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Self-Maps of Low Rank Lie Groups at Odd Primes

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Abstract. Let G be a simple, compact, simply-connected Lie group localized at an odd prime p. We study the group of homotopy classes of self-maps $[G, G]$ when the rank of G is low and in certain cases describe the set of homotopy classes of multiplicative self-maps $H[G, G]$. The low rank condition gives G certain structural properties which make calculations accessible. Several examples and applications are given.

1 Introduction

When studying any mathematical object it is a natural to ask what its self-maps are, as this often reveals an interesting structure. In homotopy theory the objects are topological spaces and the self-maps are homotopy classes of pointed, continuous self-maps. One collection of spaces that is interesting to study is Lie groups, as they are fundamental to many areas of mathematics. However, little is known about their homotopy classes of self-maps. It is classical that for $\mathrm{SU}(2) \simeq S^3$ the set of homotopy classes of self-maps is $\pi_3(S^3) \cong \mathbb{Z}$. Mimura and Oshima [MO] determined the set of homotopy classes of $\mathrm{SU}(3)$ and $\mathrm{Sp}(2)$. But for higher rank Lie groups the calculations quickly become overwhelming, largely due to an inability to control the 2-primary information.

In this paper we invert the prime 2 in order to get more information. To be precise, assume that all spaces are pointed, connected, topological spaces with the homotopy types of finite type CW-complexes. Assume that all spaces and maps have been localized at an odd prime p and homology is taken with mod-p coefficients. For spaces X and Z, let $[X, Z]$ be the set of homotopy classes of pointed, continuous maps. If X and Z are H-spaces, a distinguished subset of $[X, Z]$ is the set $H[X, Z]$ of homotopy classes of H-maps between X and Z. Let G be a simple, compact, simply-connected Lie group. In this paper we study $[G, G]$ when the rank of G is low and in certain cases describe $H[G, G]$. For example, we consider $[\mathrm{SU}(n), \mathrm{SU}(n)]$ when $n \leq (p-1)^2 + 1$ and $H[\mathrm{SU}(n), \mathrm{SU}(n)]$ when $2n < p$. The methods we use are also applicable in other cases, some of which will be indicated as we proceed.

We begin with a general theorem that decomposes certain homotopy classes of maps.

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Theorem 1.1 *Let Z be a homotopy associative H-space. Let X be a space such that $\Sigma X \simeq \bigvee_{i=1}^{t} \Sigma X_i$. Then there is an isomorphism of sets*

$$[X, Z] \cong \prod_{i=1}^{t} [X_i, Z].$$

Furthermore, if Z is also homotopy commutative, then the isomorphism is of abelian groups.

Theorem 1.1 is most useful when Z is not known to be a loop space. For if $Z = \Omega Z'$, the set isomorphism in Theorem 1.1 is a straightforward consequence of adjunction, although the group isomorphism in the commutative case requires a bit of work. An example which is not a loop space is S^{2n+1}, which is both homotopy associative and homotopy commutative when localized at $p \geq 5$. More generally, families of p-local finite torsion free H-spaces are constructed in [CHZ, CN] and work of the second author [Th1] gives conditions for when they are homotopy associative and homotopy commutative. Many interesting spaces satisfy the suspension condition on the domain in Theorem 1.1. For example, if $X = S_g$ is a surface of genus g, then $\Sigma S_g \simeq (\bigvee_{i=1}^{2g} \Sigma S^1) \vee \Sigma S^2$, or if $X = M$ is a simply-connected 4-manifold, then (at odd primes) $\Sigma M \simeq (\bigvee_{i=1}^{d} \Sigma S^2) \vee \Sigma S^4$ for some nonnegative integer d.

The spaces of primary interest that satisfy the suspension condition in Theorem 1.1 are torsion-free simple, compact, simply-connected Lie groups. The list of such groups and the relevant primes is: Spin(n) for $n \geq 3$ and $p \geq 3$; SU(n) for $n \geq 2$ and $p \geq 3$; Sp(n) for $n \geq 1$ and $p \geq 3$; G_2 for $p \geq 3$; F_4, E_6, and E_7 for $p \geq 5$, and E_8 for $p \geq 7$. In all such cases, we have $H_*(G) \cong \Lambda(x_{2n_1+1}, \ldots, x_{2n_t+1})$ where $n_1 < n_2 < \cdots < n_t$, the degree of x_{2n_i+1} is $2n_i + 1$, and t is the rank of G. Let $q = 2(p-1)$. It is well known that there is an algebra decomposition

$$H_*(G) \cong \bigotimes_{i=1}^{p-1} \Lambda(V_i),$$

where V_i consists of those generators in $\{x_{2n_1+1}, \ldots, x_{2n_t+1}\}$ whose degrees are of the form $2i + jq + 1$ for some $j \geq 0$. Note that, depending on G, it may be the case that $V_i = \varnothing$ for some i. Mimura, Nishida, and Toda [MNT2] realized this algebra decomposition geometrically by showing that there is a homotopy equivalence $G \simeq \prod_{i=1}^{p-1} \overline{B}_i$, where $H_*(\overline{B}_i) \cong \Lambda(V_i)$. This can be pushed further. Let l_i be the cardinality of V_i. For $0 \leq k \leq l_i$, let $\Lambda_k(V_i)$ be the submodule of $\Lambda(V_i)$ consisting of the elements of tensor length k. Then there is a module isomorphism

$$\Lambda(V_i) \cong \bigoplus_{k=0}^{l_i} \Lambda_k(V_i).$$

Thus there is a module isomorphism

$$H_*(G) \cong \bigoplus_{k_1, \ldots, k_{p-1}=0}^{l_1, \ldots, l_{p-1}} \Lambda_{k_1}(V_1) \otimes \cdots \otimes \Lambda_{k_{p-1}}(V_{p-1}).$$

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We will show that in low rank this module decomposition can be realized geometrically, in the following sense.

Theorem 1.2 *Let G be one of the following:* $\mathrm{SU}(n)$ *if* $n \leq (p-1)^2 + 1$*;* $\mathrm{Sp}(n)$ *if* $2n \leq (p-1)^2$*;* $\mathrm{Spin}(2n+1)$ *if* $2n \leq (p-1)^2$*;* $\mathrm{Spin}(2n)$ *if* $2(n-1) \leq (p-1)^2$*; G_2 if $p \geq 3$; F_4 or E_6 if $p \geq 5$; E_7 or E_8 if $p \geq 7$. Then for each $1 \leq i \leq p-1$ and $0 \leq k_i \leq l_i$ there are spaces S_{k_i} such that $\widetilde{H}_*(S_{k_i}) \cong \Lambda_{k_i}(V_i)$, and there is a homotopy decomposition*

$$\Sigma G \simeq \bigvee_{k_1,\dots,k_{p-1}=0}^{l_1,\dots,l_{p-1}} \Sigma S_{k_1} \wedge \cdots \wedge S_{k_{p-1}}.$$

(Here $\Lambda_{k_i}(V_i) = \{1\}$ if $k_i = 0$, in which case $S_{k_i} = *$, and then the smash product is interpreted as excluding *S_{k_i} rather than smashing with a point.)*

It should be emphasized that the new information contained in Theorem 1.2 is not the existence of a wedge decomposition of ΣG that geometrically realizes the module decomposition of $\Sigma H_*(G)$, it is the fact that a decomposition can be chosen so that each of the wedge summands is a suspension. This suspension property will be shown to be a consequence of the fact that the smash of a co-H-space with itself is homotopy equivalent to a suspension [GTW].

Theorem 1.2 is reminiscent of a p-local stable decomposition of $U(n)$ by Nishida and Yang [NY], which is an odd primary refinement of Miller's [Mil] integral stable decomposition of $U(n)$. Nishida and Yang showed that the mod-p module decomposition for $H_*(U(n))$ can be geometrically realized stably. So in this sense, after replacing $U(n)$ with $\mathrm{SU}(n)$, Theorem 1.2 can be regarded as a maximal desuspension of Nishida and Yang's stable decomposition, at least in low rank.

Theorems 1.1 and 1.2 combine to give the following decomposition of $[G, G]$ which is useful for calculations.

Corollary 1.3 *Let G be one of the Lie groups listed in Theorem* 1.2*. Then there is an isomorphism of sets*

$$[G, G] \cong \prod_{k_1,\dots,k_{p-1}=0}^{l_1,\dots,l_{p-1}} [S_{k_1} \wedge \cdots \wedge S_{k_{p-1}}, G],$$

which is an isomorphism of abelian groups if the loop multiplication on G is homotopy commutative.

The cases when the loop multiplication on G is (p-locally) homotopy commutative are known. McGibbon [Mc] showed that homotopy commutativity holds in precisely the following cases[1]:

(1.1)	$\mathrm{SU}(n)$ if $2n < p$;	G_2 if $p \geq 13$;	G_2 if $p = 5$;
	$\mathrm{Sp}(n)$ if $4n < p$;	F_4, E_6 if $p \geq 29$;	$\mathrm{Sp}(2)$ if $p = 3$.
	$\mathrm{Spin}(2n+1)$ if $4n < p$;	E_7 if $p \geq 37$;	
	$\mathrm{Spin}(2n)$ if $4(n-1) < p$;	E_8 if $p \geq 61$;	

[1]The second author would like to apologize for omitting the groups in the second column in [Th2], to which the results in that paper also apply.

