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CONTINUOUS RINGS AND RINGS OF QUOTIENTS

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I. **Introduction.** Throughout R will denote an associative ring with identity. Let $Z_\ell(R)$ be the left singular ideal of R. It is well known that $Z_\ell(R)=0$ if and only if the left maximal ring of quotients of R, $Q(R)$, is Von Neumann regular. When $Z_\ell(R)=0$, $q(R)$ is also a left self injective ring and is, in fact, the injective hull of R. A natural generalization of the notion of injective is the concept of left continuous as studied by Utumi [4]. One of the major obstacles to studying the relationships between $Q(R)$ and R is a description of $J(Q(R))$, the Jacobson radical of $Q(R)$. When a ring is left continuous, then its left singular ideal is its Jacobson radical. This facilitates the study of the cases when either $Q(R)$ is continuous or R is continuous.

Since, in case R is left self-injective, $Q(R)=R$, it is natural to ask what happens in case R is left continuous. It does seem to be true that left continuity of R implies that of $Q(R)$.

Generalizations of all the above questions can be placed in the setting of torsion theories of Gabriel [1], and so this is the setting in which we work.

If a ring R is both right and left continuous we know from Utumi, [4], that $Z_\ell(R)=J(R)=Z_r(R)$. When $Z_\ell(R)\neq 0$, then $Z_\ell(R)$ is not the torsion part of R relative to the torsion theory determined by the essential left ideals, but $Z_\ell^2(R)$ is. In the final section of this note we show that if R has enough non-singular primitive idempotents ($e=e^2$ is non singular if $Z_\ell(Re)=0$) then $Z_r^2(R)=Z_\ell^2(R)$. From this we deduce that continuous rings with enough non-singular idempotents are products of regular rings and rings in which the Jacobson radical is essential as a left ideal and as a right ideal.

II. **Preliminaries.** Throughout R will denote a ring with identity and, unless otherwise stated or obvious, all modules will be unitary left modules.

For torsion theory and localization we will use the following notation and terminology due mainly to Goldman [2]. A torsion theory will be denoted by σ and σ will also denote the subfunctor of the identity functor which picks out the torsion submodule, $\mathscr{F}_\sigma$ will denote the corresponding filter of left ideals, Q_σ the localization functor, elements of $\mathscr{F}_\sigma$ will be called σ-open left ideals and a submodule, N, of a module M such that $\sigma(M/N)=M/N$ will be called a σ-open submodule. All σ will be assumed to be idempotent.

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Definition and Theorem [2]. *Let σ be a torsion theory. The following are equivalent*

(i) *all $Q_\sigma(R)$ modules are σ-torsion free*

(ii) *For $i(R)$ = the image of R in $Q_\sigma(R)$ and $U \in \mathcal{F}_\sigma$, $Q_\sigma(R)i(U) = Q_\sigma(R)$*

(iii) *$Q_\sigma(R) \otimes_R (M)$ for all R modules M.*

(iv) *Q_σ preserves direct sums and is right exact.*

We say a torsion theory σ is perfect if σ satisfies any of (i)–(iv) in the above. We will also identify R with $i(R)$ whenever $\sigma(R) = 0$.

Definition. The left singular submodule of a module M is the set of elements annihilated by essential left ideals of R, and will be denoted by $Z_\ell(M)$. $Z_\ell^2(M)$ will denote the inverse image in M under the natural map of the left singular submodule of $M/Z_\ell(M)$.

We will say that a submodule N of a module M misses a submodule N' of M if $N \cap N' = 0$, that N is a complement if N is maximal w.r.t. missing some N' and that a pair (N, N') are mutual complements if N is maximal w.r.t. missing N' and N' is maximal w.r.t. missing N.

Definition. An R-module M is called continuous if (i) each submodule has an essential extension in M which is a direct summand of M, (ii) each submodule isomorphic to a direct summand is a direct summand.

A ring is said to be left continuous if it is continuous as a left module over itself. Note that all left self-injective rings are left continuous.

Lemma. *If M is a continuous R module and N a complement in M, then N is a direct summand of M.*

Proof. Let N' be such that N is maximal w.r.t. missing N'. Then N is contained in a direct summand which is an essential extension of N, but by maximality N is this summand.

