every irreducible component of the exceptional locus of $X \to X_0$ is horizontal over $C$. Hence, such components intersect both $F_0$ and $F_\infty$. We can apply the connectedness of log canonical centres to conclude that all fibres of $X \to X_0$ appear in $\lfloor B \rfloor$. Note that $X \to X_0$ has at most two irreducible components contained in its exceptional locus. Furthermore, each such exceptional curve intersects $F_0$ and $F_\infty$ exactly once. Otherwise, the induced pair $(X_0, B_0)$ would have negative local complexity at the image of such an exceptional curve.

Consider the following diagram obtained by quotienting by the finite group action:

$$
\begin{array}{ccc}
(X, B) & \xrightarrow{\;/G\;} & (Y, B_Y) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\
C & \xrightarrow{\;/G_b\;} & C_Y.
\end{array}
$$

Let $(C_Y, B_Y)$ be the pair obtained by the canonical bundle formula. By assumption, $(C_Y, B_Y) \simeq \left( \mathbb{P}^1, \{0\} + \{\infty\} \right)$ and $\rho(Y) = 2$. Furthermore, all fibres of $\pi_Y$ are irreducible. We will consider three different cases depending on the number and coefficients of the horizontal components of $B$.

*Case 1:* We assume that $\lfloor B \rfloor$ has a unique horizontal prime component which is unramified over $C$.

Then $B$ has at least one other horizontal component. Furthermore, $\lfloor B \rfloor_{\mathrm{hor}}$ is $G$-invariant. Note that $F_0, F_\infty$ and $\lfloor B \rfloor_{\mathrm{hor}}$ are $G$-invariant divisors. Then $\gamma^G(X, B) < 1$ and $\gamma(Y, B_Y) < 1$. By [BMSZ18, Theorem 1.2], we conclude that $(Y, \lfloor B_Y \rfloor)$ is a toric pair.

**Claim 2.26.** *There is a prime divisor $P$ on $X$ such that $(X, \lfloor B \rfloor + P)$ is a G-invariant Calabi–Yau toric pair.*

*Proof.* Assume that $X \to Y$ is unramified over $Y \setminus \mathrm{supp}(\lfloor B_Y \rfloor)$. Let $P_Y$ be the prime divisor on $Y$ such that $(Y, \lfloor B_Y \rfloor + P_Y)$ is log Calabi–Yau toric. Let $(X, D)$ be the log pullback of $(Y, \lfloor B_Y \rfloor + P_Y)$. Note that the preimage $P$ of $P_Y$ is prime. Since $X \to Y$ is unramified over the torus, we conclude that $(X, D)$ is a log Calabi–Yau toric pair.

From now on, we assume that $X \to Y$ ramifies over $Y \setminus \mathrm{supp}(\lfloor B_Y \rfloor)$. Observe that the finite morphism $X \to Y$ can ramify over at most one horizontal prime divisor $P$ on $Y \setminus \mathrm{supp}(\lfloor B_Y \rfloor)$. Otherwise, we would have $\gamma(Y, B_Y) < 0$. Indeed, any ramification divisor will appear with coefficient at least $\frac{1}{2}$. Let $P_Y$ be the image of $P$ on $Y$. Note that $P_Y$ does not intersect $\lfloor B_Y \rfloor_{\mathrm{hor}}$, or else the induced contraction $Y \to Y_0$ would have negative local complexity at the image of $\lfloor B_Y \rfloor_{\mathrm{hor}}$. We conclude that $P_Y$ is the closure of $\lambda \times \mathbb{G}_m \subset \mathbb{A}^1 \times \mathbb{G}_m \simeq Y \setminus \mathrm{supp}(\lfloor B_Y \rfloor)$ on $Y$. In particular, the log pair $(Y, \lfloor B_Y \rfloor + \delta P_Y)$ is log canonical for every $0 \leq \delta \leq 1$.

Hence the pair $(X, \lfloor B \rfloor + \epsilon P)$ is antinef, log canonical and $G$-invariant. Indeed, the push-forward of this pair to $Y$ has the form

$$
\left( Y, \lfloor B_Y \rfloor + \left( 1 - \frac{1 - \epsilon}{m} \right) P_Y \right),
$$

where $m$ is the multiplicity index at $P$. This pair has complexity strictly less than $\frac{1}{2}$. By [BMSZ18, Theorem 1.2], we conclude that $P_Y$ is a torus invariant divisor, and hence $(Y, \lfloor B_Y \rfloor + P_Y)$ is a log Calabi–Yau pair. We conclude that $(X, \lfloor B \rfloor + P)$ is a $G$-invariant log Calabi–Yau pair. We set $D := \lfloor B \rfloor + P$, which has two $G$-invariant horizontal components. By Theorem 2.21, we conclude that $(X, D)$ is a log Calabi–Yau toric pair. If $|G_f| \geq N$, then $(X, B)$ is a log Calabi–Yau toric pair as well. Indeed, in this case, the divisors $B$ and $D$ coincide. $\qquad\square$

We now prove the existence of the subgroup $H$. Let $g_b$ be the generator of $G_b$ and let $g_0 \in G$ be an element whose image on $G_b$ is $g_b$. Note that the divisors $\lfloor B \rfloor$, $P$, $F_0$ and $F_\infty$ are $G$-prime divisors. Up to replacing $g_0$ with $g_0^2$, we may assume that $g_0$ acts as $g_0 \cdot (t_1, t_2) = (\mu^a t_1, \mu^b t_2)$ on the big torus $\mathbb{G}_m^2$, where $\mu$ is an $n := |G|$-root of unity and $\mu^a$ is a $|G_b|$-root of unity. We distinguish two possible cases. Let $N_0$ be as in the statement of Lemma 2.14. If $n/\gcd(b, n)$ is larger than $N_0$, then $B = D$ and $(X, B)$ is a $G$-equivariant log Calabi–Yau toric pair. Indeed, in such a case, the only possible invariant horizontal divisors are $\lfloor B \rfloor$ and $P$. If $n/\gcd(b, n)$ is bounded by $N_0$, then we can consider the subgroup $H := \left\langle g_0^{n/\gcd(b,n)} \right\rangle \leq G$. Note that $H$ acts as $(\mu' t_1, t_2)$ on the big torus and furthermore has bounded index on $G$. Note that $H$ embeds naturally on the horizontal $\mathbb{G}_m$-action on the big torus given by $\lambda \cdot (t_1, t_2) = (\lambda t_1, t_2)$. It is clear that in this case we have $H \trianglelefteq \mathbb{G}_m H$. Observe that no horizontal components of $B_{\mathrm{hor}} \setminus \mathrm{supp}(\lfloor B \rfloor_{\mathrm{hor}})$ intersect $\lfloor B \rfloor_{\mathrm{hor}}$ nontrivially. Otherwise, the log pair $(X_0, B_0)$ would have a point of negative local complexity. We conclude that the boundary $B$ must be invariant with respect to the horizontal torus action, and hence the log pair $(X, B)$ is $\mathbb{G}_m$-invariant.