We now turn to H-maps. In general, it is difficult to determine when a self-map of G is an H-map, so the set $H[G,G]$ can be mysterious. However, when G is homotopy commutative, there is a tractable description of $H[G,G]$. To state this, it is well known (see, for example [Th2]) that when G is torsion free there is a space A such that $H_*(G) \cong \Lambda(\widetilde{H}_*(A))$ and a map $A \to G$ that induces the inclusion of the generating set in homology. In [Th2] it was shown that if G is homotopy commutative, then there is an isomorphism of abelian groups

$$H[G,G] \cong [A,G]. \tag{1.2}$$

Using this in combination with Corollary 1.3 gives interesting results. In particular, we prove the following theorem, which identifies cases when *every* self-map of G is homotopic to an H-map.

Theorem 1.4 *Let p be an odd prime and let G be a homotopy commutative Lie group. There is a group isomorphism $[G,G] \cong H[G,G]$ in the following cases:*

(i) $G = \mathrm{SU}(n)$ *and* $n \leq 7$, $2n < p$, *and* $n^2 - 1 < 2p$;
(ii) $G = \mathrm{Sp}(n)$ *and* $n \leq 13$, $4n < p$, *and* $2n^2 + n < 2p$;
(iii) $G = \mathrm{Spin}(2n+1)$ *and* $n \leq 13$, $4n < p$, *and* $2n^2 + n < 2p$;
(iv) $G = \mathrm{Spin}(2n)$ *and* $n \leq 6$, $4(n-1) < p$, *and* $2n^2 - n < 2p$;
(v) $G = G_2$ *and* $p = 5$.

For example, the conditions on n and p in Theorem 1.4 (i) hold for $n = 2$ and $p \geq 5$; $n = 3$ and $p \geq 7$; $n = 4$ and $p \geq 11$; $n = 5$ and $p \geq 13$; $n = 6$ and $p \geq 19$; $n = 7$ and $p \geq 29$. It should be noted that there may be cases for which the conclusion of the theorem holds but which fall outside the hypotheses, for example, when $G = \mathrm{SU}(n)$, $n = 6$, and $p = 17$. We go on in Section 6 to give an explicit generating set of $H[G_2, G_2]$.

This paper is organized as follows. Section 2 gives general results on H-maps. Section 3 proves Theorem 1.1. Section 4 discusses low rank torsion free finite H-spaces, which establishes some of the background for the following section on Lie groups, as well as providing interesting examples of Theorem 1.1 in action. Section 5 discusses low rank Lie groups and proves Theorem 1.2. Section 6 gives examples and applications of the preceeding theorems, and in particular proves Theorem 1.4.

2 Preliminary Results on H-Maps

This section gives some general results on H-maps. To begin, let Z be an H-space. Recall that we are assuming that all spaces have the homotopy type of a CW-complex. So by [J2], Z has a left homotopy inverse and a right homotopy inverse. Further, these coincide if Z is homotopy associative and there is a unique homotopy inverse. Thus if Z is homotopy associative, then $[A,Z]$ is a group for any space A. If Z is homotopy commutative as well, then $[A,Z]$ is an abelian group.

Now suppose X and Z are H-spaces, and consider the subset $H[X,Z]$ of $[X,Z]$. If Z is homotopy associative, then $[X,Z]$ is a group, but the restriction to $H[X,Z]$ need not preserve the group structure. To see this, let $f, g \colon X \to Z$ represent homotopy classes in $H[X,Z]$. The sum $f+g$ is given by the composite

$$f+g\colon X \xrightarrow{\Delta} X\times X \xrightarrow{f\times g} Z\times Z \xrightarrow{\mu} Z,$$

where Δ is the diagonal map and μ is the multiplication on Z. This sum, however, need not be an H-map. For if $f+g$ were an H-map, there would be a homotopy commutative diagram

$$\begin{array}{ccc} X\times X & \xrightarrow{(f+g)\times(f+g)} & Z\times Z \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ X & \xrightarrow{f+g} & Z. \end{array}$$

But the definition of $f+g$ implies that the upper direction around the diagram sends a pair of points (a,b) to $f(a)g(a)f(b)g(b)$, while the lower direction around the diagram sends (a,b) to $f(a)f(b)g(a)g(b)$. Thus, to have the two directions around the diagram homotopic, some commutativity condition is needed. The following lemma shows that if Z is homotopy associative and homotopy commutative, then the restriction of $[X,Z]$ to $H[X,Z]$ does preserve the group structure.

Lemma 2.1 *Let X be an H-space and let Z be a homotopy associative, homotopy commutative H-space. Then the multiplication on Z gives $H[X,Z]$ the structure of an abelian group and the inclusion $I\colon H[X,Z]\to[X,Z]$ is a group homomorphism.*

Proof Consider $H[X,Z]$ as a subset of $[X,Z]$. The assertions of the lemma follow if we show that $H[X,Z]$ is a subgroup of $[X,Z]$. The identity element in the group $[X,Z]$ is the trivial map, which is an H-map, and so is in $H[X,Z]$. It is well known (and easy to verify) that the homotopy associativity and homotopy commutativity of Z implies that the multiplication $Z\times Z\xrightarrow{\mu} Z$ is an H-map, and that the inverse $Z\xrightarrow{-1} Z$ is also an H-map. Thus if $f,g\colon X\to Z$ are H-maps, then the composite $f+g\colon X\xrightarrow{\Delta} X\times X\xrightarrow{f\times g} Z\times Z\xrightarrow{\mu} Z$ is an H-map as each of Δ, $f\times g$, and μ are. Thus $H[X,Z]$ is closed under addition. As well, the composite $-f\colon X\xrightarrow{f} Z\xrightarrow{-1} Z$ is an H-map as are each of f and -1. Thus $H[X,Z]$ is closed under inverses. Hence $H[X,Z]$ is a subgroup of $[X,Z]$. ∎

In general, if X and Z are H-spaces, then it is difficult to determine when a map $X\to Z$ is an H-map. However, there are special cases when H-maps can be determined by using a certain universal property. The best known case is due to James [J1]. For a space A, let $E\colon A\to\Omega\Sigma A$ be the suspension map.

Theorem 2.2 *Let A be a path-connected space, and let Z be a homotopy associative H-space. Let $f\colon A\to Z$ be a map. Then there is a unique H-map $\overline{f}\colon\Omega\Sigma A\to Z$ such that $\overline{f}\circ E\simeq f$. Therefore, the map $\theta\colon[A,Z]\to H[\Omega\Sigma A,Z]$ defined by $\theta(f)=\overline{f}$ is a bijection*

Thus Theorem 2.2 gives a means of determining the set $H[\Omega\Sigma A,Z]$. Assuming Z is also homotopy commutative, the correspondence can be strengthened to one of abelian groups.

Lemma 2.3 *Let A be a path-connected space, and let Z be a homotopy associative, homotopy commutative H-space. Then the one-to-one correspondence*

$$[A, Z] \xrightarrow{\theta} H[\Omega\Sigma A, Z]$$

of Lemma 2.2 *is an isomorphism of abelian groups.*

Proof First note that the H-structure on Z implies that $[A, Z]$ is an abelian group and by Lemma 2.1 $H[\Omega\Sigma A, Z]$ is also an abelian group. By Theorem 2.2, θ is a bijection, so it only remains to show that θ is a group homomorphism. Suppose $f, g \colon A \to Z$ represent homotopy classes in $[A, Z]$. Then $\theta(f + g) = \overline{f + g}$ has the property that it is the unique H-map such that $(\overline{f + g}) \circ E \simeq f + g$. On the other hand, $\theta(f) = \overline{f}$ and $\theta(g) = \overline{g}$, where $\overline{f}$ and $\overline{g}$ are H-maps such that $\overline{f} \circ E \simeq f$ and $\overline{g} \circ E \simeq g$. Since Z is homotopy associative and homotopy commutative, Lemma 2.1 says that $\overline{f} + \overline{g}$ is an H-map. Moreover, $(\overline{f} + \overline{g}) \circ E \simeq (\overline{f} \circ E) + (\overline{g} \circ E) \simeq f + g$. Thus $\overline{f} + \overline{g}$ is another H-map which precomposes with E to give $f + g$. The uniqueness property of $\overline{f + g}$ therefore implies that $\overline{f + g} \simeq \overline{f} + \overline{g}$. That is, $\theta(f + g) \simeq \theta(f) + \theta(g)$ and so θ is a group homomorphism. ■

A similar notion of universality can be defined with respect to homotopy associative, homotopy commutative H-spaces. A homotopy associative, homotopy commutative H-space B is *universal* for a space A if there is a map $i \colon A \to B$ with the following property: whenever Z is a homotopy associative, homotopy commutative H-space and $f \colon A \to Z$ is a map, then there is a unique H-map $\overline{f} \colon B \to Z$ such that $\overline{f} \circ i \simeq f$. There is no known functorial construction which starts with a space A and produces its universal space B. However, there are many special cases of interesting spaces whose universal spaces have been constructed by *ad hoc* methods [Gra, Grb1, Grb2, Th1, Th2]. The analogue of Theorem 2.2 and Lemma 2.3 is the following.

