REGULAR SEMIGROUPS WHICH ARE SUBDIRECT PRODUCTS OF A BAND AND A SEMILATTICE OF GROUPS

by MARIO PETRICH

(Received 27 April, 1971)

1. Introduction and summary. In the study of the structure of regular semigroups, it is customary to impose several conditions restricting the behaviour of ideals, idempotents or elements. In a few instances, one may represent them as subdirect products of some much more restricted types of regular semigroups, e.g., completely (0-) simple semigroups, bands, semilattices, etc. In particular, studying the structure of completely regular semigroups, one quickly distinguishes certain special cases of interest when these semigroups are represented as semilattices of completely simple semigroups. In fact, this semilattice of semigroups may be built in a particular way, idempotents may form a subsemigroup, $\mathscr{H}$ may be a congruence, and so on.

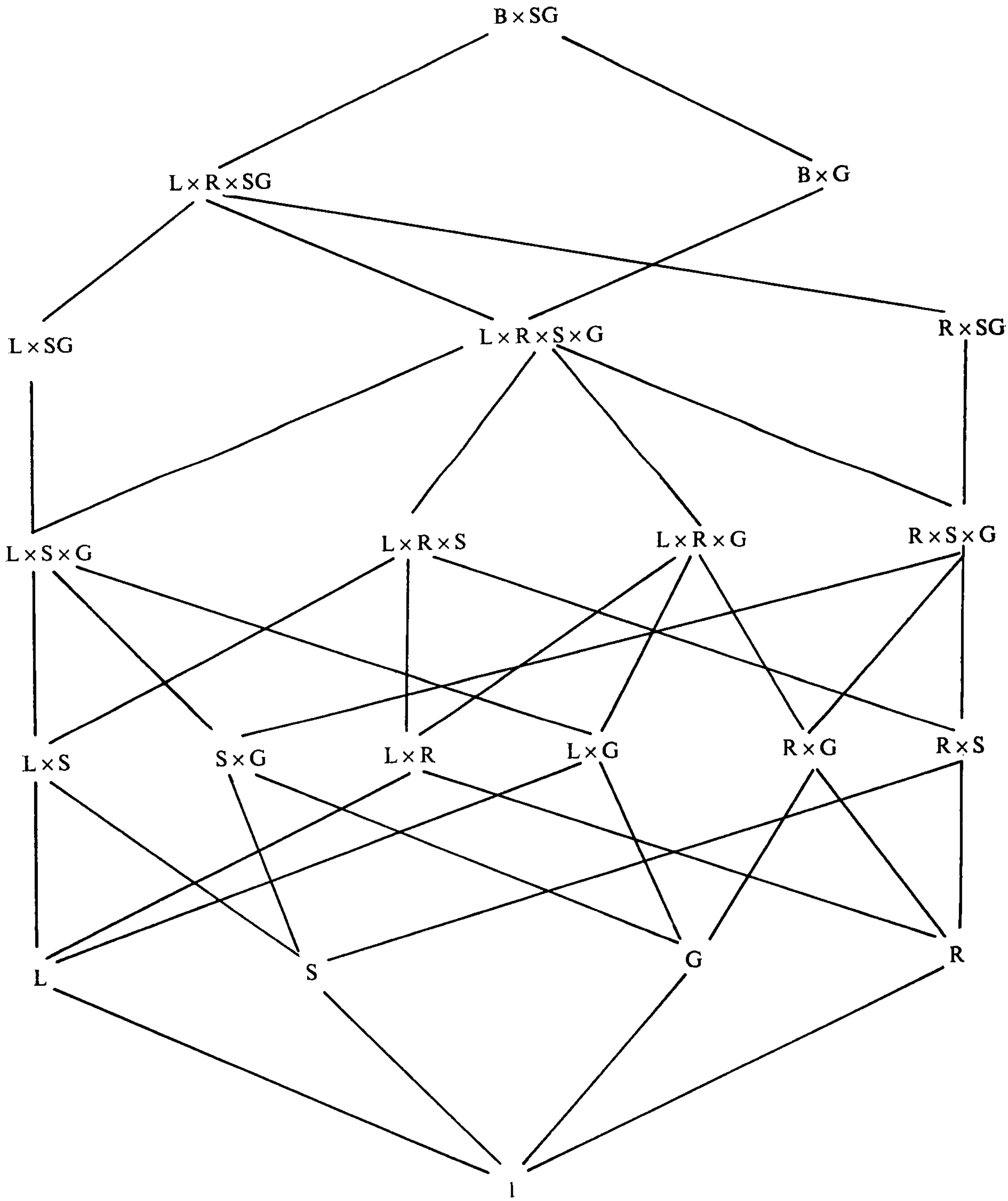
Instead of making some arbitrary choice of these conditions, we consider, in a certain sense, the converse problem, by starting with regular semigroups which are subdirect products of a band and a semilattice of groups. This choice, in turn, may seem arbitrary, but it is a natural one in view of the following abstract characterization of such semigroups: they are bands of groups and their idempotents form a subsemigroup. It is then quite natural to consider various special cases such as regular semigroups which are subdirect products of: a band and a group, a rectangular band and a semilattice of groups, etc. For each of these special cases, we establish (a) an abstract characterization in terms of completely regular semigroups satisfying some additional restrictions, (b) a construction from the component semigroups in the subdirect product, (c) an isomorphism theorem in terms of the representation in (b), (d) a relationship among the congruences on an arbitrary regular semigroup which yield a quotient semigroup of the type under study.