III. **Quotient Rings.** If R is a ring with zero singular left ideal and $\mathcal{F}_\sigma$ is the filter of dense left ideals, $Q_\sigma(R)$ is a regular (Von-Neumann) ring and can be identified with the injective hull of R. In case $Z_\ell(R) \neq 0$, then $Q_\sigma(R)$ is not regular, is not necessarily self-injective and can not be identified with the injective hull of R. In general, if R is self-injective and σ is a torsion theory for which $\sigma(R) = 0$, then $Q_\sigma(R) = R$, so is self-injective.

We first examine the quotient rings of left continuous rings.

Proposition. *Let R be a left continuous ring and σ a torsion theory with $\sigma(R) = 0$. Then every left ideal of $Q_\sigma(R)$ is contained in a $Q_\sigma(R)$-essential extension which is a $Q_\sigma(R)$ summand of $Q_\sigma(R)$.*

Proof. Let I be a nonzero left ideal of $Q_\sigma(R)$. Then $I \cap R \neq 0$. Let X be any complement of $I \cap R$ and let Y be maximal w.r.t. missing X and containing

$I \cap R$. Such X and Y exist by Zorn's lemma. Then, because R is continuous $R = X \oplus Y$ and $Q_\sigma(R) = Q_\sigma(X) \oplus Q_\sigma(Y)$. Now $R \cap I$ is R-essential in $\mathscr{Y}$ and Y is R-essential in $Q_\sigma(Y)$ so $R \cap I$ is R-essential in $Q_\sigma(Y)$. It follows that $Q_\sigma(R \cap I) \supset I$ is $Q_\sigma(R)$-essential in $Q_\sigma(Y)$.

LEMMA. *Let R be a ring and σ a perfect torsion theory with $\sigma(R) = 0$. Then if I is a left ideal of $Q_\sigma(R)$, $Q_\sigma(R \cap I) = I$.*

Proof. $I/R \cap I$ is σ-torsion and, since σ is perfect, I is σ-torsion free and $Q_\sigma(R \cap I) \cong Q_\sigma(R) \otimes R \cap I \cong Q_\sigma(R)(R \cap I) \subset I$. But by the definition of $Q_\sigma(R \cap I)$ we must have equality since all $Q_\sigma(R)$ modules are σ-torsion free.

We now can prove:

THEOREM. *Let R be a ring, σ a perfect torsion theory with $\sigma(R) = 0$. If R is left continuous then $Q_\sigma(R)$ is left continuous.*

Proof. By the previous proposition all we need show is that if A and B are left ideals of $Q_\sigma(R)$ with $A = Qe$, $e^2 = e$ and $A \cong B$, then $B = Qf$ for some $f = f^2$. First note that $Qe \cap R$ is a complement of $Q(1-e) \cap R$ so is a direct summand of R, hence we have $Qe \cap R = Re'$, $(e')^2 = e'$, and $Qe' = Qe$. Therefore, without loss of generality, we let $e \in R$. Let $\psi : Qe \to B$ be a Q_σ-isomorphism. The image of $Re = Qe \cap R$ under ψ is a σ-open R-submodule of B, B_1 say. Now $B_1 \cap R = B_2$ is an open submodule of B_1 hence of B. Since σ is perfect $Q_\sigma(R)B_2 = B$. Now take, by left continuity of R, $B_2 \subset Rg$, $g^2 = g$, and B_2 essential in Rg. Then $Q_\sigma(Rg) = Q_\sigma(R)g$, by perfectness of σ and $\sigma(R) = 0$. Also $Q_\sigma(R)g$ is an essential extension of B_2 hence of B. Let $E = \{r \in R : rg \in B_2\}$. Then since $B_2 \subset Q_\sigma(R)g$, $Eg = B_2$. This says $Q_\sigma(R)Eg = B$. Since $B \cong Q_\sigma(R)e$, $B = Q_\sigma(R)b$ for some b. Let $b = \sum_{i=1}^n q_i x_i g$ where $q_i \in Q_\sigma(R)$ and $x_i \in E$. Let B_3 be the left ideal of $Q_\sigma(R)$ generated by $\{x_i g\}_{i=1}^n$. $B_3 \subset B_2$ and $Q_\sigma(R)B_3 = B$, so B_3 is σ-open in B_2. Let A_0 be the left ideal of R generated by $\psi^{-1}(x_i g)$ $i = 1, \ldots, n$. Note that A_0 is σ-open in Re. Note also that if $x \in Z_l(R)$, then $x \in Z_l'(Q_\sigma(R))$, the $Q_\sigma(R)$-singular submodule of $Q_\sigma(R)$. Since R is left continuous, $Z_l(R)$ is the Jacobson radical of R and indempotents of $R/Z_l(R)$ lift. Now letting $\bar{Y}$ be the image of Y in $R/Z_l(R)$ for Y any subset of R we see that $\bar{A}_0$ is generated by an idempotent. Lemma 3.1 of [4] implies we have an idempotent h in Re such that $\bar{R}\bar{h} = \bar{A}_0$. Then $h - z \in A_0$ for some $z \in Z_l(R)$. Then $h - hz \in A_0$ and $R(h - hz) = Rh(1-z)$. Now $1 - z$ is a unit so $R(h - hz) = Re_1$, where $e_1^2 = e_1 \in A_0$. It follows that A_0 is generated by $e_1, z_2, \ldots, z_n$, with $z_i \in Z_l(R) \cap A_0$, $i = 2, \ldots, n$. If $Re \neq Re_1$, then $Re = Re_1 \oplus Re_2$ with $e_1 e_2 = 0$ and $e_2^2 = e_2$. Since A_0 is σ-open, $e_2 = q_1 e_1 + \sum_{1=2}^n q_i z_i$ which gives $e_2 = \sum_{i=2}^n q_i z_i e_2 \in Z_l'(Q)$ so $e_2 = 0$. If $A_0 = Re$ then $\psi(Re) \subset R$ and is idempotent generated by $f = f^2$ say. It follows that $Qf = B$.