Lemma 2.4 *Let B be a homotopy associative, homotopy commutative H-space which is universal for a space A. Let Z be a homotopy associative, homotopy commutative H-space. Then the map $\Theta \colon [A, Z] \to H[B, Z]$ defined by $\Theta(f) = \overline{f}$ is an isomorphism of abelian groups.*

Proof Since Z is homotopy associative and homotopy commutative, $[A, Z]$ is an abelian group, and by Lemma 2.1, $H[B, Z]$ is also an abelian group. The bijectivity of Θ is built into the definition of universality through the existence and uniqueness conditions on the H-map $\overline{f}$. The proof that Θ is a group homomorphism is exactly the same as in Lemma 2.3. ■

3 Homotopy Classes of Maps

The purpose of this section is to prove Theorem 1.1. To motivate this, suppose Z is a loop space, $Z = \Omega Z'$. Suppose X has the property that $\Sigma X \simeq \Sigma X_1 \vee \cdots \vee \Sigma X_t$. This decomposition is only assumed to be a homotopy equivalence of spaces, not

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co-H-spaces. Consider the string of isomorphisms

$$(3.1)\qquad [X,\Omega Z'] \cong [\Sigma X, Z'] \cong \Big[\bigvee_{i=1}^{t} \Sigma X_i, Z'\Big] \cong \prod_{i=1}^{t}[\Sigma X_i, Z'] \cong \prod_{i=1}^{t}[X_i,\Omega Z'].$$

Numbering from left to right, the first and fourth isomorphisms are adjunctions, and so are group isomorphisms. The third isomorphism is a categorical identification and so is a group isomorphism. The second isomorphism is due to the homotopy decomposition $\Sigma X \simeq \bigvee_{i=1}^{t} X_i$; as this is a homotopy equivalence of spaces, the isomorphism is as sets. Thus $[X,\Omega Z']$ is isomorphic to $\prod_{i=1}^{t}[X_i,\Omega Z']$ as sets.

This can be improved if Z is a double loop space, so that $Z = \Omega^2 Z''$. Consider the string of isomorphisms

$$(3.2)\qquad [X,\Omega^2 Z''] \cong [\Sigma^2 X, Z''] \cong \Big[\bigvee_{i=1}^{t} \Sigma^2 X_i, Z''\Big] \cong \prod_{i=1}^{t}[\Sigma^2 X_i, Z''] \cong \prod_{i=1}^{t}[X_i,\Omega^2 Z''].$$

Again, the first and fourth isomorphisms are adjunctions and so are group isomorphisms, while the third isomorphism is a categorical identification and so is a group isomorphism. The second isomorphism is now also an isomorphism of groups because it comes from the homotopy equivalence of co-H-spaces $\Sigma^2 X \simeq \bigvee_{i=1}^{t} \Sigma^2 X_i$. Thus $[X,\Omega^2 Z'']$ is isomorphic to $\prod_{i=1}^{t}[X_i,\Omega^2 Z'']$, as groups. Moreover, this is an isomorphism of abelian groups because $\Omega^2 Z''$ is homotopy commutative.

Theorem 1.1 is a generalization of the isomorphism in (3.1) when Z is not a loop space but only a homotopy associative H-space, and of the isomorphism in (3.2) when Z is not a double loop space but only a homotopy associative, homotopy commutative H-space.

Proof of Theorem 1.1 We will regard $\prod_{i=1}^{t}[X_i, Z]$ equivalently as $[\bigvee_{i=1}^{t} X_i, Z]$. Let $f\colon X \to Z$ represent a homotopy class in $[X,Z]$. Since Z is homotopy associative, Theorem 2.2 states that there is a unique H-map $\overline{f}\colon \Omega\Sigma X \to Z$ such that $\overline{f}\circ E \simeq f$. By hypothesis, there is a homotopy equivalence $e\colon \Sigma X \to \Sigma(X_1 \vee \cdots \vee X_t)$. Define

$$\pi\colon [X,Z] \longrightarrow [X_1 \vee \cdots \vee X_t, Z]$$

by $\pi(f) = \overline{f}\circ \Omega(e^{-1}) \circ E$. That is, $\pi(f)$ is the composite

$$\pi(f)\colon X_1 \vee \cdots \vee X_t \xrightarrow{E} \Omega\Sigma(X_1 \vee \cdots \vee X_t) \xrightarrow{\Omega(e^{-1})} \Omega\Sigma X \xrightarrow{\overline{f}} Z.$$

Similarly, let $g\colon X_1 \vee \cdots \vee X_t \to Z$ represent a homotopy class in $[X_1 \vee \cdots \vee X_t, Z]$. Since Z is homotopy associative, Theorem 2.2 says that there is a unique H-map $\overline{g}\colon \Omega\Sigma(X_1 \vee \cdots \vee X_t) \to Z$ such that $\overline{g}\circ E \simeq g$. Define

$$\rho\colon [X_1 \vee \cdots \vee X_t, Z] \to [X,Z]$$

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by $\rho(g) = \overline{g} \circ \Omega e \circ E$. That is, $\rho(g)$ is the composite

$$\rho(g)\colon X \xrightarrow{E} \Omega\Sigma X \xrightarrow{\Omega e} \Omega\Sigma(X_1 \vee \cdots \vee X_t) \xrightarrow{\overline{g}} Z.$$

The asserted isomorphism of sets $[X, Z] \cong [\bigvee_{i=1}^{t} X_i, Z]$ will be proved by showing that π is a bijection. This is equivalent to showing that $\rho\circ\pi$ and $\pi\circ\rho$ are the respective identity maps. Given $f\colon X \to Z$, let $g = \pi(f) = \overline{f} \circ \Omega(e^{-1}) \circ E$. Then by definition we have $(\rho\circ\pi)(f) = \rho(\pi(f)) = \rho(g) = \overline{g}\circ\Omega e\circ E$, where $\overline{g}$ is the unique H-map such that $\overline{g} \circ E \simeq g$. On the other hand, $\overline{f} \circ \Omega(e^{-1})$ is an H-map as it is the composite of H-maps, and by definition, $g = \pi(f) = (\overline{f} \circ \Omega(e^{-1})) \circ E$. Thus $\overline{f} \circ \Omega(e^{-1})$ is another H-map such that $(\overline{f} \circ \Omega(e^{-1})) \circ E \simeq g$. The uniqueness property of $\overline{g}$ therefore implies that $\overline{g} \simeq \overline{f} \circ \Omega(e^{-1})$. Hence

$$(\rho \circ \pi)(f) \simeq \overline{g} \circ \Omega e \circ E \simeq (\overline{f} \circ \Omega(e^{-1})) \circ \Omega e \circ E \simeq \overline{f} \circ E \simeq f,$$

and so $\rho \circ \pi$ is the identity map on $[X, Z]$. Similarly, $\pi \circ \rho$ is the identity map on $[X_1 \vee \cdots \vee X_t, Z]$. Thus π is a bijection.

Now suppose that Z is also homotopy commutative. To prove that there is a group isomorphism $[X, Z] \cong [\bigvee_{i=1}^{t} X_i, Z]$ it remains to show that ϕ is a group homomorphism. Let $f_1, f_2\colon X \to Z$ represent homotopy classes in $[X, Z]$. Then by definition, $\pi(f_1 + f_2) = \overline{(f_1 + f_2)} \circ \Omega(e^{-1}) \circ E$, where $\overline{(f_1 + f_2)}$ is the unique H-map such that $\overline{(f_1 + f_2)} \circ E \simeq f_1 + f_2$. On the other hand, since Z is homotopy associative, Theorem 2.2 applied to each of f_1 and f_2 individually gives H-maps $\overline{f}_1$ and $\overline{f}_2$ such that $\overline{f}_1 \circ E \simeq f_1$ and $\overline{f}_2 \circ E \simeq f_2$. Since Z is also homotopy commutative, Lemma 2.1 implies that the sum $\overline{f}_1 + \overline{f}_2$ is also an H-map. As well, we have $(\overline{f}_1 + \overline{f}_2) \circ E \simeq \overline{f}_1 \circ E + \overline{f}_2 \circ E \simeq f_1 + f_2$. The uniqueness property of $\overline{(f_1 + f_2)}$ therefore implies that $\overline{(f_1 + f_2)} \simeq \overline{f}_1 + \overline{f}_2$. Thus, with $t = \Omega(e^{-1}) \circ E$, we have

$$\pi(f_1 + f_2) \simeq \overline{(f_1 + f_2)} \circ t \simeq (\overline{f}_1 + \overline{f}_2) \circ t \simeq (\overline{f}_1 \circ t) + (\overline{f}_2 \circ t) \simeq \pi(f_1) + \pi(f_2)$$

and so π is a group homomorphism. ■

Example 3.1 By [J1], if A is a path-connected space then $\Sigma\Omega\Sigma A \simeq \bigvee_{i=1}^{\infty} \Sigma A^{(i)}$, where $A^{(i)}$ is the i-fold smash of A with itself. If Z is a homotopy associative H-space, then Theorem 1.1 says that there is a bijection of sets $[\Omega\Sigma A, Z] \cong \prod_{i=1}^{\infty}[A^{(i)}, Z]$ which is a group isomorphism if Z is also homotopy commutative. In particular, if $A = S^m$, there is a bijection of sets

$$[\Omega S^{m+1}, Z] \cong \prod_{i=1}^{\infty}[(S^m)^{(i)}, Z] \cong \bigoplus_{i=1}^{\infty} \pi_{mi}(Z)$$

which is a group isomorphism if Z is homotopy commutative.