Using the following abbreviations: B—bands; SG—semilattices of groups; L—left zero semigroups; R—right zero semigroups; S—semilattices; G—groups; 1—one element semigroups, we consider regular semigroup subdirect products of these according to the diagram below.

This leads us to classes of semigroups for which we introduce the following abbreviations: V—a variety of bands; UVG—semigroups S for which $\mathscr{H}$ is a congruence, $S/\mathscr{H} \in V$, and the idempotents form a unitary subset of S; M—rectangular bands; CRISN—completely regular semigroups whose idempotents form a strongly normal subband (defined in §2); CRILSN—are CRISN with left added in front of " strongly "; CRUSN—are CRISN whose idempotents form a unitary subset; CRULSN—conjunction of the last two. In addition we adopt from [5]: ISBG—bands of groups whose idempotents form a subsemigroup. Thus, we call a congruence ρ on a semigroup S an X-congruence if S/ρ is in class X, e.g., for $X = G$, a group congruence, for $X = S$, a semilattice congruence (should not be confused with the letter S which usually denotes a semigroup).

By sections, this work is divided as follows. §2 is devoted to results which are needed

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press



https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

either for an understanding of the general system of study or are needed in the proofs of the main sections. This also takes care of the following parts of the diagram: $\mathrm{L}\times\mathrm{G}$, $\mathrm{R}\times\mathrm{G}$, $\mathrm{L}\times\mathrm{R}$, $\mathrm{L}\times\mathrm{R}\times\mathrm{G}$. §3 takes care of $\mathrm{B}\times\mathrm{SG}$; §4 of $\mathrm{B}\times\mathrm{G}$, $\mathrm{S}\times\mathrm{G}$ (more generally of $\mathrm{V}\times\mathrm{G}$); §5 of $\mathrm{L}\times\mathrm{R}\times\mathrm{SG}$, $\mathrm{L}\times\mathrm{SG}$, $\mathrm{R}\times\mathrm{SG}$, $\mathrm{L}\times\mathrm{R}\times\mathrm{S}$; §6 of $\mathrm{L}\times\mathrm{R}\times\mathrm{S}\times\mathrm{G}$, $\mathrm{L}\times\mathrm{S}\times\mathrm{G}$, $\mathrm{R}\times\mathrm{S}\times\mathrm{G}$, $\mathrm{L}\times\mathrm{S}$, $\mathrm{R}\times\mathrm{S}$.

Some of our results overlap with those proved by Yamada in **[12]** in a somewhat different guise; on the other hand, there is some overlapping with the results of Howie and Lallement **[5]**, proved there by different methods. We make frequent use of several results established in the last mentioned paper, and add to the list of interesting connections among special congruences on a regular semigroup.

2. Preliminary results. We discuss in this section several known results and establish some new ones which will serve as a basis for the main body of the paper comprising all the remaining sections. First we briefly mention bands of groups, then introduce strong and sturdy semilattices of semigroups, then discuss several results concerning normal bands and rectangular groups, and finally establish some lemmas to be used throughout the paper. All undefined terminology and notation can be found in **[3]**.

Bands of groups. Recall that a semigroup S is a *band of groups* if S has a congruence σ such that each σ-class is a group. In such a case, S/σ is a band, which justifies the terminology, and $\sigma = \mathscr{H}$. Since every semigroup we shall study in this paper is a band of groups, the following theorem, due to Clifford **[2,** Theorem 7] in a slightly different form, is of particular interest.

THEOREM 2.1. *The following conditions on a semigroup S are equivalent.*

(i) *S is a band of groups.*
(ii) *S is regular and $abS = a^2bS$, $Sab = Sab^2$ for all $a, b \in S$.*
(iii) *S is completely regular and $\mathscr{H}$ is a congruence.*

Recall that " completely regular " is a synonym for " union of groups " or " Clifford semigroup ". We shall not use part (ii), but shall use the equivalence of (i) and (iii) without express mention. Further recall that a semigroup S is a *semilattice of groups* if S is completely regular, $\mathscr{H}$ is a congruence and $S/\mathscr{H}$ is a semilattice. A construction of these semigroups, due to Clifford (see **[3,** Theorem 4.11]) can be generalized in an obvious manner as follows (it plays an implicit but basic role in most of our investigations).

Strong semilattices of semigroups. Let $\{S_\alpha\}_{\alpha\in Y}$ be a family of pairwise disjoint semigroups indexed by a semilattice Y, and, for any pair $\alpha \geqq \beta$ of elements of Y, let there be given a homomorphism $\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying the conditions (a) $\varphi_{\alpha,\alpha}$ is the identity mapping, (b) if $\alpha > \beta > \gamma$, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$, where the functions are written on the right. On $S = \bigcup_{\alpha\in Y} S_\alpha$ define a multiplication $*$ by

$$a * b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) \quad \text{if} \quad a \in S_\alpha,\ b \in S_\beta.$$

Then S is a semigroup, to be called a *strong semilattice of semigroups* S_α and to be denoted $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$.

