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A REMARK ON MAPS BETWEEN CLASSIFYING SPACES OF COMPACT LIE GROUPS

BY
ZDZISŁAW WOJTKOWIAK

ABSTRACT. We show that two maps between classifying spaces of compact, connected Lie groups are homotopic after inverting the order of the Weyl group of the source if and only if they induce the same maps on rational cohomology. We shall also give some results on maps from classifying spaces of finite groups to classifying spaces of compact Lie groups. Among other things we construct a map from $B(Z/2 + Z/2 + Z/3)$ into $BSO(3)$ which is not induced by a homomorphism.

0. In this note we prove some results about maps between classifying spaces. The results are corollaries of the Dwyer and Zabrodsky Theorem (see [2] 1.1 Theorem) and two results of us (see [3] Theorem 2 and Theorem 3).

1. In the first section we give a proof of the following theorem which we announced in the Barcelona Conference on Algebraic Topology 1986.

THEOREM 1. *Let G and H be connected, compact Lie groups and let n be the order of the Weyl group of G. Let $f{:}BG \to BH$ and $g{:}BG \to BH$ be two maps such that $H^*(f; Q) = H^*(g; Q)$. Let $c_p{:}BH \to BH_{(p)}$ be a localization map. Then for any prime p not dividing n, the maps*

$$BG \xrightarrow{f} BH \xrightarrow{c_p} BH_{(p)}$$

and

$$BG \xrightarrow{g} BH \xrightarrow{c_p} BH_{(p)}$$

are homotopic.

The proof of Theorem 1 is based on the following results.

PROPOSITION 1. (see [3] Theorem 2). *Let G be a connected, compact Lie group,*

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T a maximal torus in G and W a Weyl group of G. If p is a prime such that $(p; |W|) = 1$ and X is nilpotent of finite $\hat{Z}_p$-type, then the natural map

$$[BG; X] \to [BT; X]^W$$

is a bijection.

PROPOSITION 2. (see [4]). *Let T be a torus and G a connected compact Lie group. Any map $f:BT \to BG$ is homotopic to the map $B\varphi:BT \to BG$, where $\varphi:T \to G$ is a homomorphism.*

The last results we need are the following lemmas.

LEMMA 1. *Let $f, g:T \to H$ be two homomorphisms from a torus T to a connected, compact Lie group H. Let us assume that $H^*(Bf, Q) = H^*(Bg, Q)$. Then the homomorphisms f and g are conjugate.*

PROOF. Let $R(G)$ be the complex representation ring of G and $K^0(BG) = [BG; Z \times BU]$. We have the following commutative diagram

$$\begin{array}{ccc}
R(T) & \xleftarrow{\quad R(f) \quad} & R(H) \\
\downarrow{\scriptstyle i_T} & & \downarrow{\scriptstyle i_H} \\
K^0(BT) & \xleftarrow{\quad K^0(Bf) \quad} & K^0(BH) \\
\downarrow{\scriptstyle ch_T} & & \downarrow{\scriptstyle ch_H} \\
\prod_{i=0}^{\infty} H^{2i}(BT, Q) & \xleftarrow{\quad H(Bf, Q) \quad} & \prod_{i=0}^{\infty} H^{2i}(BH, Q)
\end{array}$$

and the similar diagram for g. The homomorphisms i_T, i_H, ch_T and ch_H are monomorphisms. Therefore the equality $H^*(Bf, Q) = H^*(Bg, Q)$ implies that $R(f) = R(g)$. The torus T is a monogenic group and let $t \in T$ be a generator of T. The equality $R(f) = R(g)$ implies that there is $a \in H$ such that $f(t) = a \cdot g(t) \cdot a^{-1}$ because the characters of a compact Lie group separate conjugacy classes. Therefore the homomorphism f is conjugate to g.