*Case 2:* We assume that $B$ has a unique horizontal component.

Furthermore, $\lfloor B \rfloor$ has a unique horizontal component which admits a ramified morphism to $C$. By Proposition 2.24, this case only happens if $|G_f| \leq N_0$. Hence $\lfloor B_Y \rfloor$ has a unique horizontal component $C_Z$ with a ramified morphism to $C_Y$. Let $Z$ be the normalisation of the main component of $Y \times_{C_Y} C_Z$, and let $(Z, B_Z)$ be the log pullback of $(Y, B_Y)$ to $Z$. Note that $Y$ is the quotient of $Z$ by an involution $\tau_Z$. We have the following commutative diagram:

$$
\begin{array}{ccc}
(Z, B_Z) & \xrightarrow{\ /\tau_Z\ } & (Y, B_Y) \\
{\scriptstyle \pi_Z}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Y} \\
C_Z & \xrightarrow{\ /\tau\ } & C_Y.
\end{array}
$$

Observe that all fibres of $Z \to C_Z$ are irreducible outside the points at zero and infinity, and note that $\lfloor B_Z \rfloor$ has $k_0 + k_\infty + 2$ prime components, where $k_0$ is the number of components over zero and $k_\infty$ is the number of components over infinity. Furthermore, $\rho(Z) \leq k_0 + k_\infty$. By Theorem 2.18, we know that $(Z, B_Z)$ is a log Calabi–Yau toric pair. Let $W$ be the normalisation of a connected component of $X \times_Y Z$, and note that $X$ is the quotient of $W$ by an involution $\tau_W$. Let $(W, B_W)$ be the log pair obtained by pulling back $(X, B)$ to $W$. We obtain the diagram

$$
\begin{array}{ccc}
(X, B) & \xleftarrow{\ /\tau_W\ } & (W, B_W) \\
{\scriptstyle /G}\big\downarrow & & \big\downarrow \\
(Y, B_Y) & \xleftarrow{\ /\tau_Z\ } & (Z, B_Z).
\end{array}
$$

Note that $B_W$ is a reduced divisor. Observe that $X \to Y$ is unramified over $Y \setminus \mathrm{supp}(\lfloor B_Y \rfloor)$ and $Z \to Y$ is unramified over $Y \setminus \mathrm{supp}(\lfloor B_Y \rfloor)$. We conclude that $W \to Z$ is unramified over $Z \setminus \mathrm{supp}(\lfloor B_Z \rfloor)$. Thus, the pair $(W, B_W)$ is a log Calabi–Yau toric pair. Observe that $W$ is equipped with a $\tau_W$-invariant fibration to a curve $C_W$. By construction, the induced log pair structure in $C_W$ is isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$.

Let $G_0$ be a cyclic subgroup of $G$ surjecting onto $G_b$ and let $g_0$ be the generator of $G_0$. Up to replacing $G_f$ and $G_0$ with an index 2 subgroup, we may assume that both of them act on $W$ as automorphism

groups. Let $(t_1, t_2)$ be the coordinates of the big torus of $W$. The involution acts as $\tau \cdot (t_1, t_2) = \left( -t_1, \lambda t_1^a t_2^{-1} \right)$, where $\lambda \neq 0$ and $a \in \mathbb{Z}$. Furthermore, up to replacing $g_f$ (resp., $g_0$) with $g_f^2$ (resp., $g_0^2$), we may assume that $g_f$ acts as $g_f \cdot (t_1, t_2) = (t_1, \mu t_2)$ and $g_0$ acts as $g_0 \cdot (t_1, t_2) = \left( \eta^{b_1} t_1, \eta^{b_2} t_2 \right)$. Here, as usual, $\mu$ and $\eta$ are roots of unity. Furthermore, the relation $g_0 \tau_W g_0^{-1} \in \langle \tau_W \rangle$ holds. We conclude that $g_0$ acts as $g_0 \cdot (t_1, t_2) = \left( \eta^{b_1} t_1, \eta^{b_2} t_2 \right)$ with $\eta^{2b_2 - ab_1} = 1$. Then there exists an index 2 subgroup $H_0$ of $G_0$ acting as $h_0 \cdot (t_1, t_2) = \left( \eta^{2b_1} t_1, t_2^{ab_1} \right)$. Note that $H$ naturally embeds in the horizontal torus $\mathbb{G}_m$ acting as $t \cdot (t_1, t_2) = \left( t^2 t_1, t^a t_2 \right)$. Furthermore, the $\mathbb{G}_m$-action commutes with $\tau_W$, $g_f$ and $g_b$. Hence, we conclude that $X$ is endowed with a horizontal $\mathbb{G}_m$-action such that $(X, B)$ is $\mathbb{G}_m$-invariant. Furthermore, we have $H \trianglelefteq \mathbb{G}_m H$, by construction. Hence $H$ admits a monomorphism into $\mathbb{G}_m$.

*Case 3:* We assume that $\lfloor B \rfloor$ has two horizontal components $S_0$ and $S_\infty$, which are not $G$-invariant.

In this case, the $G$-orbit of $S_0$ contains $S_\infty$, and hence the effective divisor $\lfloor B_Y \rfloor$ has a unique horizontal component $C_Z$ with a ramified morphism to $C_Y$. Let $Z$ be the normalisation of the main component of $Y \times_{C_Y} C_Z$ and let $(Z, B_Z)$ be the log pullback of $(Y, B_Y)$ to $Z$. Note that $Y$ is the quotient of $Z$ by an involution $\tau_Z$. Analogous to case 2, we deduce that $(Z, B_Z)$ is a log Calabi–Yau toric pair.