Example 3.2 When localized at an odd prime $p \geq 5$, S^{2n+1} is a homotopy associative, homotopy commutative H-space. So Example 3.1 implies that there is a

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(p-local) group isomorphism $[\Omega S^{m+1}, S^{2n+1}] \cong \bigoplus_{i=1}^{\infty} \pi_{mi}(S^{2n+1})$. A curious instance of this is when $m = 1$, in which case there is a (p-local) group isomorphism

$$[\Omega S^2, S^{2n+1}] \cong \bigoplus_{i=1}^{\infty} \pi_i(S^{2n+1}).$$

Thus calculating $[\Omega S^2, S^{2n+1}]$ is equivalent to calculating the p-local homotopy groups of spheres.

4 Low Rank Torsion Free Finite H-Spaces

We will see in Section 5 that if G is a low rank Lie group, then it decomposes as a product of certain torsion-free finite H-spaces which have nice properties. The purpose of this section is to introduce these finite H-spaces, and discuss their relevant properties. In particular, we will see that when they are suspended, they decompose as a wedge of suspensions, allowing us to apply Theorem 1.1.

In all that follows, we start with a space X which has l odd dimensional cells, and then localize at a prime p. Homology is taken with mod-p coefficients. We consider p-local H-spaces Y such that $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$. The *rank* of Y is the number of generators it has in rational cohomology, so in this case Y has rank l. For $1 \leq k \leq l$, let $\Lambda_k(\widetilde{H}_*(X))$ denote the submodule of length k tensor elements in $\Lambda(\widetilde{H}_*(X))$. The following theorem was proved in [CN].

Theorem 4.1 *Let X be a CW-complex consisting of l odd-dimensional cells, where $l \leq p-1$. Suppose there is an H-space Y such that $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$ and a map $X \to Y$ that induces the inclusion of the generating set in homology. Then there is a homotopy equivalence $\Sigma Y \simeq \bigvee_{k=1}^{l} R_k(X)$, where $R_k(X)$ is a space such that $\widetilde{H}_*(R_k(X)) \cong \Sigma\Lambda_k(\widetilde{H}_*(X))$, and $R_1(X) = \Sigma X$.*

In [CN, CHZ] it was shown that if X has l odd-dimensional cells, where $l < p-1$, then it is guaranteed that there is an H-space Y such that $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$ and a map $X \to Y$ that induces the inclusion of the generating set in homology. If X has $p-1$ odd-dimensional cells, then it may be the case that such an H-space Y exists, but there is no guarantee of it.

We would like to apply Theorem 1.1 with one of the H-spaces Y in Theorem 4.1 as the domain. This requires that each of the wedge summands $R_k(X)$ be a suspension. So we wish to find conditions on X that guarantee that $R_k(X)$ is a suspension for each k. To do so we must first consider how $R_k(X)$ was constructed in [CN].

Let Σ_k be the symmetric group on k letters, $\mathbb{Z}_{(p)}$ be the p-local integers, and let $\mathbb{Z}_{(p)}[\Sigma_k]$ be the group ring. Let

$$\bar{s}_k = \sum_{\sigma \in \Sigma_k} \sigma \in \mathbb{Z}_{(p)}[\Sigma_k].$$

It is a standard fact that $\bar{s}_k \circ \bar{s}_k = k!\bar{s}_k$. If $k < p$, then $k!$ is invertible in $\mathbb{Z}_{(p)}$ and so $s_k = \frac{1}{k!}\bar{s}_k$ is an idempotent in $\mathbb{Z}_{(p)}[\Sigma_k]$. In terms of topology, let $X^{(k)}$ be the k-fold smash product of X with itself. An element $\sigma \in \Sigma_k$ gives a map $\sigma\colon X^{(k)} \to X^{(k)}$

defined by permuting the smash factors. Suspending, we can add, giving a map $s_k\colon \Sigma X^{(k)} \to \Sigma X^{(k)}$ corresponding to the idempotent $s_k \in \mathbb{Z}_{(p)}[\Sigma_k]$. The space $R_k(X)$ in Theorem 4.1 is defined as the mapping telescope: $R_k(X) = \operatorname{hocolim}_{s_k} \Sigma X^{(k)}$. In homology, $(s_k)_*$ is an idempotent and we have $\widetilde{H}_*(R_k(X)) \cong \operatorname{Im}(s_k)_* \cong \Sigma\Lambda_k(V)$. Moreover, if $R'_k(X) = \operatorname{hocolim}_{1-s_k} \Sigma X^{(k)}$, then $H_*(\Sigma X^{(k)}) \cong H_*(R_k(X)) \oplus H_*(R'_k(X))$ because $(s_k)_*$ and $(1-s_k)_*$ are orthogonal idempotents. This homology decomposition can be realized geometrically. Using the co-H structure on $\Sigma X^{(k)}$, we can add the telescope maps, giving a composite $\Sigma X^{(k)} \to \Sigma X^{(k)} \vee \Sigma X^{(k)} \to R_k(X) \vee R'_k(X)$, which is an isomorphism in homology and so is a homotopy equivalence. (It does not play a role in what follows, but it may be helpful to observe that what we have done is show that ΣY is a retract of $\Sigma\Omega\Sigma X \simeq \bigvee_{k=1}^{\infty} \Sigma X^{(k)}$, where for $1 \le k \le l$, $R_k(X)$ is a retract of $\Sigma X^{(k)}$ corresponding to the suspension of the submodule of symmetric tensors of length k in $H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$.)

We now consider conditions which guarantee that $R_k(X)$ is a suspension for $1 \le k \le l$. Note that $R_1(X) = \Sigma X$, so the issue is for $k > 1$. First, suppose that X is a suspension, $X = \Sigma\overline{X}$. Then $X^{(k)} \simeq \Sigma^k \overline{X}^{(k)}$ and the map $\Sigma X^{(k)} \xrightarrow{s_k} \Sigma X^{(k)}$ is essentially (up to shuffling suspension coordinates) the k-fold suspension of the map $\Sigma\overline{X}^{(k)} \xrightarrow{s_k} \Sigma\overline{X}^{(k)}$. Thus there is a homotopy equivalence of mapping telescopes $R_k(X) \simeq \Sigma^k R_k(\overline{X})$. So in this case, the decomposition of ΣY becomes

$$\Sigma Y \simeq \bigvee_{k=1}^{l} \Sigma^k R_k(\overline{X}).$$

More generally, suppose X is a co-H-space. Now $X^{(k)}$ is not apparently a suspension. However, by [GTW] it is in fact a $(k-1)$-fold suspension for $k \ge 2$, and satisfies the appropriate properties.

Proposition 4.2 *Let X be a co-H-space, and let $V = \Sigma^{-1}\widetilde{H}_*(X)$. For $k \ge 2$, there is a homotopy equivalence $X^{(k)} \simeq \Sigma^{k-1} M_k(X)$, where $M_k(X)$ is a space such that $\widetilde{H}_*(M_k(X)) \cong \Sigma V^{\otimes k}$. Furthermore, there is a homotopy equivalence $R_k(X) \simeq \Sigma^k MR_k(X)$, where $MR_k(X)$ is a space such that $\widetilde{H}_*(MR_k(X)) \cong \Sigma\Lambda_k(V)$.*

The space $MR_k(X)$ is defined as the mapping telescope of a map

$$\tilde{s}_k\colon M_k(X) \to M_k(X),$$

which is, essentially, a $(k-1)$-fold desuspension of $X^{(k)} \xrightarrow{s_k} X^{(k)}$. To normalize the number of suspensions, for $k \ge 2$ let $S_k(X) = \Sigma^{k-1} MR_k(X)$. Then there is a homotopy equivalence $\Sigma Y \simeq \bigvee_{k=1}^{l} \Sigma S_k(X)$, where

$$\widetilde{H}_*(S_k(X)) \cong \Sigma^{k-1}\widetilde{H}_*(MR_k(X)) \cong \Sigma^k \Lambda_k(V) \cong \Lambda_k(\widetilde{H}_*(X)).$$

Summarizing, we have the following.