Of course, many non-left continuous rings have left continuous quotient rings. Consequences of having a left continuous quotient ring are what we take

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up next. The easiest and most straightforward observation is the following:

PROPOSITION. *If R is a ring, σ a torsion theory for which $\sigma(R)=0$ and $Q_\sigma(R)$ is left continuous, then for any pair (A, B) of mutual left complements, $A\oplus B\in \mathcal{F}_\sigma$.*

Proof. If A and B are mutual complements then $Q_\sigma(A)\cap Q_\sigma(B)=0$ in $Q_\sigma(R)$. Suppose C is a left ideal of $Q_\sigma(R)$ and $C\supset Q_\sigma(A)$ and $C\cap Q_\sigma(B)=0$. Then $C\cap R\supset A$ so $C\cap R=A$ for $C\cap B\cap R=0$. But $C=Q_\sigma(A)$ is a complement in $Q_\sigma(R)$. The same holds for $Q_\sigma(B)$ w.r.t. $Q_\sigma(A)$ and because $Q_\sigma(R)$ is left continuous $Q_\sigma(A)\oplus Q_\sigma(B)=Q_\sigma(R)$. We also have $Q_\sigma(A\oplus B)=Q_\sigma(A)\oplus Q_\sigma(B)$ so that $A\oplus B\in\mathcal{F}_\sigma$.

COROLLARY. *If R, σ, and $Q_\sigma(R)$ are as above and A is open in a complement of B in R then $Q_\sigma(A)$ is a direct summand in $Q_\sigma(R)$.*

Utumi showed in [4] that for left continuous rings, R, $Z_\ell(R)=J(R)$. In case $Q_\sigma(R)$ is left continuous we gain an alternative description of $J(Q_\sigma(R))$.

DEFINITION. Let σ be a torsion theory for a ring R. A left ideal L is called σ-critical if for every $A\supset L$, $A\in\mathcal{F}_\sigma$ and R/L is σ torsion free.

DEFINITION. Let $J_\sigma(R)=$ the intersection of the σ-critical left ideals.

THEOREM [3]: *Let R be a ring and σ be a perfect torsion theory. Then $J(Q_\sigma(R))=Q_\sigma(R)(J_\sigma(R))$.*

Proof. When σ is perfect L is critical iff $Q_\sigma(R/L)$ is simple. Since $J(Q_\sigma(R))$ is the intersection of the left annihilators of simple $Q_\sigma(R)$ modules, one can deduce easily that $x\in(R\cap J(Q_\sigma(R)))$ iff x annihilates every R/L where L is critical, iff $x\in L$ for every critical left ideal, L.