Let $W$ be the normalisation of a connected component of $X \times_Y Z$. Note that $X$ is the quotient of $W$ by an involution $\tau_W$. Let $(W, B_W)$ be the log pair obtained by pulling back $(X, B)$ to $W$. We know that $W \to Z$ is unramified over $Z \setminus \operatorname{supp}(\lfloor B_Z \rfloor)$, and hence $(W, B_W)$ is a log Calabi–Yau toric pair. Note that the involution $\tau_W$ acts on the log pair $(W, B_W)$. In particular, it restricts to an involution of $\mathbb{G}_m^2$. Moreover, since $B$ has two horizontal components over $C$, we conclude that $\tau_W$ must act on $\mathbb{G}_m^2$ as multiplication by roots of unity. Hence, we deduce that $(X, B)$ is a toric pair as well, and thus is a $G$-equivariant log Calabi–Yau toric pair. $\qquad\square$

## 3. Proof of the main theorems

In this section we prove the two main theorems of this article. We start by proving the case in which the finite automorphism group has large 2-generation. Theorem 2 is a particular case of the following theorem for $G$-invariant log pairs:

**Theorem 3.1.** *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $N := N(\Lambda)$, depending only on $\Lambda$, satisfying the following: Let $X$ be a Fano-type surface and $\Delta$ be a boundary on $X$ such that the following conditions hold:*

1. *$G \leq \operatorname{Aut}(X)$ is a finite subgroup with $g_2(G) \geq N$.*
2. *$(X, \Delta)$ is log canonical and $G$-invariant.*
3. *The coefficients of $\Delta$ belong to $\Lambda$.*
4. *$-(K_X + \Delta)$ is $\mathbb{Q}$-complemented.*

*Then there exist*

1. *a normal abelian subgroup $A \leq G$ of index at most $N$,*
2. *a boundary $B \geq \Delta$ on $X$ and*
3. *an $A$-equivariant birational map $X \dashrightarrow X'$*

*satisfying the following conditions:*

1. *The pair $(X, B)$ is log canonical and $G$-invariant, and $(K_X + B) \sim 0$.*
2. *The log crepant transform $(X', B')$ of $(X, B)$ on $X'$ is a log pair.*
3. *$(X', B')$ is a log Calabi–Yau toric pair.*
4. *There are group monomorphisms $A < \mathbb{G}_m^2 \leq \operatorname{Aut}(X', B')$.*

*In particular, $B'$ is the reduced toric boundary of $X'$.*

*Proof.* We prove the statement of the theorem in several steps. We either reduce to the case of del Pezzo surfaces or induce a $G$-equivariant Mori fibre space. In the latter case, we will have large $G_f$ and $G_b$

actions, and hence we can use the techniques developed in Sections 2.7 and subsection 2.8. We deduce that in some crepant equivalent model, the effective divisor $\lfloor B \rfloor$ has several prime components. Then we can finish the proof using complexity.

*Step 1:* We construct a normal abelian subgroup of $G$ of bounded index.

We can apply the Jordan property for finite birational automorphism groups of Fano-type varieties [PS16] . There exists a normal abelian subgroup $G_0 \leq G$ whose index is bounded by $J$, a constant that depends only on the dimension of $X$. By Lemma 2.8, we have $g_2(G_0) \geq g_2(G)/J$.

*Step 2:* We produce a $G$-equivariant complement and run a $G_0$-equivariant minimal model program.

By Theorem 2.6, we know that there exists a $G$-invariant log canonical $M$-complement $(X, B)$ for $(X, \Delta)$. Note that the positive integer $M$ depends only on the set $\Lambda$. By definition, $(X, B)$ is $G$-invariant and log canonical, and $M(K_X + B) \sim 0$. In particular, $(X, B)$ is also a $G_0$-invariant log canonical $M$-complement of $(X, \Delta)$. Let $Y$ be a $G_0$-equivariant log resolution of $(X, \Delta)$, and let $K_Y + B_Y$ be the log pullback of $K_X + B$ to $Y$. We may assume that $Y \to X$ does not extract divisors with log discrepancy larger than 1 with respect to the pair $(X, \Delta)$. Hence, $Y$ remains a Fano-type variety. Observe that the log pair $(Y, B_Y)$ is $G_0$-invariant as well. We run a $G_0$-equivariant minimal model for $K_Y$. All the steps of this minimal model program are $G_0$-equivariant crepant equivalent contractions for $(Y, B_Y)$. Since $Y$ is rational, this minimal model program terminates with a $G_0$-equivariant Mori fibre space $Y' \to W$. Note that $W$ is smooth, since the minimal model program for smooth surfaces always terminates with smooth surfaces. We denote by $(Y', B_{Y'})$ the log crepant transform of the log pair $(Y, B_Y)$ to $Y'$.

*Step 3:* We prove the statement in the case where the $G_0$-Mori fibre space has a 0-dimensional base.

Assume that this minimal model program terminates with a $G_0$-equivariant Mori fibre space $Y' \to W = \mathrm{Spec}(\mathbb{K})$. Then $\rho^{G_0}(Y') = 1$ and $Y'$ is a smooth Fano-type surface. We conclude that $-K_{Y'}$ is an ample divisor, and thus $Y'$ is a $G_0$-equivariant del Pezzo surface. By Proposition 2.15, there exists a constant $N_0 := N_0(M)$ such that $(Y', B_{Y'})$ is a log Calabi–Yau toric surface, provided that $g_2(G_0) \geq N_0$. In particular, we have $K_{Y'} + B_{Y'} \sim 0$. Thus, we conclude that $K_Y + B_Y \sim 0$ and $K_X + B \sim 0$ hold as well. Moreover, by Proposition 2.15, there is a subgroup $A \leq G_0$ with a monomorphism $A < \mathbb{G}_m^2 \leq \mathrm{Aut}(Y', B_{Y'})$ and the index of $A$ in $G_0$ is bounded by $N_0$. We conclude that $A$ is a normal subgroup of $G$ with index bounded by $N_0 J$. We set $X' := Y'$. Provided that $N \geq N_0 J$, the statement of the theorem holds for the $A$-equivariant birational map $X \dashrightarrow X'$.

*Step 4:* We assume that the $G_0$-Mori fibre space maps to a curve.