Proposition 4.3 *Let X be a CW-complex consisting of l odd-dimensional cells, where $l \le p-1$. Suppose X is a co-H-space and there is an H-space Y such that $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$, together with a map $X \to Y$ that induces the inclusion of the generating set in homology. Then there is a homotopy equivalence $\Sigma Y \simeq \bigvee_{k=1}^{l} \Sigma S_k(X)$, where $S_k(X)$ is a space such that $\widetilde{H}_*(S_k(X)) \cong \Lambda_k(\widetilde{H}_*(X))$, and $S_1(X) = X$.*

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Applying Theorem 1.1 immediately gives the following.

Corollary 4.4 *Given X and Y as in Proposition 4.3, if Z is a homotopy associative H-space, then there is an isomorphism of sets $[Y,Z] \cong \prod_{k=1}^{l}[S_k(X),Z]$ and if Z is also homotopy commutative, then this isomorphism is of abelian groups.*

We have already remarked that if X has l odd-dimensional cells, where $l < p-1$, then it is known that there is an H-space Y with $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$ and a map $X \to Y$ that induces the inclusion of the generating set in homology. In [Th1] it was shown that if X is a suspension and $l < p-2$, then Y is homotopy associative and homotopy commutative. This was generalized in [Th2] to the case when X is a co-H-space and $l < p-2$. Thus Proposition 4.3 and Corollary 4.4 imply the following.

Corollary 4.5 *Let X be a co-H-space consisting of l odd-dimensional cells, $l < p-2$, and let Y be the corresponding homotopy associative, homotopy commutative H-space with $H_*(Y) \cong \Lambda(\widetilde{H}_*(X))$. Then there is an isomorphism of abelian groups*

$$[Y,Y] \cong \prod_{k=1}^{l}[S_k(X),Y].$$

5 Lie Groups

In this section we prove Theorem 1.2. The following proposition is well known, although not usually stated this way in the literature. An explicit proof can be found in [Th2] and is based on work in [H, MT, MNT1, MNT2]. Let $q = 2(p-1)$.

Proposition 5.1 *Let G be a torsion free, compact, simply-connected, simple Lie group. Then there is a co-H-space A such that $H_*(G) \cong \Lambda(\widetilde{H}_*(A))$ and a map $A \to G$ that induces the inclusion of the generating set in homology. Furthermore, there is a homotopy decomposition $A \simeq \bigvee_{i=1}^{p-1} A_i$, where $\widetilde{H}_*(A_i)$ consists of those elements in $\widetilde{H}_*(A)$ in degrees $2i + kq + 1$ for some $k \geq 0$.*

For example, if $G = SU(n)$, then $A = \Sigma\mathbb{C}P^{n-1}$. In this case the space A and the map $A \to G$ exist integrally, but in other cases the existence of the map $A \to G$ occurs only after localization at p. Also note that for a given G it may be possible that there are no elements in $H_*(A)$ in degrees of the form $2i+kq+1$, in which case $\widetilde{H}_*(A_i) = 0$ and so $A_i \simeq *$.

The decomposition $A \simeq \bigvee_{i=1}^{p-1} A_i$ in Proposition 5.1 results in a homology decomposition $H_*(G) \cong \bigotimes_{i=1}^{p-1} \Lambda(\widetilde{H}_*(A_i))$. This homology decomposition was realized geometrically by Mimura, Nishida, and Toda [MT, MNT2], incorporating work of Harris [H]. They showed that there is a homotopy decomposition

$$G \simeq \prod_{i=1}^{p-1} \overline{B}_i, \tag{5.1}$$

where $H_*(\overline{B}_i) \cong \Lambda(\widetilde{H}_*(A_i))$, and each $\overline{B}_i$ is indecomposable. Let l_i be the number of cells in A_i.

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Proof of Theorem 1.2 First consider $\Sigma\overline{B}_i$ for each i. Observe that $\overline{B}_i$ is an H-space because it is a retract of the H-space G. Theorem 5.1 and the decomposition in (5.1) give a map $A_i \longrightarrow \overline{B}_i$ which induces the inclusion of the generating set in homology. All the homology generators of $H_*(G)$, and therefore of $H_*(\overline{B}_i)$, are in odd dimensions, so the cells of A_i are in odd dimensions. By [MNT2], the hypotheses on the rank and the prime in the statement of the theorem guarantee that the number l_i of cells in A_i satisfies $l_i \leq p-1$. Thus all the hypotheses of Proposition 4.3 are fulfilled with respect to A_i and $\overline{B}_i$, and so we obtain a homotopy decomposition

$$\Sigma\overline{B}_i \simeq \bigvee_{k_i=1}^{l_i} \Sigma S_{k_i}(A_i),$$

where $S_{k_i}(A_i)$ is a space such that $\widetilde{H}_*(S_{k_i}(A_i)) \cong \Lambda_{k_i}(\widetilde{H}_*(A_i)) \cong \Lambda_{k_i}(V_i)$, and $S_1(A_i) = A_i$.

In general, for any spaces X and Y there is a homotopy decomposition

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y).$$

If $\Sigma X \simeq \Sigma X_1 \vee \Sigma X_2$ and $\Sigma Y \simeq \Sigma Y_1 \vee \Sigma Y_2$, then this homotopy decomposition can be refined to $\Sigma(X \times Y) \simeq \bigvee_{i=0}^{2}\bigvee_{j=1}^{2} \Sigma X_i \wedge Y_j$ where $X_0 \wedge Y_j$ is regarded as Y_j and $X_i \wedge Y_0$ is regarded as X_i. Applied iteratively to $\Sigma G \simeq \Sigma(\prod_{i=1}^{p-1} \overline{B}_i)$ and each of the wedge decompositions of $\Sigma\overline{B}_i$ gives the decomposition in the statement of the theorem. ∎

Theorem 1.2 is useful for calculations. Consider $G = \mathrm{SU}(n)$ as an example. There are homotopy fibrations

$$\begin{gathered} \mathrm{SU}(n-1) \longrightarrow \mathrm{SU}(n) \longrightarrow S^{2n-1}, \\ \mathrm{SU}(n-2) \longrightarrow \mathrm{SU}(n-1) \longrightarrow S^{2n-3}, \\ \vdots \\ S^3 \longrightarrow \mathrm{SU}(3) \longrightarrow S^5, \end{gathered}$$

which determine long exact sequences

$$[\mathrm{SU}(n), S^{2m-1}] \longrightarrow [\mathrm{SU}(n), \mathrm{SU}(m)] \longrightarrow [\mathrm{SU}(n), \mathrm{SU}(m-1)] \longrightarrow [\mathrm{SU}(n), \Omega S^{2m-1}] \longrightarrow \cdots$$

for $3 \leq m \leq n$. This gives an inductive approach to calculating $[\mathrm{SU}(n), \mathrm{SU}(n)]$ by first calculating the cohomotopy groups $[\mathrm{SU}(n), S^{2m-1}]$ and then assembling this information using the long exact sequences. To calculate the cohomotopy groups, observe that if $p \geq 5$, then S^{2m-1} is homotopy associative and homotopy commutative, so the decomposition of $\Sigma\,\mathrm{SU}(n)$ in Theorem 1.2 together with Theorem 1.1 gives an isomorphism

$$[\mathrm{SU}(n), S^{2m-1}] \cong \prod_{k_1,\dots,k_{p-1}=0}^{l_1,\dots,l_{p-1}} [S_{k_1}(A_1) \wedge \cdots \wedge S_{k_{p-1}}(A_{p-1}), S^{2m-1}],$$

where $A_1 \vee \cdots \vee A_{p-1} \simeq \Sigma\mathbb{C}P^{n-1}$. The factors on the right are easier to calculate in the sense that they are determined by smaller spaces.

6 Examples

In this section we prove Theorem 1.4 and give several concrete calculations. As many of these will involve calculations of specific homotopy groups, we give some preliminary information first, together with two lemmas that will allow us to identify when certain sets of homotopy classes of maps are zero.

Toda [To] calculated the low-dimensional odd primary homotopy groups of spheres. They are as follows.

Theorem 6.1 *Let p be an odd prime and let $q = 2(p-1)$. Fix $m \geq 2$ and let $t \leq 2m + pq - 4$. Then the following hold:*

$$\pi_{2m-1+rq-1}(S^{2m-1}) = \mathbb{Z}/p\mathbb{Z} \quad \text{for } 1 \leq r \leq p-1;$$
$$\pi_{2m-1+rq-2}(S^{2m-1}) = \mathbb{Z}/p\mathbb{Z} \quad \text{for } 2 \leq r \leq p-1 \text{ and } r \geq m;$$
$$\pi_t(S^{2m-1}) = 0 \quad \text{otherwise.}$$

The elements in $\pi_{2m-1+rq-1}(S^{2m-1})$ for $1 \leq r \leq p-1$ are stable for all $m \geq 2$. The remaining elements are unstable.