LEMMA. *If R is a ring and σ is a perfect torsion theory then $Z_\ell({}_RQ_\sigma(R))=Z_\ell({}_{Q_\sigma(R)}Q_\sigma(R))=Q_\sigma(R)Z_\ell(R)$.*

Proof. Let $Q=Q_\sigma(R)$. Let $x\in Z_\ell({}_QQ)$. Then $Ex=0$ for some essential left ideal E of Q. Then $E\cap R$ is an essential left ideal of R so $x\in Z_\ell({}_RQ)$. If E' is an essential left ideal of R, then QE' is an essential left ideal of Q, so the reverse inclusion holds and this gives the first equality. The second equality holds since σ is perfect and $Z_\ell(Q)\cap R=Z_\ell(R)$.

LEMMA. *If R is a ring and σ is perfect, then $Z_\ell(R)$ finitely generated implies $Z_\ell(Q_\sigma(R))$ is finitely generated.*

THEOREM. *If R is a ring, σ is a perfect torsion theory, and $Q_\sigma(R)$ is left continuous, then $J_\sigma(R)=Z_\ell(R)$.*

Proof. Since $J(Q_\sigma(R))=Z_\ell(Q_\sigma(R))=Q_\sigma(R)Z_\ell(R)$ and $J(Q_\sigma(R))=Q_\sigma(R)J_\sigma(R)$ it follows that $J(Q_\sigma(R))\cap R=Z_\ell(R)=J_\sigma(R)$.

This gives another description of Quasi-Frobenius quotient rings.

THEOREM. *Let R be a ring and σ torsion theory for which $\sigma(R)=0$. Then $Q_\sigma(R)$ is Quasi-Frobenius iff*

(i) *For any pair (A, B) of mutual left complements $A \oplus B \in \mathcal{F}_\sigma$*

(ii) *If A is σ-open in a left complement and $A \cong B$ then B is open in a left complement.*

(iii) *R has finite Goldie dimension and $Z_\ell(R)$ contains a f.g. open submodule*

(vi) *σ is perfect.*

Proof. Given conditions (i)–(iv) to show that $Q_\sigma(R)$ is Quasi-Frobenius first note that (i), (ii), and (iv) insure that $Q_\sigma(R)$ is continuous so that $Q_\sigma(R)/J(Q_\sigma(R))$ is regular and idempotents lift, [4]. Now $Q_\sigma(R)/JQ_\sigma(R)$ is a regular ring and the Goldie dimension of $Q_\sigma(R)/JQ_\sigma(R)$ is less than the Goldie dimension of $Q_\sigma(R)$ because idempotents lift. The Goldie dimension of $Q_\sigma(R)$ equals the Goldie dimension of R so $Q_\sigma(R)/JQ_\sigma(R)$ is a regular ring of finite Goldie dimension hence semi-simple Artinian. By the above lemmas $J(Q_\sigma(R))$ is finitely generated so $Q_\sigma(R)$ is quasi Frobenius by Utumi [4].

For the converse we know, [3], that σ is perfect. The other conditions follow easily from this and are left to the reader.

Along the same lines we have

THEOREM. *Let R be a ring and σ a perfect torsion theory. If $J_\sigma(R)$ is right T-nilpotent and $R/Z_\ell(R)$ has finite Goldie dimension, then $Q_\sigma(R)$ is left perfect whenever $Q_\sigma(R)$ is continuous.*

Proof. As above $Q_\sigma(R)/JQ_\sigma(R)$ is semi-simple Artinian. $J(Q_\sigma(R))$ is a two-sided ideal. If $JQ_\sigma(R)$ is not right T-nilpotent choose $\{a_k\}_{k=1}^\infty$ so that a_n, $a_{n-1}, \ldots, a_1 \neq 0$ for all n. Write $a_1 = \sum_{i=1}^{n_1} q_{1i}x_{1i}x_j \in J_\sigma$, and $a_2q_{1i} = \sum_{j=1}^{n_{2i}} q_{2ji}x_{2ji}$, $x_{2j} \in J_\sigma$ $a_2a_1 = \sum_{i,j} q_{2ji}x_{2ji}x_{1i}$. Letting $a_3q_{2ji} = \sum_m q_{3mji}x_{3mji}$, $x_{mji} \in J_\sigma$ gives $a_3a_2a_1 = \sum_{i,j,m} q_{3mji}x_{3mji}x_{2ji}x_{1i} \neq 0$. In this manner we create longer and longer chains of products in J_σ. By the Konig graph theorem we contradict the right T-nilpotence of J_σ.