Recall that a semigroup S is a *subdirect product* of a family of semigroups $\{S_\alpha\}_{\alpha\in A}$ if there exists an isomorphism χ of S into the direct product $\prod_{\alpha\in A} S_\alpha$ of semigroups S_α such that, for every $\alpha\in A$, the projection homomorphism maps $S\chi$ onto S_α.

PROPOSITION 2.2. *Every strong semilattice of semigroups S_α is a subdirect product of semigroups S_α with a zero possibly adjoined.*

Proof. Let $S=[Y; S_\alpha, \varphi_{\alpha,\beta}]$ and, for every $\alpha\in Y$, define a function ψ_α on S by

$$\psi_\alpha : b\to b\varphi_{\beta,\alpha} \quad \text{if} \quad b\in S_\beta \quad \text{and} \quad \beta\geqq\alpha, \quad \text{and} \quad 0_\alpha \quad \text{otherwise.}$$

It is easy to verify that ψ_α is a homomorphism of S onto $T_\alpha=S_\alpha$ if α is the zero of Y and onto $T_\alpha=S_\alpha\cup 0_\alpha$ otherwise, where 0_α acts as the zero of $S_\alpha\cup 0_\alpha$ and $0_\alpha\notin S_\alpha$. Furthermore, the function

$$\psi : b\to(b\psi_\alpha)_{\alpha\in Y} \qquad (b\in S)$$

is easily seen to be an isomorphism of S onto a subsemigroup of $\prod_{\alpha\in Y} T_\alpha$, all of whose projections into components T_α are onto, making S a subdirect product of the family $\{T_\alpha\}_{\alpha\in Y}$.

COROLLARY 2.3. *A semigroup S is regular and a subdirect product of groups with a zero possibly adjoined, if and only if S is a semilattice of groups.*

Proof. Necessity. It is immediate that S is a regular semigroup with all idempotents in the centre, making it a semilattice of groups.

Sufficiency. This is a direct consequence of Proposition 2.2.

Sturdy semilattices of semigroups. A semigroup $S=[Y; S_\alpha, \varphi_{\alpha,\beta}]$ with all homomorphisms $\varphi_{\alpha,\beta}$ one-to-one will be called a *sturdy semilattice of semigroups* S_α and will be denoted by $S=\langle Y; S_\alpha, \varphi_{\alpha,\beta}\rangle$.

This special case is of particular interest in view of the following basic result.

THEOREM 2.4. *On $S=\langle Y; S_\alpha, \varphi_{\alpha,\beta}\rangle$ define a relation ω by*

$$a\,\omega\, b \Leftrightarrow a\varphi_{\alpha,\alpha\beta}=b\varphi_{\beta,\alpha\beta} \quad \textit{if} \quad a\in S_\alpha,\, b\in S_\beta.$$

Then ω is a congruence, S is a subdirect product of Y and S/ω, and, if all S_α are (left, right) simple, so is S/ω.

Proof. It is clear that ω is reflexive and symmetric. Let $a\in S_\alpha$, $b\in S_\beta$, $c\in S_\gamma$, $\delta=\alpha\beta\gamma$. If $a\,\omega\, b$ and $b\,\omega\, c$, then

$$\begin{aligned} a\varphi_{\alpha,\alpha\gamma}\varphi_{\alpha\gamma,\delta} &= a\varphi_{\alpha,\delta}=a\varphi_{\alpha,\alpha\beta}\varphi_{\alpha\beta,\delta}=b\varphi_{\beta,\alpha\beta}\varphi_{\alpha\beta,\delta}=b\varphi_{\beta,\delta}=b\varphi_{\beta,\beta\gamma}\varphi_{\beta\gamma,\delta} \\ &= c\varphi_{\gamma,\beta\gamma}\varphi_{\beta\gamma,\delta}=c\varphi_{\gamma,\delta}=c\varphi_{\gamma,\alpha\gamma}\varphi_{\alpha\gamma,\delta}, \end{aligned}$$

which by the hypothesis implies that $a\varphi_{\alpha,\alpha\gamma}=c\varphi_{\gamma,\alpha\gamma}$, so that $a\,\omega\, c$, proving that ω is transitive.

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

If $a \omega b$, then

$$\begin{aligned}(a*c)\varphi_{\alpha\gamma,\delta} &= [(a\varphi_{\alpha,\alpha\gamma})(c\varphi_{\gamma,\alpha\gamma})]\varphi_{\alpha\gamma,\delta} = (a\varphi_{\alpha,\delta})(c\varphi_{\gamma,\delta}) = (a\varphi_{\alpha,\alpha\beta}\varphi_{\alpha\beta,\delta})(c\varphi_{\gamma,\delta}) \\ &= (b\varphi_{\beta,\alpha\beta}\varphi_{\alpha\beta,\delta})(c\varphi_{\gamma,\delta}) = (b\varphi_{\beta,\delta})(c\varphi_{\gamma,\delta}) = (b_{\beta,\beta\gamma}\varphi_{\beta\gamma,\delta})(c\varphi_{\gamma,\beta\gamma}\varphi_{\beta\gamma,\delta}) \\ &= [(b\varphi_{\beta,\beta\gamma})(c\varphi_{\gamma,\beta\gamma})]\varphi_{\beta\gamma,\delta} = (b*c)\varphi_{\beta\gamma,\delta},\end{aligned}$$