From now on, we may assume that this minimal model program terminates with a $G_0$-equivariant Mori fibre space to a curve $Y' \to C$. We prove that the log pairs induced on the base and the general fibre are log Calabi–Yau toric curves. Since $Y'$ is rational, we conclude that $C \simeq \mathbb{P}^1$. We denote by $F$ the general fibre, which is isomorphic to $\mathbb{P}^1$. Denote by $B_F$ the restriction of the boundary to the general fibre. We have an exact sequence

$$1 \to G_f \to G_0 \to G_b \to 1,$$

where $G_f$ is the subgroup of $G_0$ acting on a general fibre $F$ and $G_b$ is the quotient group of $G_0$ acting on the base $C$. We may assume that $G_f$ (resp., $G_b$) is a cyclic group acting as multiplication of a root unity on $F \simeq \mathbb{P}^1$ (resp., $C \simeq \mathbb{P}^1$). In particular, $G_0$ is a finite abelian group of rank 2. By Lemma 2.9, we conclude that $\min\{|G_f|, |G_b|\} \geq g_2(G_0)$. Note that $M(K_F + B_F) \sim 0$ is $G_0$-invariant. On the other hand, by Proposition 2.7 we can write $q(K_C + B_C) \sim 0$, where $q$ depends only on $M$. By Lemma 2.14, we conclude that for $g_2(G_0) \geq \max\{4M, 4q\}$, the pairs $(F, B_F)$ and $(C, B_C)$ are isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$.

*Step 5:* We reduce to the case in which the second $G_0$-equivariant extremal contraction of $Y'$ is a fibration.

Assume that the second $G_0$-equivariant extremal contraction of $Y'$ is not a fibration; then it is a $G_0$-equivariant birational contraction. Passing to a $G_0$-equivariant crepant equivalent birational model of $(Y', B_{Y'})$, we may assume that the fibres over zero and infinity are $G_0$-prime and contained in $\lfloor B_{Y'} \rfloor$. By Proposition 2.25, we know that $(Y', B_{Y'})$ is a log Calabi–Yau toric pair, provided that

$g_2(G_0) \geq N_1 := N_1(M)$. In particular, $K_{Y'} + B_{Y'} \sim 0$, so $K_X + B \sim 0$. Note that $(Y', B_{Y'})$ is a log Calabi–Yau toric pair and $G_0$ fixes $\lfloor B_{Y'} \rfloor$. Then we have a monomorphism $G_0 < \mathrm{Aut}\left(\mathbb{G}_m^2\right) \simeq \mathbb{G}_m^2 \rtimes \mathrm{Gl}_2(\mathbb{Z})$. By Lemma 2.12, there exists a subgroup $A \leq G_0$ of index bounded by $l := l(J)$ such that $A < \mathbb{G}_m^2 \leq \mathrm{Aut}(Y', B_{Y'})$. We conclude that $A \leq G$ is a normal subgroup of index bounded by $lJ$ for which $A \leq \mathbb{G}_m^2$. We set $X' := Y'$. We conclude that the statement of the theorem holds for the $A$-equivariant birational map $X \dashrightarrow X'$.

*Step 6:* We prove the statement in the case where the second $G_0$-equivariant extremal contraction is a fibration.

In this case, we have two $G_0$-equivariant fibrations $\pi_1 : Y' \to C_1$ and $\pi_2 : Y' \to C_2$. Let $G_{f_i}$ and $G_{b_i}$ be the groups acting on a general fibre and the base of each contraction. By assumption, $\min\left\{|G_{f_i}|, |G_{b_i}|\right\} \geq N$ for each $i$. Proceeding as in the third step for both contractions, we conclude that the log pairs induced on $C_1$ and $C_2$ are isomorphic to $\left(\mathbb{P}^1, \{0\} + \{\infty\}\right)$. $\lfloor B_{Y'} \rfloor$ has two $G$-prime components $F_0$ and $F_\infty$ over $\{0\}$ and $\{\infty\}$ of $C_1$, respectively. These $G$-prime components are disjoint. On the other hand, since $\{0\}$ is a log canonical centre on $C_2$, we conclude that $\lfloor B_{Y'} \rfloor$ has a log canonical centre which maps to $0 \in C_2$. We can apply the connectedness of log canonical centres to conclude that $\lfloor B_{Y'} \rfloor$ contains a sequence of curves in $\pi_2^{-1}(\{0\})$ connecting the two vertical components $F_0$ and $F_\infty$. In particular, $\lfloor B_{Y'} \rfloor$ contains a $G$-prime component $S_0$ over $\{0\} \in C_2$. Analogously, $\lfloor B_{Y'} \rfloor$ contains a $G$-prime component $S_\infty$ over $\{\infty\} \in C_2$. We conclude that $\lfloor B_{Y'} \rfloor$ has at least four $G_0$-prime components of coefficient 1: $F_0$, $F_\infty$, $S_0$ and $S_\infty$. Note that $\gamma^{G_0}(Y', B_{Y'}) = 0$. By Theorem 2.21, we conclude that $(Y', B_{Y'})$ is a $G_0$-equivariant toric log Calabi–Yau pair. Indeed, $\lfloor B_{Y'} \rfloor = F_0 + F_\infty + S_0 + S_\infty$. In particular, $K_{Y'} + B_{Y'} \sim 0$, so $K_X + B \sim 0$. Note that $(Y', B_{Y'})$ is a log Calabi–Yau toric pair and $G_0$ fixes $B_{Y'}$. Then we have a monomorphism $G_0 < \mathrm{Aut}\left(\mathbb{G}_m^2\right) \simeq \mathbb{G}_m^2 \rtimes \mathrm{Gl}_n(\mathbb{Z})$. By Lemma 2.12, there exists a subgroup $A \leq G_0$ of index bounded by $l := l(J)$ such that $A < \mathbb{G}_m^2 \leq \mathrm{Aut}(Y', B_{Y'})$. In particular, $A \leq G$ is a normal subgroup of index bounded by $lJ$. We set $X' := Y'$, and conclude that the statement of the theorem holds for the $A$-equivariant birational map $X \dashrightarrow X'$. □

Now we prove a characterisation of Fano-type surfaces with large cyclic automorphism groups. Theorem 1 is a particular case of the following theorem for $G$-invariant log pairs:

**Theorem 3.2.** *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $N := N(\Lambda)$, depending only on $\Lambda$, satisfying the following: Let $X$ be a Fano-type surface and $\Delta$ be a boundary on $X$ such that the following conditions hold:*

1. *$G := \mathbb{Z}_m \leq \mathrm{Aut}(X)$ with $m \geq N$.*
2. *$(X, \Delta)$ is log canonical and $G$-invariant.*
3. *The coefficients of $\Delta$ belong to $\Lambda$.*
4. *$-(K_X + \Delta)$ is $\mathbb{Q}$-complemented.*

*Then there exist*

1. *a subgroup $A \leq G$ of index at most $N$,*
2. *a boundary $B \geq \Delta$ on $X$ and*
3. *an $A$-equivariant birational map $X \dashrightarrow X'$*

*satisfying the following conditions:*

1. *The pair $(X, B)$ is log canonical and $G$-invariant, and $N(K_X + B) \sim 0$.*
2. *The log crepant transform $(X', B')$ of $(X, B)$ on $X'$ is a log pair.*
3. *$(X', B')$ admits a $\mathbb{G}_m$-action.*
4. *There are group monomorphisms $A < \mathbb{G}_m \leq \mathrm{Aut}(X', B')$.*

*Proof.* We prove the statement in several steps. The aim is to reduce the statement to the case in which we have a $G$-equivariant Mori fibre space with a large fibre-wise action. In such a case, we will use Proposition 2.22 to prove that such a $G_f$ is embedded in a fibre-wise $\mathbb{G}_m$-action.