LEMMA 2. *Let f and g be two maps from BG to BH where H and G are connected, compact Lie groups. Let $\hat{c}_p:BH \to BH_p^\wedge$ be the p-completion map. Let us suppose that $\hat{c}_p \circ f$ is homotopic to $\hat{c}_p \circ g$. Then $c_p \circ f$ is homotopic to $c_p \circ g$.*

PROOF. The rationalization of BH, the space BH_0 is an H-space with a multiplication μ. Therefore we have a fibration

$$BH_{(p)} \xrightarrow{i = (\hat{c}_p, c_0)} BH_p^\wedge \times BH_0 \xrightarrow{j} (BH_p^\wedge)_0,$$

where j is the composition:

$$BH_p^\wedge \times BH_0 \to (BH_p^\wedge)_0 \times (BH_p^\wedge)_0 \to (BH_p^\wedge)_0.$$

Hence we get a long exact sequence of sets

$$[BG, \Omega(BH_p^\wedge)_0] \to [BG, BH_{(p)}] \xrightarrow{i_*} [BG, BH_p^\wedge] \times [BG, BH_0] \to [BG, (BH_p^\wedge)_0].$$

The abelian group $[BG, \Omega(BH_p^\wedge)_0]$ acts on the set $[BG, BH_{(p)}]$ and $i_*(f) = i_*(g)$ if and only if there is an element $z \in [BG, \Omega(BH_p^\wedge)_0]$ such that $z * f = g$ where $*$ denotes the action. The group $[BG, \Omega(BH_p^\wedge)_0]$ equals 0, hence $c_p \circ f$ is homotopic to $c_p \circ g$ if and only if $\hat{c}_p \circ f$ is homotopic to $\hat{c}_p \circ g$.

PROOF OF THEOREM 1. Let $i:BT \to BG$ be a map induced by the inclusion of a maximal torus T of G into G. We have $H^*(f \circ i, Q) = H^*(g \circ i, Q)$. It follows from Corollary 1 and Lemma 1 that $f \circ i$ is homotopic to $g \circ i$. Therefore also $\hat{c}_p \circ f \circ i$ is homotopic to $\hat{c}_p \circ g \circ i$. Proposition 1 implies for any prime p not dividing $|W|$ that $\hat{c}_p \circ f$ is homotopic to $\hat{c}_p \circ g$. Hence it follows from Lemma 2 that $c_p \circ f$ is homotopic to $c_p \circ g$.

2. In this section we shall investigate maps from $B\pi$ to BH where π is a finite group and H is a connected compact Lie group.

We recall the definition and notations from [3]. Let π be a finite group, let π_p be a p-Sylow subgroup of π, let $N(\pi_p)$ be a normalizer of π_p in π and $W_p = N(\pi_p)/\pi_p$.

DEFINITION 1. (see [3]). We say that π satisfies W_p-condition if the map

$$H^*(\pi, Z_{(p)}) \to H^*(\pi_p, Z_{(p)})^{W_p} \text{ is an isomorphism.}$$

PROPOSITION 3. (see [3] Theorem 3). *If a finite group π satisfies the W_p-condition, then for any nilpotent p-local space X of $Z_{(p)}$-finite type the natural map*

$$[B\pi, X] \to [B\pi_p, X]^{W_p}$$

is a bijection.

PROPOSITION 4. (see [2] 1.1 Theorem). *If P is a finite p-group and G is a connected, compact Lie group then any map $f:BP \to BG$ is homotopic to the map $B\varphi:BP \to BG$ where $\varphi:P \to G$ is a homomorphism.*

Let f and g be two homomorphisms from p-group P to a connected, compact Lie group H such that Bf is homotopic to Bg. Then f is conjugate to g i.e. there exists $h \in H$ such that for each $x \in P$, $f(x) = h \cdot g(x) \cdot h^{-1}$.

DEFINITION 2. Let P and H be two groups. $\mathrm{Hom}_{\mathrm{conj}}(P, H)$ is the set of equivalence classes on the set $\mathrm{Hom}(P, H)$ defined by the equivalence relation $f \sim g$ if and only if f and g are conjugate.

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We state now main results of this section.