proving that $a*c \ \omega\ b*c$. A dual proof shows that $c*a \ \omega\ c*b$ also and thus ω is a congruence. It is then immediate that the function ψ defined on S by

$$\psi : a \to (\alpha, \bar{a}) \quad \text{if} \quad a \in S_\alpha,$$

where $\bar{a}$ is the ω-class containing a, is an isomorphism of S into $Y \times S/\omega$, making S a subdirect product of Y and S/ω.

Suppose next that every S_α is simple. Let $a \in S_\alpha$, $b \in S_\beta$; then $a\varphi_{\alpha,\alpha\beta}$, $b\varphi_{\beta,\alpha\beta} \in S_{\alpha\beta}$ and the hypothesis implies the existence of $x, y \in S_{\alpha\beta}$ such that $a\varphi_{\alpha,\alpha\beta} = x(b\varphi_{\beta,\alpha\beta})y$. But then $a\varphi_{\alpha,\alpha\beta} = (x*b*y)\varphi_{\alpha\beta,\alpha\beta}$, which implies that $a \ \omega\ x*b*y$, proving that S/ω is simple. Left and right simplicity are treated similarly.

Throughout, if S is a semigroup, E_S denotes its (partially ordered) set of idempotents.

Recall the following definition: A nonempty subset A of a semigroup S is *left unitary* if $s \in S$, $a, as \in A$ implies $s \in A$; *right unitary* is defined dually, *unitary* is the conjunction of the two. Since we shall frequently encounter the condition that E_S is unitary for a regular semigroup S, the next proposition is of particular interest.

PROPOSITION 2.5. *The following conditions on a regular semigroup S are equivalent.*

(i) *E_S is right (resp. left) unitary.*
(ii) *E_S is unitary.*
(iii) *$e, ese \in E_S$ implies $s \in E_S$.*
(iv) *$e, aeb \in E_S$ implies $ab \in E_S$.*

If S is a regular semigroup satisfying these conditions, then E_S is a subsemigroup of S.

Proof. That E_S left unitary implies that E_S is right unitary and a subsemigroup of S is the content of [5, Lemma 2.1]. That (ii) implies both (i) and (iii) is trivial in any semigroup. Suppose that (iii) holds and let $e, se \in E_S$. Then $se, (se)(ese)(se) \in E_S$, which implies that $ese \in E_S$, and this together with $e \in E_S$ implies that $s \in E_S$; so (i) holds. Again the hypothesis of regularity is not needed.

Suppose next that (ii) holds and let $e, aeb \in E_S$. It follows easily that $(ebae)^2 \in E_S$ and hence $e, e(baeba)e \in E_S$, which implies that $baeba \in E_S$. Thus $baeba, (baeba)(ebae) = (baeba)e \in E_S$, since E_S is a subsemigroup of S, which implies that $ebae \in E_S$. But then $e, ebae \in E_S$, so that $ba \in E_S$ and [5, Lemma 2.2] yields $ab \in E_S$, proving (iv).

Finally, suppose that (iv) holds. For $e, f \in E_S$, let x be an inverse of fe. Then exf, $e(exf)f \in E_S$, so that $ef \in E_S$, i.e., E_S is a subsemigroup of S. Now let $e, se \in E_S$, and let s' be any inverse of s. Then $(se)(s's) \in E_S$, since E_S is a subsemigroup of S. By (iv), $s = ss's \in E_S$, proving (i).

We shall use the preceding proposition without express mention. The next result will be quite useful.

PROPOSITION 2.6. *The following conditions on a semigroup S are equivalent.*

(i) *S is a sturdy semilattice of groups.*
(ii) *S is a regular semigroup subdirect product of a semilattice and a group.*
(iii) *S is a semilattice of groups and E_S is a unitary subset of S.*

Proof. (i) implies (ii) by Theorem 2.4. If S is as in (ii), we may suppose that S is a subsemigroup of $Y \times G$, where Y is a semilattice and G is a group. For every $\alpha \in Y$, let $G_\alpha = \{g \in G \mid (\alpha, g) \in S\}$. It is quite easy to verify that S is a semilattice of groups G_α and that E_S is unitary. Finally the implication " (iii) implies (i) " follows directly from **[5**, Corollary 2.4].

Normal bands. Recall **[14]** that a band B is *normal* if it satisfies the identity $axya = ayxa$; B is *left* (resp. *right*) *normal* if it satisfies the identity $axy = ayx$ (resp. $xya = yxa$).

The smallest semilattice congruence on any semigroup S will be denoted throughout by $\mathcal{N}$, the $\mathcal{N}$-class containing an element a by N_a; $S/\mathcal{N}$ is a semilattice with $N_a N_b = N_{ab}$ (see **[8]** for an extensive discussion).