*Step 1:* We produce a $G$-equivariant complement and run a $G$-equivariant minimal model program.

By Theorem 2.6, we know that there exists a $G$-invariant log canonical $M$-complement $(X, B)$ for $(X, \Delta)$. Recalll that $M$ depends only on $\Lambda$. By definition, $(X, B)$ is log canonical and $G$-equivariant, and $M(K_X + B) \sim 0$. We will denote by $Y$ a $G$-equivariant resolution of singularities of $(X, \Delta)$. Denote by $K_Y + B_Y$ the log pullback of $K_X + B$. Note that $(Y, B_Y)$ is a log canonical pair which is $G$-invariant. We run a $G$-equivariant minimal model program for $K_Y$. This minimal model program terminates with a $G$-equivariant Mori fibre space $Y' \to W$. We denote by $(Y', B_{Y'})$ the log crepant transform of the log Calabi–Yau pair $(Y, B_Y)$ to $Y'$. Note that $(Y', B_{Y'})$ is still a log Calabi–Yau pair.

*Step 2:* We prove the statement in the case that the $G$-Mori fibre space has a 0-dimensional base.

Assume that this minimal model program terminates with a $G$-equivariant Mori fibre space with 0-dimensional base $Y' \to W = \mathrm{Spec}(\mathbb{K})$. Then we have $\rho^G(Y') = 1$ and $Y'$ is a smooth Fano-type surface. Hence, $-K_{Y'}$ is an ample divisor, so $Y'$ is a $G$-equivariant del Pezzo surface. By Proposition 2.15, there exists a constant $N_0 := N_0(M)$ such that $|G| \geq N_0$ implies that the pair $(Y', B_{Y'})$ is a Calabi–Yau toric surface with a $\mathbb{G}_m$-action. Moreover, by Proposition 2.15, there exists a subgroup $A \leq G$ of index at most $N_0$ satisfying $A < \mathbb{G}_m^2 \leq \mathrm{Aut}(Y', B_{Y'})$. We set $X' := Y'$ and conclude that the statement of the theorem holds for the $G$-equivariant birational map $X \dashrightarrow X'$.

*Step 3:* We assume that the $G$-Mori fibre space maps to a curve. We prove that either the base or a general fibre is a log Calabi–Yau toric curve.

From now on, we may assume that this $G$-equivariant minimal model program terminates with a $G$-equivariant Mori fibre space to a curve $C$ which we denote by $Y' \to C$. Then, given that $Y'$ is rational, we conclude that $C \simeq \mathbb{P}^1$. On the other hand, the general fibre $F$ is also isomorphic to $\mathbb{P}^1$. We have an exact sequence

$$1 \to G_f \to G \to G_b \to 1,$$

where $G_f$ is the subgroup of $G$ acting on the general fibre $F$ and $G_b$ is the quotient group acting on the base $C \simeq \mathbb{P}^1$. We denote by $(F, B_F)$ the restriction of our log pair $(Y', B_{Y'})$ to the general fibre. We denote by $(C, B_C)$ the log pair obtained by the $G$-equivariant canonical bundle formula (Proposition 2.7). We may assume that $M(K_C + B_C)$, up to replacing $M$ with a bounded multiple. Since we are assuming that $|G| \geq N$, one of these groups must be of order at least $N/2$. Replacing $N$ with $2N$, we can just assume $\max\{|G_f|, |G_b|\} \geq N$. If $|G_f| \geq N$ and $N \geq 4M$, then $(F, B_F)$ is isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$. On the other hand, if $|G_b| \geq N$ and $N \geq 4M$, then $(C, B_C)$ is isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$. We will divide the proof into three cases: horizontal, vertical and mixed. In the horizontal (resp., vertical) case we assume that $|G_f| \leq N$ and $|G_b| \geq N$ (resp., $|G_b| \leq N$ and $|G_f| \geq N$). Finally, in the mixed case, we assume that both groups are large – that is, $\min\{|G_f|, |G_b|\} \geq N$. One of these cases must hold.

*Step 4:* We prove the statement in the horizontal case, where $|G_b| \geq N$ and $|G_f|$ is bounded by $N$.

We are assuming that $(C, B_C) \simeq (\mathbb{P}^1, \{0\} + \{\infty\})$. By performing a $G$-equivariant crepant equivalent birational modification of the pair $(Y', B_{Y'})$, we may assume that the fibres over zero and infinity are contained in $\lfloor B_{Y'} \rfloor$. By Proposition 2.25, we may assume that one of the following conditions is satisfied:

1. Both extremal rays of the $G$-equivariant cone of curves define fibrations.
2. The pair $(Y', B_{Y'})$ is a log Calabi–Yau toric pair.
3. The pair $(Y', B_{Y'})$ admits a $\mathbb{G}_m$-action, with $H < \mathbb{G}_m \leq \mathrm{Aut}(Y', B_{Y'})$, and $H \leq G$ has index bounded by $N$.

Assume that (1) is satisfied. In this case, we have two $G$-equivariant fibrations $\pi_1 \colon Y' \to C_1$ and $\pi_2 \colon Y' \to C_2$. Let $G_{f_i}$ and $G_{b_i}$ be the groups acting on a general fibre and base of each contraction. If $|G_{f_2}| \geq N$, then we reduce to the vertical case. We may assume that $\min\{|G_{b_1}|, |G_{b_2}|\} \geq N$. We conclude that the log pairs induced on $C_1$ and $C_2$ are isomorphic to $(\mathbb{P}^1, \{0\} + \{\infty\})$. Applying the connectedness of the log canonical centres for $\pi_1$ and $\pi_2$, we conclude that $B_{Y'}$ has four $G$-invariant irreducible components of coefficient 1. By Theorem 2.21, we conclude that $(Y', B_{Y'})$ is a $G$-equivariant toric log Calabi–Yau pair. Furthermore, a subgroup of $G$, of bounded index, acts as multiplication by a root of unity of a 1-dimensional subtorus of $\mathbb{G}_m^2$. We set $X' := Y'$ and conclude that the statement of

the theorem holds for the $G$-equivariant birational map $X \dashrightarrow X'$. The statement holds if (2) or (3) is satisfied.