It is often useful in practise to know when homotopy groups of spheres are zero. Theorem 6.1 is helpful as it shows that the odd primary low-dimensional homotopy groups of spheres are relatively sparse. The next two lemmas can be thought of as systematically taking advantage of homotopy groups of spheres which are zero.

A space B is said to be spherically resolved by $S^{2n_1+1}, \dots, S^{2n_t+1}$ if there is a sequence of homotopy fibrations

$$\begin{array}{c} B_2 \longrightarrow B_1 \longrightarrow S^{2n_1+1}, \\ B_3 \longrightarrow B_2 \longrightarrow S^{2n_2+1}, \\ \vdots \\ B_t \longrightarrow B_{t-1} \longrightarrow S^{2n_{t-1}+1}, \\ * \longrightarrow B_t \longrightarrow S^{2n_t+1}, \end{array}$$

where $B_1 = B$. A standard example is $SU(n)$ which is spherically resolved by

$$S^{2n-1}, S^{2n-3}, \dots, S^3.$$

Lemma 6.2 *Let B be a space which is spherically resolved by $S^{2n_1+1}, \dots, S^{2n_t+1}$. Fix $m \geq 1$. If $\pi_m(S^{2n_j+1}) = 0$ for each $1 \leq j \leq t$, then $\pi_m(B) = 0$.*

Proof Induct on the number t of resolving spheres. If $t = 1$, then $B = S^{2n_1+1}$ and the hypothesis $\pi_m(S^{2n_1+1}) = 0$ equivalently says that $\pi_m(B) = 0$. Suppose the lemma is true when B is resolved by $t-1$ spheres. Let $f\colon S^m \to B$ represent a homotopy class in $\pi_m(B)$. Consider the homotopy fibration $B_2 \to BS^{2n_1+1}$. As $\pi_m(S^{2n_1+1}) = 0$, composing f to S^{2n_1+1} is null homotopic, and so f lifts to a map $f'\colon S^m \to B_2$. Since B_2 is resolved by the $t-1$ spheres $S^{2n_2+1}, \dots, S^{2n_t+1}$, the inductive hypothesis says that $\pi_m(B_2) = 0$. Thus f' is null homotopic, and so f is null homotopic. Hence $\pi_m(B) = 0$. ■

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Lemma 6.3 *Let X be a finite CW-complex with cells in dimensions $m_1 < m_2 < \cdots < m_s$. Let B be a space which is spherically resolved by spheres $S^{2n_1+1}, \dots, S^{2n_t+1}$. If $\pi_{m_i}(S^{2n_j+1}) = 0$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, then $[X, B] = 0$.*

Proof Induct on the number of dimensions for which X has cells. If $s = 1$, then X is a wedge of copies of S^{m_1}. The hypothesis that $\pi_{m_1}(S^{2n_j+1}) = 0$ for $1 \leq j \leq t$ lets us apply Lemma 6.2 to say that $\pi_{m_1}(B) = 0$. Thus $[X, B] = 0$.

Suppose the lemma is true for finite CW-complexes with cells in $s - 1$ different dimensions. Let X' be the m_1-skeleton of X. So X' is a wedge of copies of S^{m_1}. Define a space X'' by the homotopy cofibration $X' \to X \to X''$. Observe that X'' is a finite CW-complex with cells in $s - 1$ different dimensions, $m_2 < \cdots < m_s$. Now let $f \colon X \to B$ represent a homotopy class in $[X, B]$. The composite $f' \colon X' \to X \xrightarrow{f} B$ represents a homotopy class in $[X', B]$. By the base case of the induction, this set is zero. Thus f' is null homotopic, and so f extends along the cofiber to a map $f'' \colon X'' \to B$. Since $\pi_{m_i}(S^{2n_j+1}) = 0$ for $2 \leq i \leq s$ and $1 \leq j \leq t$, the inductive hypothesis applied to X'' says that $[X'', B] = 0$. Thus f'' is null homotopic and therefore so is f. Hence $[X, B] = 0$. ∎

6.1 Cases When All Self-Maps Are Homotopic to H-Maps

In this subsection we establish cases of homotopy commutative Lie groups G for which every self-map of G is homotopic to an H-map. Lemmas 6.4 through 6.11 will collectively prove Theorem 1.4.

Lemma 6.4 *Let p be an odd prime. Suppose*

(i) $n \leq 7$,
(ii) $2n < p$,
(iii) $n^2 - 1 < 2p$.

Then there is a group isomorphism $[\mathrm{SU}(n), \mathrm{SU}(n)] \cong H[\mathrm{SU}(n), \mathrm{SU}(n)]$.

Proof The hypothesis $2n < p$ implies that $\mathrm{SU}(n)$ is homotopy commutative. So by (1.2) there is a group isomorphism $H[\mathrm{SU}(n), \mathrm{SU}(n)] \cong [A, \mathrm{SU}(n)]$, where $A = \Sigma\mathbb{C}P^{n-1}$. Thus, to prove the lemma, it is equivalent to show that there is a group isomorphism $[\mathrm{SU}(n), \mathrm{SU}(n)] \cong [A, \mathrm{SU}(n)]$.

The hypothesis $2n < p$ also implies that there is a homotopy equivalence $\mathrm{SU}(n) \simeq S^3 \times \cdots \times S^{2n-1}$. Suspending therefore gives a homotopy equivalence

$$\Sigma\,\mathrm{SU}(n) \simeq \bigvee_{k=2}^{n} \Big(\bigvee_{2 \leq i_1 < \cdots < i_k \leq n} \Sigma S^{2i_1-1} \wedge \cdots \wedge S^{2i_k-1} \Big).$$

To compress notation, let I be an index set consisting of sequences

$$\alpha = (2i_1 - 1, 2i_2 - 1, \dots, 2i_k - 1)$$

with $2 \leq k \leq n$ and $2 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let $t_\alpha = \Sigma_{s=2}^{k}(2i_s - 1)$. Then $\Sigma\,\mathrm{SU}(n) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma S^{t_\alpha}$. This decomposition, together with the fact that $\mathrm{SU}(n)$ is a

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homotopy associative, homotopy commutative H-space, lets us apply Proposition 1.1 to show that there is a group isomorphism

$$[\mathrm{SU}(n), \mathrm{SU}(n)] \cong \prod_{\alpha \in \mathcal{I}} [S^{t_\alpha}, \mathrm{SU}(n)].$$

The multiplicative homotopy equivalence $\mathrm{SU}(n) \simeq S^3 \times \cdots \times S^{2n-1}$ lets us refine this to a group isomorphism

$$[\mathrm{SU}(n), \mathrm{SU}(n)] \cong \prod_{j=2}^{n} \prod_{\alpha \in \mathcal{I}} [S^{t_\alpha}, S^{2j-1}]. \tag{6.1}$$

Observe that the cell of highest dimension in $\mathrm{SU}(n)$ is in dimension

$$3 + 5 + \cdots + 2n - 1 = (2n + 2)(n - 1)/2 = n^2 - 1.$$

The hypothesis $n^2 - 1 < 2p$ therefore implies that the cells of $\mathrm{SU}(n)$ are of dimension less than $2p$, and so $t_\alpha < 2p$ for all α. For $j \geq 2$, the least dimensional torsion homotopy group of S^{2j-1} occurs in dimension $(2j - 1) + (2p - 3) \geq 2p$. Thus $[S^{t_\alpha}, S^{2j-1}] = \pi_{t_\alpha}(S^{2j-1}) = 0$ in all cases except when $t_\alpha = 2j - 1$.

We are left to consider the cases when a $\mathbb{Z}_{(p)}$ summand may appear in (6.1). If α is the sequence $(2j - 1)$ of length 1, then $S^{t_\alpha} = S^{2j-1}$ and so $[S^{t_\alpha}, S^{2j-1}] \cong \mathbb{Z}_{(p)}$. If α is a sequence of even length, then S^{t_α} has even dimension, so $[S^{t_\alpha}, S^{2j-1}] \neq \mathbb{Z}_{(p)}$. If α is a sequence of odd length ≥ 3, there are many possible ways that S^{t_α} can have odd dimension. The possibility of least dimension is $S^{t_\alpha} = S^3 \wedge S^5 \wedge S^7$. Thus we have to avoid the possibility of having $[S^3 \wedge S^5 \wedge S^7, S^{15}]$ in (6.1). The hypothesis that $n \leq 7$ does this, as it implies that S^{15} cannot appear on the right.