Let A and B be semigroups, let η be an isomorphism of $A/\mathcal{N}$ onto $B/\mathcal{N}$, and let

$$S = \{(a, b) \in A \times B \mid N_a \eta = N_b\}$$

with the multiplication induced by the direct product $A \times B$. Then S is the spined product of A and B relative to η, or simply a *spined product* of A and B (see **[6]**). One may define a spined product of any family of semigroups; for a finite family one may do it by induction, since $S/\mathcal{N} \simeq A/\mathcal{N} \simeq B/\mathcal{N}$.

THEOREM 2.7. *The following conditions on a band B are equivalent.*

(i) *B is normal.*
(ii) *B is a strong semilattice of rectangular bands.*
(iii) *B is a spined product of a left normal and a right normal band.*
(iv) *In B, $e > f$, $e > g$, $f \mathcal{N} g$ imply $f = g$.*

Proof. The equivalence of (i), (ii), and (iii) was announced by Yamada and Kimura **[14]**; the entire theorem follows easily from **[11**, Theorems 4.3 and 4.4].

COROLLARY 2.8. *A band B is left normal if and only if B is a strong semilattice of left zero semigroups.*

A further case of interest for our purposes is represented by

PROPOSITION 2.9. *The following conditions on a band B are equivalent.*

(i) *B is normal and $e < f$, $e < g$, $f \mathcal{N} g$ imply $f = g$.*
(ii) *B is a sturdy semilattice of rectangular bands.*
(iii) *B is a subdirect product of a semilattice and a rectangular band.*

Proof. (i) implies (ii). By Theorem 2.7 we may take $B = [Y; B_\alpha, \varphi_{\alpha,\beta}]$, where each B_α is a rectangular band. If $f, g \in B_\alpha$, $\alpha > \beta$, and $f\varphi_{\alpha,\beta} = g\varphi_{\alpha,\beta}$, then it follows quickly that $e = f\varphi_{\alpha,\beta}$ has the property $e < f$, $e < g$ and, since $f\mathcal{N}g$, we infer that $f = g$, proving that $\varphi_{\alpha,\beta}$ is one-to-one.

(ii) implies (iii). By Theorem 2.4 we have that B is a subdirect product of the semilattice $B/\mathcal{N}$ and the band B/ω, which is simple and thus must be a rectangular band.

(iii) implies (i). Let $B \subseteq Y \times M$ be a subdirect product, where Y is a semilattice and M is a rectangular band. Since both Y and M are normal bands, so is B. One verifies without difficulty that, in B, $(\alpha, e)\mathcal{N}(\beta, f)$ if and only if $\alpha = \beta$; further $(\alpha, e) < (\beta, f)$, $(\alpha, e) < (\beta, g)$ implies $e = f = g$. Combining these two yields the validity of (i).

We call a band B satisfying the conditions of Proposition 2.9 *strongly normal*; if, in addition, B is left (right) normal, we call it a *left (right) strongly normal band.* We immediately have

COROLLARY 2.10. *The following conditions on a band B are equivalent.*

(i) *B is a left strongly normal band.*
(ii) *B is a sturdy semilattice of left zero semigroups.*
(iii) *B is a subdirect product of a semilattice and a left zero semigroup.*

Rectangular groups. Recall that a semigroup S is a *rectangular group* if S is isomorphic to the direct product of a rectangular band and a group. The next statement appears in [**3**, exercise 2(b), p. 97]; a proof is given in [**7**, Lemma 1].

PROPOSITION 2.11. *A completely simple semigroup S in which E_S is a semigroup is a rectangular group and conversely.*

PROPOSITION 2.12. *Let L, R and G be a left zero semigroup, a right zero semigroup and a group, respectively. Then $L \times R \times G$ is the only subdirect product of L, R and G contained in $L \times R \times G$.*

Proof. Let $S \subseteq L \times R \times G$ be a subdirect product, and let $(l, r, g) \in L \times R \times G$. Then $(l, r', h) \in S$ for some $r' \in R$, $h \in G$; further, $(u, v, h^{-1}) \in S$ for some $u \in L$, $v \in R$. Letting 1 denote the identity of G, we obtain

$$(l, r', 1) = (l, r', h)(u, v, h^{-1})^2(l, r', h) \in S.$$

Analogously $(l', r, 1) \in S$ for some $l' \in L$, and $(l'', r'', g) \in S$ for some $l'' \in L$, $r'' \in R$, which yields

$$(l, r, g) = (l, r', 1)(l'', r'', g)(l', r, 1) \in S$$

as desired.

COROLLARY 2.13. *Let L, R, G be as in Proposition* 2.12, *and let A be any semigroup. Then the subdirect products of the following combinations of semigroups coincide*: (i) L, R, G, A, (ii) $L \times R, G, A$, (iii) $L \times G, R, A$, (iv) $R \times G, L, A$, (v) $L \times R \times G, A$.

Proof. For example, if $S \subseteq L \times R \times G \times A$ is a subdirect product, then its projection in $L \times R$ is a subdirect product of L and R and hence, by Proposition 2.12, must coincide with $L \times R$. Hence (i) implies (ii). The converse is trivial.