*Step 5:* We prove the statement in the vertical case, where $\left| G_f \right| \geq N$ and $(F, B_F)$ is a $G_f$-equivariant pair.

We denote by $\pi \colon Y' \to C$ the $G$-equivariant Mori fibre space. As before, $(F, B_F)$ is the log Calabi–Yau toric curve. By [Amb05, Theorem 3.3], we conclude that the moduli part of the $G$-equivariant canonical bundle formula is a $\mathbb{Q}$-trivial birational divisor. In particular, there exist an open set $U \subset C \simeq \mathbb{P}^1$ and a Galois cover $V \to U$, such that

$$V \times_U (Y', B_{Y'}) \simeq V \times (F, B_F).$$

In $V \times (F, B_F)$ we have a $\mathbb{G}_m$-action on the second component. On the other hand, in $V \times (F, B_F) \simeq V \times (\mathbb{P}^1, \{0\} + \{\infty\})$ we have a Galois action $G'$. Observe that $G'$ fixes $V \times \{0\}$ and $V \times \{\infty\}$; otherwise, $\lfloor B_{Y'} \rfloor$ would have a unique horizontal component over the base. Hence, $G'$ must act as multiplication of a root of unity on the fibres, so it commutes with $\mathbb{G}_m$. Thus we have an induced $\mathbb{G}_m$-action over $U$ which acts fibre-wise. We may replace $(Y', B_{Y'})$ with a $G$-equivariant dlt modification $(Y'', B_{Y''})$. Denote by $E$ the vertical components of $\lfloor B_{Y''} \rfloor$. Note that $E$ is $G$-invariant. Hence, we may run a $G$-equivariant $(K_{Y''} + B_{Y''} - \epsilon E)$-minimal model program over $C$ which terminates with a model $(Z, B_Z - \epsilon E_Z)$ over $C$. Here, as usual, $E_Z$ is the push-forward of $E$ to $Z$. Note that $\mathbb{G}_m$ is still acting fibre-wise over an open set of $C$. On the other hand, by Theorem 2.20 and Proposition 2.22, we conclude that the fibration $(Z, B_Z) \to C$ is everywhere $G$-equivariantly formally toric over the base – that is, for each point $c \in C$ there is a neighbourhood of $c \in C$ over which the morphism is toric. Hence, we conclude that the $\mathbb{G}_m$-action on the general fibres extends to all fibres, and thus the fibration $(Z, B_Z) \to C$ is the $\mathbb{G}_m$-quotient of a $\mathbb{G}_m$-action on the pair $(Z, B_Z)$.

Finally, observe that $G_f$ embeds in the $\mathbb{G}_m$ of the general fibre. Thus, it suffices to define $X' := Z$ and consider the $G$-equivariant map $X \dashrightarrow X'$ which satisfies all the conditions on the statement of the theorem.

*Step 6:* We prove the statement in the mixed case.

The mixed case follows from the proof of Theorem 3.1. In this case, using the same argument as before, we conclude that the model $(Z, B_Z)$ is indeed a toric pair. Thus, it suffices to define $X' := Z$ and consider the $G$-equivariant map $X \dashrightarrow X'$ which satisfies all the conditions on the statement of the theorem. □

## 4. Fano-type surfaces with large fundamental group of the smooth locus

In this section, we prove the first application of the main theorems. We give a characterisation of Fano-type surfaces with large fundamental group of the log smooth locus. First, we recall the definition of the fundamental group of the log smooth locus of a Fano-type variety.

**Definition 25.** Let $(X, \Delta)$ be a simple normal crossing pair with standard coefficients and $\mathcal{X}$ be the smooth orbifold which realises $(X, \Delta)$ as the coarse moduli space of $(\mathcal{X}, 0)$. We have a surjection of fundamental groups

$$\pi_1(X \setminus \mathrm{supp}(\Delta)) \to \pi_1(\mathcal{X}) \to 1.$$

The kernel of this homomorphism is generated by elements of the form $\gamma^n$, where $\gamma$ is the loop around a prime component of $D$ with coefficient $1 - \frac{1}{n}$. We may denote the group $\pi_1(\mathcal{X})$ by $\pi_1(X, \Delta)$ and call it the *fundamental group of the pair* $(X, \Delta)$. Note that the elements of $\pi_1(X, \Delta)$ correspond to finite covers of $X$ on which the log pullback of $K_X + \Delta$ remains a log pair.

Given a log pair $(X, \Delta)$ with standard coefficients, we denote by $\pi_1(X, \Delta)$ the fundamental group $\pi_1(X^0, \Delta^0)$, where $X^0$ is a big open set of $X$ on which the pair $(X, \Delta)$ has simple normal crossing singularities and $\Delta^0$ is the restriction of $\Delta$ to $X^0$. Our definition is independent of $X^0$.

Given an effective divisor $\Delta$ on a variety $X$, we denote by $\Delta_s$ the *lower standard approximation of* $\Delta$. This is the largest effective divisor $\Delta_s$ with standard coefficients such that $\Delta_s \le \Delta$. We define $\pi_1(X, \Delta)$ to be $\pi_1(X, \Delta_s)$. Note that $\pi_1(X, \Delta)$ corresponds to finite covers of $X$ on which the log pullback of $(X, B)$ remains a log pair.