Therefore every term $[S^{t_\alpha}, S^{2j-1}]$ in (6.1) is zero except for the $\mathbb{Z}_{(p)}$ summands that arise from the length 1 sequences $\alpha = (2j - 1)$. Thus there is a group isomorphism

$$[\mathrm{SU}(n), \mathrm{SU}(n)] \cong \prod_{j=2}^{n} [S^{2j-1}, S^{2j-1}].$$

Phrased differently, the summands that arise from the length 1 sequences $\alpha = (2j-1)$ arise from the inclusion $A = S^3 \vee \cdots \vee S^{2n-1} \to S^3 \times \cdots \times S^{2n-1} \simeq \mathrm{SU}(n)$, and so $[\mathrm{SU}(n), \mathrm{SU}(n)] \subseteq [A, \mathrm{SU}(n)]$. On the other hand, the same argument regarding torsion homotopy classes shows that there is a group isomorphism

$$[A, \mathrm{SU}(n)] \cong \prod_{j=2}^{n} [S^{2j-1}, S^{2j-1}].$$

Hence there is a group isomorphism $[\mathrm{SU}(n), \mathrm{SU}(n)] \cong [A, \mathrm{SU}(n)]$. ■

Remark 6.5 Observe that the three hypotheses on n in Lemma 6.4 are satisfied in the following cases: $n = 2$ and $p \geq 5$; $n = 3$ and $p \geq 7$; $n = 4$ and $p \geq 11$; $n = 5$ and $p \geq 13$; $n = 6$ and $p \geq 19$; $n = 7$ and $p \geq 29$. There is one additional case that falls outside the hypotheses. A direct calculation shows that the conclusion of Lemma 6.4 also holds when $n = 6$ and $p = 17$.

Lemma 6.6 *Let p be an odd prime. Suppose*

(i) $n \leq 13$,
(ii) $4n < p$,
(iii) $2n^2 + n < 2p$.

Then there is a group isomorphism $[\mathrm{Sp}(n), \mathrm{Sp}(n)] \cong H[\mathrm{Sp}(n), \mathrm{Sp}(n)]$.

Proof Argue as in Lemma 6.4. The hypothesis $4n < p$ implies that $\mathrm{Sp}(n) \simeq S^3 \times \cdots \times S^{4n-1}$ and $\mathrm{Sp}(n)$ is homotopy commutative. The cell of highest dimension is in dimension $3 + 7 + \cdots + 4n - 1 = 4(1 + 2 + \cdots + n) - n = 2n^2 + n$ that, when compared to torsion in the homotopy groups of spheres, is the origin of the hypothesis $2n^2 + n < 2p$. Note that the analogous spheres S^{t_α} in the homotopy decomposition of $\Sigma\,\mathrm{Sp}(n)$ have dimensions $t_\alpha = \Sigma_{s=1}^{t} 4i_s - 1$. The least dimensional possibility of a term $[S^{t_\alpha}, S^{4j-1}]$ being $\mathbb{Z}_{(p)}$ with α a sequence of length > 1 is $[S^3 \wedge S^7 \wedge \cdots \wedge S^{19}, S^{55}]$. The hypothesis $n \leq 13$ avoids this possibility. ■

Remark 6.7 The hypotheses on n in Lemma 6.6 for $1 \leq n \leq 7$ are satisfied in the following cases: $n = 1$ and $p \geq 5$; $n = 2$ and $p \geq 11$; $n = 3$ and $p \geq 13$; $n = 4$ and $p \geq 19$; $n = 5$ and $p \geq 23$; $n = 6$ and $p \geq 41$; $n = 7$ and $p \geq 53$. Again, there may be cases that fall outside the hypotheses but for which the conclusion of the lemma holds. One example is when $n = 4$ and $p = 17$.

By [H] there is a homotopy equivalence $\mathrm{Spin}(2n+1) \simeq \mathrm{Sp}(n)$, and the list in (1.1) shows that $\mathrm{Spin}(2n + 1)$ is homotopy commutative for the same values of n as $\mathrm{Sp}(n)$. Thus Lemma 6.6 implies the following.

Corollary 6.8 *Let p be an odd prime. Suppose*

(i) $n \leq 13$,
(ii) $4n < p$,
(iii) $2n^2 + n < 2p$.

Then there is a group isomorphism

$$[\mathrm{Spin}(2n + 1), \mathrm{Spin}(2n + 1)] \cong H[\mathrm{Spin}(2n + 1), \mathrm{Spin}(2n + 1)].$$

The $Spin(2n)$ cases are as follows.

Lemma 6.9 *Let p be an odd prime. Suppose*

(i) $n \leq 6$,
(ii) $4(n - 1) < p$,
(iii) $2n^2 - n < 2p$.

Then there is a group isomorphism $[\mathrm{Spin}(2n), \mathrm{Spin}(2n)] \cong H[\mathrm{Spin}(2n), \mathrm{Spin}(2n)]$.

Proof By [H], $\mathrm{Spin}(2n) \simeq \mathrm{Sp}(n - 1) \times S^{2n-1}$. So we modify the calculations in Lemma 6.6 to take into account the extra factor of S^{2n-1}. As stated in (1.1), the hypothesis $4(n - 1) < p$ implies that $\mathrm{Spin}(2n)$ is homotopy commutative. The cell of highest dimension in $\mathrm{Spin}(2n)$ is in dimension

$$(3 + 7 + \cdots + 4n - 1) + 2n - 1 = 4(1 + \cdots + n) - n + (2n - 1) = 2n^2 - n,$$

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which, when compared to torsion in the homotopy groups of spheres, is the origin of the hypothesis $2n^2 - n < 2p$. The presence of S^{2n-1} as a factor of Spin$(2n)$ means that when checking for $\mathbb{Z}_{(p)}$ summands one must take into account more cases. The sequence α of length > 1 that gives the least dimensional occurance of a $\mathbb{Z}_{(p)}$ summand is $[S^3 \wedge S^7 \wedge S^{13}, S^{23}]$ in $[\mathrm{Spin}(14), \mathrm{Spin}(14)]$. The hypothesis $n \leq 6$ avoids this case. ∎

Remark 6.10 The hypotheses on n in Lemma 6.6 are satisfied in the following cases: $n = 2$ and $p \geq 5$; $n = 3$ and $p \geq 11$; $n = 4$ and $p \geq 17$; $n = 5$ and $p \geq 23$; $n = 6$ and $p \geq 37$. Again, it may be possible that there are other cases where the conclusion of the Lemma hold which fall outside the hypotheses.

Lemma 6.11 *Let $p = 5$. Then there is a group isomorphism $[G_2, G_2] \cong H[G_2, G_2]$.*

Proof We assume throughout that homology is taken with mod-5 coefficients. It is well known that $H_*(G_2) \cong \Lambda(x_3, x_{11})$ and $\mathcal{P}^1_*(x_{11}) = x_3$. By Theorem 5.1, there is a space A and a map $A \to G_2$ such that $H_*(G) \cong \Lambda(\widetilde{H}_*(A))$. In particular, A is a two-cell complex with its cells attached by a $\mathcal{P}^1$. By Theorem 1.2, $\Sigma G_2 \simeq \Sigma S_1(A) \vee \Sigma S_2(A)$, where $\widetilde{H}_*(S_2(A)) \cong \Lambda_2(\widetilde{H}_*(A))$ and $S_1(A) \simeq A$. Observe that $\Lambda_2(\widetilde{H}_*(A)) \cong \{x_{14}\}$, so $S_2(A) \simeq S^{14}$. As stated in (1.1), G_2 is homotopy commutative when $p = 5$, so by Corollary 1.3 there is a group isomorphism $[G_2, G_2] \cong [A, G_2] \oplus [S^{14}, G_2]$. By [Mim], $\pi_{14}(G_2) = 0$ at 5 and so $[G_2, G_2] \cong [A, G_2]$. On the other hand, by (1.2) there is a group isomorphism $H[G_2, G_2] \cong [A, G_2]$. Hence there is a group isomorphism $[G_2, G_2] \cong H[G_2, G_2]$. ∎

6.2 Two-Cell Co-H-Spaces

In this subsection we give an explicit calculation of $H[B, B]$ when B is a homotopy associative, homotopy commutative H-space which is universal for a certain two-cell complex. In our case both cells are in odd dimensions; similar calculations were done in [Grb2] when there is both an odd- and an even-dimensional cell.

Let A be a co-H-space with two odd-dimensional cells. So there is a homotopy cofibration sequence

$$S^{2n} \xrightarrow{\epsilon} S^{2m+1} \xrightarrow{j} A \xrightarrow{q} S^{2n+1},$$

where ϵ is the attaching map, j is the inclusion, and q is the pinch map onto the top cell. If $p \geq 5$, then by [Th1, 4.3] there is a homotopy associative, homotopy commutative H-space B which is universal for A and has the property that $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$. In this case the map $i\colon A \to B$ in the definition of the universal property induces the inclusion of the generating set in homology. Let $\iota\colon B \to B$ be the identity map on B. It is an H-map which extends i. In fact, since B is homotopy associative and homotopy commutative, the universal property implies that ι is the unique H-map extending i. Suppose ϵ has order p^r. Then as in [Th2, §5], there is a

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factorization of the p^r-power map on B as

$$\begin{array}{ccc} B & \xrightarrow{p^r} & B \\ \big\downarrow{\scriptstyle \overline{a}\times\overline{q}} & & \big\| \\ S^{2m+1}\times S^{2n+1} & \xrightarrow{\overline{j}\cdot\overline{c}} & B, \end{array}$$

where all maps are H-maps. Here $\overline{q}$ is an extension of the pinch map q, $\overline{j} \simeq i \circ j$; $\overline{a}$ and $\overline{c}$ are of degree p^r in $\mathbb{Z}_{(p)}$ homology; and $\overline{j}\cdot\overline{c}$ is the product of $\overline{j}$ and $\overline{c}$ given by the multiplication on B. Since B is homotopy associative and homotopy commutative, Lemma 2.1 implies that $H[B,B]$ is an abelian group. The homotopy commutativity of the diagram implies that in the group $H[B,B]$ we have $p^r = \gamma + \delta$ where γ and δ are the homotopy classes of $\overline{c} \circ \overline{q}$ and $\overline{j} \circ \overline{a}$, respectively.