THEOREM 2. *Let π be a finite group which satisfies W_p-condition for each prime p, let H be a connected, compact Lie group and let n be the order of π. Then the obvious maps*

$$[B\pi, BH] \to \prod_{p|n} [B\pi_p, BH]^{W_p} \leftarrow \prod_{p|n} (\mathrm{Hom}_{\mathrm{conj}}(\pi_p, H))^{W_p}$$

$$\prod_{p|n} [(Z/p)_\infty(B\pi), BH] \to \prod_{p|n} [B\pi_p, BH]^{W_p}$$

are bijective.

COROLLARY 1. Let π *be a finite group which satisfies W_p-condition for each prime p. Let H be a connected, compact Lie group. The maps f, $g: B\pi \to BH$ are homotopic if and only if*

$$K^0(f) = K^0(g): K^0(BH) \to K^0(B\pi).$$

THEOREM 3. *Let G be a connected, compact Lie group such that any finite abelian p-group of G is conjugated to a subgroup of a maximal torus T of G. Then any map $f: BA \to BG$ where A is a finite abelian group is induced by a homomorphism $\rho: A \to G$.*

We note that the assumption is satisfied by groups $U(n)$ and $SU(n)$.

PROOFS.

LEMMA 3. *Let P be a p-group and let X be a simply connected space of finite type. Then we have bijections*

$$[BP, X] \overset{\approx}{\to} [BP, X_{(p)}] \overset{\approx}{\to} [BP, X_p^\wedge]$$

induced by natural maps

$$X \to X_{(p)} \to X_p^\wedge.$$

We omit the trivial proof.

PROOF OF THEOREM 2. It follows from the standard properties of localization that

$$(1) \qquad [B\pi, BH] \approx \prod_{p|n} [B\pi, BH_{(p)}].$$

Proposition 3 and Lemma 3 imply that

$$(2) \qquad \prod_{p|n} [B\pi, BH_{(p)}] \approx \prod_{p|n} [B\pi_p, BH]^{W_p}.$$

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Proposition 4 implies that

$$(3) \qquad \prod_{p|n} [B\pi_p, BH]^{W_p} \approx \prod_{p|n} (\mathrm{Hom}_{\mathrm{conj}}(\pi_p, H))^{W_p}.$$

The natural maps

$$B\pi \to \prod_{p|n} (Z/p)_\infty(B\pi) \leftarrow \bigvee_{p|n} (Z/p)_\infty(B\pi)$$

are homology equivalences. We recall that $(Z/p)_\infty(\ )$ is the Bousfield-Kan completion (see [1]). Therefore we have a bijection

$$(4) \qquad [B\pi, BH] \approx \prod_{p|n} [(Z/p)_\infty(B\pi), BH].$$

The theorem follows from (1), (2), (3) and (4).

PROOF OF COROLLARY 1. It follows from Theorem 2 that it is enough to show the result for p-groups. Let us assume first that $H = U(n)$. Then if $\varphi:P \to U(n)$ $\psi:P \to U(n)$ are such that $K^0(B\varphi) = K^0(B\psi)$ then the representations φ and ψ are conjugated. In general case we imbed H in $U(n)$ and we repeat the arguments from [4] pp. 5 and 6.

PROOF OF THEOREM 2. Let A_p be a p-Sylow subgroup of A. It follows from Proposition 4 that $f_{|BA_p}$ is induced by $\rho_p:A_p \to H$. We can assume that im $\rho_p \subset T$ for each p. Then one defines $\rho:A \to G$ by the formula $\rho\{(a_p)_p\} = \prod_p \rho_p(a_p)$. It is clear that $B\rho$ is homotopic to f.

LEMMA 4. *Let G be a finite nilpotent group i.e. $G = \Pi P_i$ where P_i's are p_i-Sylow subgroups of G. Then for any simply-connected space X we have*

$$[BG, X] \simeq \Pi[BP_i, X].$$

We omit the trivial proof.

EXAMPLE 1. There is a map $f:B(Z/2 \times Z/2 \times Z/3) \to BSO(3)$ which is not induced by a homomorphism.