C

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

Proposition 2.14. *Let L and L' be left zero semigroups, R and R' right zero semigroups, G and G' groups, $S = L \times R \times G$, $S' = L' \times R' \times G'$. Further, let $\varphi : L \to L'$, $\psi : R \to R'$, $\omega : G \to G'$ be onto isomorphisms. Then a function χ defined by*

$$(l, r, g)\chi = (l\varphi, r\psi, g\omega) \qquad ((l, r, g) \in S)$$

is an isomorphism of S onto S'. Conversely, every isomorphism of S onto S' can be expressed in this way.

Proof. The direct part is obvious; the converse follows without difficulty from [**3**, Theorem 3.11] and its proof is omitted.

We prove now a few auxiliary statements which will be useful later. The first one will be used quite frequently and without express reference.

Lemma 2.15. *Let B be a band, T be a semilattice of groups, and S be a regular subsemigroup of $B \times T$. If $(e, g) \in S$ and 1 is the identity of the maximal subgroup of T containing g, then $(e, 1) \in S$ and (e, g^{-1}) is an inverse of (e, g) in S.*

Proof. Let (f, h) be an inverse of $(e, g) \in S$. Then h is the inverse of g in T and we may write $h = g^{-1}$, and f is an inverse of e. Further, $(ef, 1) = (e, g)(f, g^{-1}) \in S$ and dually $(fe, 1) \in S$, so that $(e, 1) = (ef, 1)(fe, 1) \in S$. But then

$$(e, g^{-1}) = (e, 1)(f, g^{-1})(e, 1) \in S$$

as required.

We shall mainly be studying semigroups S which fit the description in Lemma 2.15. If T is periodic, S is automatically regular since, for $(e, g) \in S$, some power of (e, g) is an inverse of (e, g) and hence every element of S has an inverse. It follows that our results for regular semigroup subdirect products of a band and a semilattice of groups are automatically applicable to all subdirect products of a band and a semilattice of periodic groups. Conversely, if S is a subdirect product of B and T and S is periodic, then T is also periodic, being a homomorphic image of S.

Lemma 2.16. *Let S be a regular subsemigroup of a semigroup T. Then Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ on S are the restrictions of those on T.*

Proof. It suffices to consider $\mathscr{L}$; let $\mathscr{L}_S$ and $\mathscr{L}_T$ denote the corresponding $\mathscr{L}$-relations. Let $a, b \in S$ and $a \mathscr{L}_T b$. Then $a = ub$ and $b = va$ for some $u, v \in T$; letting a' and b' be inverses in S of a and b, respectively, we obtain $a = (ub)b'b = ab'b$ and $b = (va)a'a = ba'a$, which shows that $a \mathscr{L}_S b$. Consequently $\mathscr{L}_T|_S \subseteq \mathscr{L}_S$. The opposite inclusion is trivial.

Corollary 2.17. *If S is a regular semigroup subdirect product of a band B and a semilattice of groups T, then S is a band of groups and E_S is a subsemigroup of S.*

Proof. It follows from Lemma 2.15 that S is completely regular, and from Lemma 2.16 that $\mathscr{H}$ is a congruence, since this holds both in B and in T and hence in $B \times T$. Since the idempotents of both B and T form a subsemigroup of each of these semigroups, this property carries over to S.

Notation. If σ is an isomorphism of a semigroup S onto a semigroup T, then $\hat{\sigma}$ denotes the isomorphism of $S/\mathcal{N}$ onto $T/\mathcal{N}$ induced by σ (i.e., $N_s\hat{\sigma} = N_{s\sigma}$ for all $s \in S$). If $e \in E_S$, then G_e denotes the maximal subgroup of S having e as its identity. The equality relation on S is denoted by ι, the universal relation by $\mathcal{U}$.

3. Subdirect products of a band and a semilattice of groups. After a preliminary result, we establish several characterizations of the semigroups under study. For their representation as a spined product, we prove an isomorphism theorem. Finally we apply certain of these results to congruences on a regular semigroup.

THEOREM 3.1. *Let S be a band of groups for which E_S is a subsemigroup of S. On S define a relation ξ by*

$$a\,\xi\,b \Leftrightarrow a = ebe,\ b = faf,\quad \text{where} \quad a \in G_e,\ b \in G_f.$$

Then ξ is the minimum SG-*congruence on S. Moreover, $\mathcal{H} \cap \xi = \iota$, $\mathcal{H}\xi = \xi\mathcal{H} = \mathcal{H} \vee \xi = \mathcal{N}$, and, on denoting by $\bar{x}$ the ξ-class containing x, $a\,\mathcal{N}\,b$ if and only if $\bar{a}\,\mathcal{N}\,\bar{b}$.*

Proof. First note that $\xi \subseteq \mathcal{N}$. Next let $a \in G_e$, $b \in G_f$, $c \in G_t$. Suppose that $a\,\xi\,b$ and $b\,\xi\,c$. Then a, b, c, e, f, t are all contained in the same $\mathcal{N}$-class of S, which is a rectangular group by Proposition 2.11, and thus

$$a = ebe = e(fcf)e = (efc)fe = e(cfe) = ece$$

and analogously $c = tat$, proving that $a\,\xi\,c$. Hence ξ is transitive and thus an equivalence relation. Now assume only that $a\,\xi\,b$. Taking into account that