REMARK. A similar type theorem holds for semi-perfect continuous quotient rings.

IV. $Z_\ell^2(R)$. If R is both right and left continuous $Z_\ell(R) = J(R) = Z_\ell(R)$. $Z_\ell^2(R)$ is the maximal left essential extension of $Z_\ell(R)$ contained in R and a similar statement holds for $Z_r^2(R)$. If R is continuous (on both the right and left) is it true that the symmetry extends to $Z_\ell^2(R)$? (i.e. does $Z_\ell^2(R) = Z_r^2(R)$?). As a partial answer to this we have the following:

THEOREM. *If R is continuous and (i) for any idempotent e such that $Z_\ell(Re) = 0$ there exists a primitive idempotent e' such that $Re \supseteq Re'$ and (ii) for any idempotent e such that $Z_r(eR) = 0$ there exists a primitive idempotent e' such that $e'R \subseteq eR$, then $Z_\ell^2(R) = Z_r^2(R)$.*

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Proof. By left continuity of R there exist left ideals L_1 and L_2 such that $L_1 \oplus L_2 = R$ and $L_1 = Z_\ell^2(R)$. Similarly, on the right $R = H_1 \oplus H_2$ where $H_1 = Z_r^2(R)$. It is easy to see that H_1 and L_1 are two sided ideals of R. Suppose L_1 is not an essential right extension of $Z_\ell(R)$. Then there exists $x \in Z_\ell^2(R)$ such that $xR \cap Z_\ell(R) = 0$, in fact $xR \cap Z_r^2(R) = 0$. Now let $\ell(x)$ be the left annihilator of x. Because $x \notin Z_\ell(R)$, $\ell(x)$ is not essential so there exists a non-zero left ideal A such that $Ax \cong A$. If A is chosen to be a complement of $\ell(x)$, then A is a direct summand of R and so is Ax, hence $Ax = Ah$ for some idempotent h. We claim $hR \cap Z_\ell(R) = 0$. To see this, note that if $n \in hR \cap Z_\ell(R)$, $hn = n$ and $hn = axn$. But $xZ_\ell(R) = 0$, for $xR \cap Z_\ell(R) = 0$. It follows that $hR \cap Z_r^2(R) = 0$. Let $Z_r^2(R) = fR$. Then $hf = 0$, so $h = h(1-f)$ and hR is a non-singular right ideal of R. Using (ii) it follows that we could have taken h to be primitive. For h primitive we claim hR is simple. Let hI be a right ideal contained in hR. If $hIh = 0$, then $hI \subseteq J(R) = Z_\ell(R)$ so $hIh \neq 0$. In case $hIh \neq 0$ we have a map $hR \to hI$ which is a monomorphism because hR is uniform. Any such map gives rise to a summand of R, hence of hI, hence of hR. But hR is indecomposable so $hI = hR$ and hR is simple. If hR is simple, hR is contained in every essential right ideal and since $J(R) = Z_r(R)$, $J(R)hR = 0$. In particular $J(R)h = 0$ and $Rh \cap Z_\ell(R) = 0$, a contradiction. A similar argument gives that $Z_r^2(R)$ is an essential left extension of $Z_\ell(R)$, so we have $Z_\ell^2(R) = Z_r^2(R)$.

Corollary. *If R is as in the theorem, then L_2 and H_2 can be taken to be equal and two sided ideals.*

Proof. Let $Z_\ell^2(R) = Re$ and $Z_r^2(R) = fR$. Now $e - f = fe - f = f(e-1) \in fR \cap R(1-e) = 0$ so $e = f$ and $Re = eR$. Also $(1-e)R(1-e) = R(1-e)$ is two sided too!

Corollary. *If R is a continuous ring and of finite Goldie dimension on both sides, then R is the ring direct product of a semi-simple Artinian ring and a semi-perfect ring with Jacobson radical an essential left ideal.*

Corollary. *If R is continuous with conditions (i) and (ii) and M is a cyclic R-module, then $M = T(M) \oplus N$ where $T(M) = Z_\ell^2(M)$. Moreover, if sums of non-singular modules are non-singular then for every module, M, $M = Z_\ell^2(M) \oplus N$ for some N.*

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