The following theorem is folklore after the finiteness on the fundamental group of the smooth locus of a weak Fano surface [FKL93, GZ94, GZ95, KM99] and the Borisov–Alexeev–Borisov conjecture in dimension 2 [Ale94] :

**Theorem 4.1.** *Let $X$ be a Fano-type surface and $B$ be a boundary on $X$ such that $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is $\mathbb{Q}$-complemented. Then $\pi_1(X, \Delta)$ is a finite group.*

*Proof.* Since $X$ is a Fano-type surface, it is a Mori dream space. Furthermore, $X$ has $\mathbb{Q}$-factorial singularities. Replacing $\Delta$ with its lower standard approximation, we may assume that $\Delta$ has standard coefficients. We run a minimal model program for $-(K_X + \Delta)$, with steps $X \to X_1 \to \cdots \to X_k$. Note that for each $i$, we have a surjective homomorphism $\pi_1(X_{i+1}, \Delta_{i+1}) \to \pi_1(X_i, \Delta_i)$. Hence, it suffices to prove that $\pi_1(X_k, \Delta_k)$ is finite, where $-(K_{X_k} + \Delta_k)$ is nef. Analogously, we have a surjective homomorphism $\pi_1(Y, \Delta_Y) \to \pi_1(X_k, \Delta_k)$, where $Y$ is the ample model of $-(K_{X_k} + \Delta_k)$. The finiteness of $\pi_1(Y, \Delta_Y)$ follows from [FKL93] . $\qquad\square$

Theorem 3 is a particular case of the following theorem for log pairs:

**Theorem 4.2.** *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $N := N(\Lambda)$, depending only on $\Lambda$, satisfying the following: Let $X$ be a Fano-type surface and $\Delta$ be a boundary on $X$ with coefficients in $\Lambda$, such that $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is $\mathbb{Q}$-complemented. If $G_0 \le \pi_1(X, \Delta)$ satisfies $g_2(G_0) \ge N$, then $X$ is a log crepant equivalent toric quotient. Furthermore, there exists a finite cover of $(X, \Delta)$ of degree at most $N$ which is log crepant equivalent toric.*

*Proof.* By Theorem 4.1, we know that the fundamental group $G := \pi_1(X, \Delta)$ is finite. By Lemma 2.11, we know that $G$ satisfies $g_2(G) \ge N$. Let $X^0 \subset X$ be a big open subset such that $\pi_1(X, \Delta) = \pi_1(X^0, \Delta_s^0)$, where $\Delta_s$ is the lower standard approximation of $\Delta^0$. Let $(Y^0, \Delta^0)$ be the universal cover of $(X^0, \Delta_s^0)$. Define $Y$ as the normal closure of $X$ on the field of fractions of $Y^0$. We have an action of $G$ on $Y$, which we note is just the regularisation of the birational action of $G$ on $Y^0$ as in [Che04] . By construction, the pullback of $(X, \Delta)$ to $Y$ is a log pair. Let $(X, B)$ be an $M$-complement for $(X, \Delta)$; then the pullback $(Y, B_Y)$ of $(X, B)$ to $Y$ is a $G$-invariant $M$-complement for the log pair $(Y, \Delta_Y)$. By Theorem 3.1, we conclude that $(Y, B_Y)$ is a crepant equivalent toric pair. Hence $(X, B)$ is a crepant equivalent toric quotient pair, and so $X$ is a log crepant equivalent toric quotient variety.

Let $Y \dashrightarrow Y'$ be a birational map, crepant equivalent for $(Y, B_Y)$, such that $(Y', B_{Y'})$ is a log Calabi–Yau toric pair. By Theorem 3.1, we know that there exists a normal abelian subgroup $A \le G$ of index at most $N$ such that $Y \dashrightarrow Y'$ is $A$-equivariant and $A < \mathbb{G}_m^2 \le \mathrm{Aut}(Y', B_{Y'})$. Let $(Z, B_Z)$ be the quotient of $(Y, B_Y)$ by $A$; then $(Z, B_Z)$ is log crepant equivalent to the quotient $(Z', B_{Z'})$ of $(Y', B_{Y'})$ by $A$. Since $A$ is a subgroup of the torus of $Y'$, the log pair $(Z', B_{Z'})$ is still a log Calabi–Yau toric pair. We conclude that $(Z, B_Z)$ is a cover of $(X, B)$ of degree at most $N$ which is log crepant equivalent toric. $\qquad\square$

**Theorem 4.3.** *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $N := N(\Lambda)$, depending only on $\Lambda$, satisfying the following: Let $X$ be a Fano-type surface and $\Delta$ be a boundary on $X$ with coefficients in $\Lambda$, such that $(X, \Delta)$ is klt and $-(K_X + \Delta)$ is $\mathbb{Q}$-complemented. If $G_0 \le \pi_1(X, \Delta)$ satisfies $|G_0| \ge N$, then $(X, \Delta)$ admits a finite cover of degree at most $N$ which has a log crepant equivalent torus action.*

*Proof.* By Theorem 4.1, we know that the fundamental group $G := \pi_1(X, \Delta)$ is finite. Let $X^0 \subset X$ be a big open subset such that $\pi_1(X, \Delta) = \pi_1(X^0, \Delta_s^0)$, where $\Delta_s^0$ is the lower standard approximation of $\Delta^0$. Let $(Y^0, \Delta^0)$ be the universal cover of $(X^0, \Delta_s^0)$ and define $Y$ to be normal closure of $X$ on the field of fractions of $Y^0$. We have an automorphism action of $G$ on $Y$, which we note is just the regularisation of

the birational action of $G$ on $Y^0$ as in [Che04] . By construction, the pullback of $(X, \Delta)$ to $Y$ is a log pair. Let $(X, B)$ be an $M$-complement for $(X, \Delta)$; then the pullback $(Y, B_Y)$ of $(X, B)$ to $Y$ is a $G$-invariant $M$-complement for the log pair $(Y, \Delta_Y)$. If $g_2(G) \geq N$, then we can apply Theorem 4.2. Otherwise, $G$ contains a cyclic subgroup of bounded index. By Theorem 3.2, we deduce that $(Y, B_Y)$ admits a crepant equivalent torus action. Let $(Y', B_{Y'})$ be the crepant equivalent model with a torus action. There exists a subgroup $A \leq G$ of index at most $N$ such that $A < \mathbb{G}_m \leq \operatorname{Aut}(Y', B_{Y'})$. We conclude that the quotient of $(Y, B_Y)$ by $A$ is a cover of $(X, B)$ of degree at most $N$ which admits a log crepant equivalent torus action. $\qquad \square$

## 5. Degenerations of klt 3-fold singularities

In this section we study degenerations of klt 3-fold singularities, provided that their local fundamental group is large enough – that is, it has high rank and order of the generators.

**Definition 26.** Let $(X, \Delta)$ be a log pair. We define $\pi_1^{\operatorname{reg}}(X, \Delta)$ to be the inverse limit of $\operatorname{Aut}_X(X_i)$ with $i \in I$, where $I$ indexes the projective system of Galois quasi-étale covers $p_i \colon X_i \to X$ such that $p_i^*(K_X + B) = K_{X_i} + B_{X_i}$, with $B_{X_i} \geq 0$.