(1) $bec \in G_{fet}$, since $\mathcal{H}$ is a congruence,
(2) $e\,\mathcal{N}\,f$, since $a \in G_e$, $b \in G_f$, $a\,\mathcal{N}\,b$,
(3) $ac \in G_{et}$, since $\mathcal{H}$ is a congruence,
(4) tb, et, ce are in the same $\mathcal{N}$-class, which by Proposition 2.11 is a rectangular group,

we obtain the following string of equalities:

$$\begin{aligned} ac &= ebec \overset{(1)}{=} e(fet)(bec) = (efe)tbec \overset{(2)}{=} etbec \overset{(3)}{=} etbec(et) = e(tb)(et)(ce)t \\ &\overset{(4)}{=} e(tb)(ce)t = (et)(bc)(et); \end{aligned}$$

analogously $bc = (ft)(ac)(ft)$, which proves that $ac\,\xi\,bc$. By a similar argument, we may show that $ca\,\xi\,cb$, and hence ξ is a congruence.

Next let $F = M \times G$, where M is a rectangular band and G is a group. A simple calculation shows that in F we have $(e, g)\,\mathcal{H}\,(f, h)$ if and only if $e = f$, and $(e, g)\,\xi\,(f, h)$ if and only if $g = h$. Consequently $\mathcal{H} \cap \xi = \iota$, $\mathcal{H}\xi = \xi\mathcal{H} = \mathcal{H} \vee \xi = \mathcal{U}$. It follows that ξ is the minimum G-congruence on F.

Now returning to S, we conclude that the restriction of ξ to each $\mathcal{N}$-class N of S is the minimum G-congruence on N, since N is a rectangular group by Proposition 2.11. But then ξ must be the minimum SG-congruence on S. It is now clear that the remaining assertions of the theorem are valid.

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

We are now in a position to prove the principal result of this section.

THEOREM 3.2. *The following statements concerning a semigroup S are equivalent.*

(i) *S is a regular semigroup subdirect product of a band and a semilattice of groups.*

(ii) *S is a regular semigroup subdirect product of bands, groups, and groups with zero (some possibly missing).*

(iii) *S is a band of groups and E_S is a subsemigroup of S.*

(iv) *S is a spined product of a band and a semilattice of groups.*

Proof. The equivalence of (i) and (ii) follows directly from Corollary 2.3 and the transitivity of subdirect products. Further (i) implies (iii) by Corollary 2.17, and (iv) implies (i) trivially.

(iii) implies (iv). The hypotheses imply that $S/\mathscr{H} \simeq E_S$, and so we infer from Theorem 3.1 that the mapping χ defined by

$$\chi : a \to (a^*, \bar{a}) \qquad (a \in S),$$

where a^* is the identity of the maximal subgroup of S containing a and $\bar{a}$ is the ξ-class containing a, makes S a subdirect product of E_S and S/ξ. To see that $S\chi$ is a spined product of E_S and S/ξ, we define η by

$$\eta : N_e \to N_{\bar{e}} \qquad (e \in E_S).$$

One verifies without difficulty that η is an isomorphism of $E_S/\mathscr{N}$ onto $(S/\xi)/\mathscr{N}$. For $(e, g) \in S\chi$, we have $(e, g) = (a^*, \bar{a})$ for some $a \in S$. Thus $e = a^*$, $g = \bar{a}$ and $\overline{a^*}\,\mathscr{N}\,\bar{a}$, so that

$$N_g = N_{\bar{a}} = N_{\overline{a^*}} = N_{a^*}\eta = N_e\eta.$$

Conversely, suppose that $N_e\eta = N_g$. Then $\bar{e}\,\mathscr{N}\,g$ which, by the last statement of Theorem 3.1, implies that in S we must have $g \subseteq N_e$. But then, again by Theorem 3.1, the set $G_e \cap g$ contains exactly one element, say a. Consequently $a^* = e$ and $\bar{a} = g$, so that $(e, g) = a\chi \in S\chi$. We have proved that

$$S\chi = \{(e, g) \in E_S \times S/\xi \mid N_e\eta = N_g\}$$

as required.

The equivalence of (iii) and (iv) was proved by Yamada [**12**, Theorem 4] and, in a slightly different form, by the author [**9**]. An extensive use of spined products for various subclasses of these semigroups can be found in [**13**]. The implication "(iii) implies (iv)" can also be obtained by specializing the main theorem in [**4**]; however, the above proof makes the use of the hypotheses much more transparent and represents a basis for several proofs in the succeeding sections. According to Theorem 3.2, every semigroup S which is a band of groups and for which E_S is a subsemigroup is isomorphic to a spined product of a band B and a semilattice of groups T determined by an isomorphism η of $B/\mathscr{N}$ onto $T/\mathscr{N}$; we use the notation (B, η, T) for the subsemigroup of $B \times T$ which is a spined product of B and T determined by η. For two such semigroups, the following result gives an isomorphism criterion. Recall the notation at the end of §2.