In Proposition 6.12 we give an explicit generating set for $H[B,B]$, provided that $\pi_{2n+1}(B) \cong \mathbb{Z}_{(p)}$ is generated by the homotopy class of $\overline{c}$. This is an easy condition to check in the applications we have in mind.

Proposition 6.12 *Let $p \geq 5$. Let A be a two-cell co-H-space with cells in dimensions $2m+1$ and $2n+1$ for $n > m$. Let B be universal for A and suppose that $\pi_{2n+1}(B) \cong \mathbb{Z}_{(p)}$ is generated by the homotopy class of $\overline{c}$. Then $H[B,B]$ is isomorphic to the free abelian group over $\mathbb{Z}_{(p)}$ generated by ι and γ. Equivalently,*

$$H[B,B] \cong \mathbb{Z}_{(p)}\langle \iota, \gamma, \delta \mid p^r \cdot \iota = \gamma + \delta \rangle.$$

Proof We will show that $H[B,B] \cong \mathbb{Z}_{(p)}\langle \iota, \gamma\rangle$. If so, then the subsequent isomorphism $H[B,B] \cong \mathbb{Z}_{(p)}\langle \iota, \gamma, \delta \mid p^r \cdot \iota = \gamma + \delta\rangle$ is immediate. We begin by using the universal property of B for A to change the problem into an equivalent one.

Combining the universal property of B for A with the fact that B is homotopy associative and homotopy commutative, Lemma 2.4 implies that there is a group isomorphism $H[B,B] \xrightarrow{\cong} [A,B]$ which is given by precomposing each H-map $B \xrightarrow{f} B$ with $A \xrightarrow{i} B$. By definition, $\iota \circ i \simeq i$. Let g be the composite $g\colon A \xrightarrow{i} B \xrightarrow{\gamma} B$. Note that the definition of γ implies that g is homotopic to the composite $A \xrightarrow{q} S^{2n+1} \xrightarrow{\overline{c}} B$. Therefore, showing that $H[B,B] \cong \mathbb{Z}_{(p)}\langle \iota, \gamma\rangle$ is equivalent to showing that $[A,B] \cong \mathbb{Z}_{(p)}\langle i, q\rangle$. That is, it is equivalent to show that the homomorphism $\mathbb{Z}_{(p)}\langle i, g\rangle \to [A,B]$ determined by sending i and g to themselves is an isomorphism.

The homotopy cofibration sequence

$$S^{2n} \xrightarrow{\epsilon} S^{2m+1} \xrightarrow{j} A \xrightarrow{q} S^{2n+1} \xrightarrow{\Sigma\epsilon} S^{2m+2} \longrightarrow \cdots$$

determines a long exact sequence

$$\cdots \longrightarrow [S^{2m+2}, B] \xrightarrow{(\Sigma\epsilon)^*} [S^{2n+1}, B] \xrightarrow{q^*} [A, B] \xrightarrow{j^*} [S^{2m+1}, B] \xrightarrow{\epsilon^*} [S^{2n}, B].$$

Observe that for dimensional reasons $[S^{2m+2}, B] \cong [S^{2m+2}, S^{2m+1}]$ and the latter group is zero at odd primes. So $[S^{2m+2}, B] = 0$. Also, $[S^{2m+1}, B] \cong [S^{2m+1}, S^{2m+1}] \cong$

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$\mathbb{Z}_{(p)}$, and a generator is determined by the inclusion $S^{2m+1} \xrightarrow{\bar{j}} B$ of the bottom cell. This inclusion extends over A to give the map $A \xrightarrow{i} B$. Thus j^* is an epimorphism. Therefore the long exact sequence above reduces to a short exact sequence

$$0 \longrightarrow [S^{2n+1}, B] \xrightarrow{q^*} [A, B] \xrightarrow{j^*} [S^{2m+1}, B] \longrightarrow 0.$$

By hypothesis, $[S^{2n+1}, B] \cong \mathbb{Z}_{(p)}$ and a choice of the generator is $\bar{c}$. We have already seen that $[S^{2m+1}, B] \cong \mathbb{Z}_{(p)}$ and a choice of generator is the inclusion $\bar{j}$. Because $q^*(\bar{c}) = g$ and $j^*(\iota) = i$, there is a short exact sequence of groups

$$0 \longrightarrow [S^{2n+1}, B] \xrightarrow{q^*} \mathbb{Z}_{(p)}\langle i, g\rangle \xrightarrow{j^*} [S^{2m+1}, B] \longrightarrow 0.$$

The Five-Lemma therefore implies that the homomorphism $\mathbb{Z}_{(p)}\langle i, g\rangle \to [A, B]$ determined by sending i and g to themselves is an isomorphism. ∎

Proposition 6.12 has several applications. By Theorem 6.1, for $1 \leq k \leq p-1$ and $m \geq 1$, we have $\pi_{2m+kq}(S^{2m+1}) \cong \mathbb{Z}/p\mathbb{Z}$, generated by a stable class commonly named α_k. Each α_k originates on S^3, so if $m > 1$, then α_k is a suspension. When $m = 1$, it is also known that α_1 and α_2 are co-H-maps. Let $A^{2m+kq+1}$ be defined by the homotopy cofibration $S^{2m+kq} \xrightarrow{\alpha_k} S^{2m+1} \to A^{2m+kq+1}$. Then $A^{2m+kq+1}$ is a co-H-space if $m > 1$ or $m = 1$ and $k = 1, 2$. Let $B^{2m+kq+1}$ be universal for $A^{2m+kq+1}$. Since $S^{2m+kq+1}$ is homotopy associative and homotopy commutative, the pinch map $A^{2m+kq+1} \to S^{2m+kq+1}$ extends to an H-map $B^{2m+kq+1} \to S^{2m+kq+1}$. A Serre spectral sequence calculation shows that the homotopy fiber of this map has the homology of S^{2m+1} and so is homotopy equivalent to S^{2m+1}. Thus there is a homotopy fibration

$$S^{2m+1} \longrightarrow B^{2m+kq+1} \longrightarrow S^{2m+kq+1}.$$

By [MNT2, §6] or an easy calculation using Lemma 6.1, $\pi_{2m+kq+1}(B^{2m+kq+1}) \cong \mathbb{Z}_{(p)}$ and $\bar{c}$ is a choice of generator. Thus Proposition 6.12 implies that there is a group isomorphism

$$H[B^{2m+kq+1}, B^{2m+kq+1}] \cong \mathbb{Z}_{(p)}\langle \iota, \gamma, \delta \mid p \cdot \iota = \gamma + \delta\rangle.$$

More specific examples are as follows.

Example 6.13 By Lemma 6.11, when $p = 5$ there is a group isomorphism

$$[G_2, G_2] \cong H[G_2, G_2].$$

We can now complete the calculation. As noted in the proof of Lemma 6.11, G_2 is universal for a two-cell complex A where $\widetilde{H}_*(A) \cong \{x_3, x_{11}\}$ and $\mathcal{P}^1_*(x_{11}) = x_3$. The Steenrod operation $\mathcal{P}^1_*$ detects the homotopy class α_1, so the space A is what was called A^{11} above. Therefore its universal space G_2 is what was called B^{11} above. Hence we have a group isomorphism

$$H[G_2, G_2] \cong \mathbb{Z}_{(5)}\langle \iota, \gamma, \delta \mid p \cdot \iota = \gamma + \delta\rangle.$$

Example 6.14 Let $p \geq 5$ and consider B^{2p+1}. Note the corresponding homotopy fibration is $S^3 \to B^{2p+1} \to S^{2p+1}$. This is an interesting space because it is homotopy equivalent to a Clark–Ewing loop space determined by a reflection group. Note, though, that the homotopy equivalence may only be as spaces rather than H-spaces. In terms of its homotopy associative, homotopy commutative multiplication, Proposition 6.12 gives a group isomorphism

$$H[B^{2p+1}, B^{2p+1}] \cong \mathbb{Z}_{(p)}\langle \iota, \gamma, \delta \mid p \cdot \iota = \gamma + \delta \rangle.$$

Moreover, an argument exactly as in Lemma 6.11 shows that there is a group isomorphism $[B^{2p+1}, B^{2p+1}] \cong H[B^{2p+1}, B^{2p+1}]$. That is, every self-map of B^{2p+1} is homotopic to an H-map, and the H-maps are described explicitly by the group presentation above.

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