Let $(X, \Delta)$ be a log pair and $x \in X$ be a closed point. We define $\pi_1^{\operatorname{alg}}((X, \Delta); x)$ to be the inverse limit of $\pi_1(U, \Delta_U)$, where $U$ runs over all the étale neighbourhoods of $x$ in $X$ and $\Delta_U$ is defined by log pullback.

In what follows, we will work with the algebraic local fundamental group instead, since the finiteness of local fundamental groups of klt 3-fold singularities is not known (see, e.g., [TX17] for the canonical case). The algebraic local fundamental group of a klt singularity is finite [Xu14, Sti17] .

**Theorem 5.1.** *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $N := N(\Lambda)$, depending only on $\Lambda$, satisfying the following: Let $x \in (X, \Delta)$ be a klt 3-fold singularity such that the coefficients of $\Delta$ belong to $\Lambda$. If $G \leq \pi_1^{\operatorname{alg}}((X, \Delta); x)$ satisfies $g_3(G) \geq N$, then $x \in (X, \Delta)$ degenerates to an lce-tq singularity.*

*Proof.* Let $\pi \colon Y \to X$ be a plt blowup of $(X, \Delta)$ at $x$ and let $(X, B)$ be an $M$-complement of $(X, \Delta)$ such that the exceptional divisor of $\pi$ is a log canonical centre of $(X, B)$. Consider the universal cover of $((X, \Delta); x)$ denoted by $\phi \colon X' \to X$. Let $\pi' \colon Y' \to X'$ be the normalisation of the fibre product of these morphisms. Note that $\pi'$ is a plt blowup of $x' \in (X', \Delta')$, where $(X', \Delta')$ is the log pair obtained by pulling back $(X, \Delta)$ to $X'$. Furthermore, the log pullback of $(X, B)$ to $X'$, denoted by $(X', B')$, is an $M$-complement. Note that $x' \in X'$, the preimage of $x \in X$, is a fixed point for the automorphism group action of $G$ on $X'$; hence $G$ acts on $Y'$ and fixes the exceptional divisor $E'$. By construction, $E'$ is a Fano-type surface. Furthermore, pulling back $(X', B')$ to $Y'$ and restricting to $E'$, we obtain a $G_{E'}$-invariant $M$-complement $(E', B_{E'})$, where $G_{E'}$ is the quotient group of $G$ acting faithfully on $E'$. By Lemma 2.10, $g_2(G_{E'}) \geq N$. By Theorem 3.1, we conclude that $(E', B_{E'})$ is log crepant equivalent toric. We conclude that the quotient $(E, B_E)$ of $(E', B_{E'})$ by $G$ is a log crepant equivalent toric quotient. By [LX20, §2.4], we conclude that $x \in (X, \Delta)$ degenerates to the cone over $E$. Hence, $(X, \Delta)$ degenerates to the cone over a log crepant equivalent toric quotient, and thus $x \in (X, \Delta)$ degenerates to an lce-tq singularity. $\qquad \square$

**Theorem 5.2.** *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a positive integer $N := N(\Lambda)$, depending only on $\Lambda$, satisfying the following: Let $x \in (X, \Delta)$ be a klt 3-fold singularity such that the coefficients of $\Delta$ belong to $\Lambda$. If $G \leq \pi_1^{\operatorname{alg}}((X, \Delta); x)$ satisfies $g_2(G) \geq N$, then $x \in (X, \Delta)$ has a finite cover of degree at most $N$ which degenerates to the cone over a projective variety with a log crepant equivalent torus action.*

*Proof.* Let $\pi \colon Y \to X$ be a plt blowup of $(X, \Delta)$ at $x$ and let $(X, B)$ be an $M$-complement of $(X, \Delta)$ such that the exceptional divisor of $\pi$ is a log canonical centre of $(X, B)$. Consider the universal cover of $((X, \Delta); x)$ denoted by $\phi \colon X' \to X$. Let $\pi' \colon Y' \to X'$ be the normalisation of the fibre product of these morphisms. Note that $\pi'$ is a plt blowup of $x' \in (X', \Delta')$, where $(X', \Delta')$ is the log pair obtained

by pulling back $(X, \Delta)$ to $X'$. Furthermore, the log pullback of $(X, B)$ to $X'$, denoted by $(X', B')$, is an $M$-complement. Note that $x' \in X'$, the preimage of $x \in X$, is a fixed point for the automorphism group action of $G$ on $X'$. $G$ acts as an automorphism group of $Y'$ and fixes the exceptional divisor $E'$. By construction, $E'$ is a Fano-type surface. Furthermore, pulling back $(X', B')$ to $Y'$ and restricting to $E'$, we obtain a $G_{E'}$-invariant $M$-complement $(E', B_{E'})$, where $G_{E'}$ is the quotient group of $G$ acting as an automorphism group of $E'$. Observe that $g_1(G_{E'}) \geq 1$. By Theorem 3.2, we conclude that $(E', B_{E'})$ admits a crepant equivalent torus action. Furthermore, there is a subgroup $A \leq G_{E'}$ of bounded index such that the quotient of $(E', B_{E'})$ by $A$ still admits a torus action. Let $A_G \leq G$ be a normal subgroup of $G$ surjecting onto $A$. By [LX20, §2.5], the cover of $x \in (X, \Delta)$ corresponding to $A_G$ degenerates to a cone over a pair with crepant equivalent $\mathbb{G}_m$-action. □

*Proof of Theorem 4.* Let $\mathrm{Cl}_0(X, x)$ be the subgroup of the local class group generated by $\mathbb{Q}$-Cartier divisors – that is, the torsion subgroup of $\mathrm{Cl}(X, x)$. Note that $G \leq \mathrm{Cl}_0(X, x)$, given that $G$ is finite. Consider

$$Y = \mathrm{Spec}\left( \bigoplus_{[D] \in \mathrm{Cl}_0(X, x)} H^0(X, \mathcal{O}_X(D)) \right).$$

Note that $Y \to X$ is a finite morphism. Indeed, it is the quotient by the finite group $\mathrm{Cl}_0(X, x)$. By Lemma 2.11, we have $g_3(\mathrm{Cl}_0(X, x)) \geq N$. Then we can proceed as in the proof of Theorem 5.1. □

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