Let A_2 be a subgroup of $SO(3)$ of diagonal matrices with $+1$ and -1 on the diagonal and let $i_2:A_2 \to SO(3)$ be an inclusion. Let $i_3:A_3 = Z/3 \to SO(3)$ be an inclusion. Let $f:B(A_2 \times A_3) \to BSO(3)$ be a map which corresponds to the map $Bi_2 \vee Bi_3:BA_2 \vee BA_3 \to BSO(3)$ by Lemma 4. The map f is not induced by a homomorphism because there is no subgroup isomorphic to $A_2 \times A_3$ in $SO(3)$.

We mention that a similar example we can always find if there is a maximal, compact, abelian subgroup in G different from a maximal torus.

EXAMPLE 2. Let $i:SU(2) \to SU(4)$ be a natural inclusion. There is a map

$f: B(Q(8) \times Z/p) \to BSU(2)$ $(p > 2)$ such that f is not induced by a homomorphism but

$$B(Q(8) \times Z/p) \xrightarrow{f} BSU(2) \xrightarrow{Bi} BSU(4)$$

is induced by a homomorphism.

Let $j_1: Q(8) \to SU(2)$ and $j_2: Z/p \to SU(2)$, $p > 2$ be obvious inclusions. Let $G = Q(8) \times Z/p$. Let $f: BG \to BSU(2)$ be a map corresponding to $B_{j_1} \vee B_{j_2}: BQ(8) \vee BZ/p \to BSU(2)$. It is clear that f is not induced by a homomorphism because there is no subgroup of the form $Q(8) \times Z/p$ in $SU(2)$.

We show that the composition $(Bi) \circ f: BG \to BSU(4)$ is induced by a homomorphism. Let $\rho: G \to SU(4)$ be given by

$$\rho(g_1, g_2) = \begin{pmatrix} j_1(g_1), & \text{zeros} \\ \text{zeros}, & j_2(g_2) \end{pmatrix}.$$

It is clear that $B\rho$ is homotopic to $B(i) \circ f$. We point out that the examples of Sullivan, Friedlander, Adams and Milnor even stably are not induced by homomorphisms.

We finish with an example illustrating Theorem 2.

EXAMPLE 3. The group Σ_3 satisfies W_2 and W_3 conditions. Therefore we have

$$[B\Sigma_3, BU(2)] \approx [BZ/2, BU(2)] \times [BZ/3, BU(2)]^{Z/2}$$
$$\approx \mathrm{Hom}_{\mathrm{conj}}(Z/2, U(2)) \times (\mathrm{Hom}_{\mathrm{conj}}(Z/3, U(2)))^{Z/2}.$$

The set $\mathrm{Hom}_{\mathrm{conj}}(Z/2, U(2))$ contains three elements: The trivial homomorphism $1: Z/2 \to U(2)$, the homomorphism $1 \oplus \theta: Z/2 \to U(2)$ and the homomorphism $\theta \oplus \theta: Z/2 \to U(2)$; where $\theta: Z/2 \to U(1)$ is the unique non-trivial homomorphism from $Z/2$ to $U(1)$. The set $(\mathrm{Hom}_{\mathrm{conj}}(Z/3, U(2)))^{Z/2}$ contains two elements: the trivial homomorphism $1: Z/3 \to U(2)$ and the homomorphism $\xi \oplus \xi^2: Z/3 \to U(2)$ where ξ and ξ^2 are two non-trivial, different 1-dimensional representations of $Z/3$. Therefore we have six maps from $B\Sigma_3$ to $BU(2)$. These six maps induce different maps on cohomology. Four of them are induced by homomorphism from Σ_3 to $U(2)$. The last two maps are not of the form Bf for any homomorphism $f: \Sigma_3 \to U(2)$. These six maps were constructed in [3], but without Proposition 4 we were unable to show that there were no other maps.

Four maps lift to $BSU(2)$. Two maps are induced by homomorphisms and two maps are not. The classification of homotopy classes of maps from $B\Sigma_3$ to $BSU(2)$ was announced by C. Wilkerson in the Barcelona Conference on Algebraic Topology 1986.

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Universitat Autònoma de Barcelona
Bellaterra, Barcelona, Spain