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

THEOREM 3.3. *Let $S = (B, \eta, T)$ and $S' = (B', \eta', T')$. Further, let σ be an isomorphism of B onto B' and τ be an isomorphism of T onto T' such that the diagram*

$$\begin{tikzcd} B/\mathcal{N} \arrow[r, "\eta"] \arrow[d, "\hat{\sigma}"'] & T/\mathcal{N} \arrow[d, "\hat{\tau}"] \\ B'/\mathcal{N} \arrow[r, "\eta'"] & T'/\mathcal{N} \end{tikzcd}$$

is commutative. Then the function χ defined by

$$(e, g)\chi = (e\sigma, g\tau) \qquad ((e, g) \in S) \tag{1}$$

is an isomorphism of S onto S'. Conversely, every isomorphism of S onto S' can be expressed in this way.

Proof. Let χ be as defined above. Then, for any $(e, g) \in B \times T$, we obtain

$$(e, g) \in S \Leftrightarrow N_e\eta = N_g \Leftrightarrow N_e\eta\hat{\tau} = N_g\hat{\tau} \Leftrightarrow N_e\hat{\sigma}\eta' = N_g\hat{\tau} \Leftrightarrow N_{e\sigma}\eta' = N_{g\tau}$$
$$\Leftrightarrow (e\sigma, g\tau) \in S' \Leftrightarrow (e, g)\chi \in S',$$

proving that χ maps S onto S'; that χ is one-to-one and a homomorphism is obvious.

Conversely, let χ be an isomorphism of S onto S'. Define the functions σ and τ by the formula

$$(e, g)\chi = ((e, g)\sigma, (e, g)\tau) \qquad ((e, g) \in S), \tag{2}$$

so that $\sigma : S \to B'$, $\tau : S \to T'$ are homomorphisms. If $(e, g), (e, h) \in S$, then $N_e\eta = N_g = N_h$; so g and h are contained in the same maximal subgroup of T and thus $(e, g)\ \mathcal{H}\ (e, h)$ in S. But then $(e, g)\sigma\ \mathcal{H}\ (e, h)\sigma$, which in a band implies that $(e, g)\sigma = (e, h)\sigma$. Consequently we may write $(e, g)\sigma = e\sigma$ and consider σ as a homomorphism of B into B'. If $(e, g), (f, g) \in S$, then

$$(e, g) = (e, 1)(f, g)(e, 1), \quad (f, g) = (f, 1)(e, g)(f, 1),$$

where 1 is the identity of the maximal subgroup of T containing g. Hence, applying τ, we obtain by Theorem 3.1 that $(e, g)\tau\, \xi\, (f, g)\tau$, which in a semilattice of groups implies that $(e, g)\tau = (f, g)\tau$. Hence we may write $(e, g)\tau = g\tau$ and consider τ as a homomorphism of T into T'. It follows that (2) takes on the form of (1).

Let $e' \in B'$; then, for any $g' \in T'$ for which $N_{e'}\eta' = N_{g'}$, we have $(e', g') \in S'$; so there exists $(e, g) \in S$ such that $(e, g)\chi = (e', g')$ and $e\sigma = e'$. Hence σ maps B onto B'; analogously τ maps T onto T'. Suppose next that $e\sigma = f\sigma$ and let 1_α be the identity of $N_e\eta$, 1_β be the identity of $N_f\eta$. Then $(e, 1_\alpha), (f, 1_\beta) \in S$ and

$$(e, 1_\alpha)\chi = (e\sigma, 1_\alpha\tau) = (f\sigma, 1_\alpha\tau) \in S',$$

and, on the other hand, $(f, 1_\beta)\chi = (f\sigma, 1_\beta\tau) \in S'$. Hence $N_{f\sigma}\eta' = N_{1_\alpha\tau} = N_{1_\beta\tau}$ and, since $1_\alpha\tau$ and $1_\beta\tau$ are idempotents, we must have $1_\alpha\tau = 1_\beta\tau$. Consequently

$$(e, 1_\alpha)\chi = (e\sigma, 1_\alpha\tau) = (f\sigma, 1_\beta\tau) = (f, 1_\beta)\chi,$$

which implies that $e=f$, proving that σ is one-to-one. Suppose that $g\tau = h\tau$ and let $1_\alpha = gg^{-1}$, $1_\beta = hh^{-1}$. Then $(e, g), (f, h) \in S$ for some $e, f \in B$, and $(e, 1_\alpha), (f, 1_\beta) \in S$. Hence

$$(e, 1_\alpha)\chi = (e, g)\chi(e, g^{-1})\chi = (e\sigma, g\tau)(e\sigma, g^{-1}\tau) = (e\sigma, h\tau)(e\sigma, h^{-1}\tau) = (e\sigma, (hh^{-1})\tau) = (e, 1_\beta)\chi,$$

so that $1_\alpha = 1_\beta$ and thus $(f, 1_\alpha) \in S$. It then follows that

$$(e, g)\chi = (e\sigma, g\tau) = (e\sigma, h\tau) = (e, h)\chi$$

which implies that $g = h$, proving that τ is one-to-one.

Commutativity of the diagram follows from the following sequence of equivalences: for $(e, g) \in B \times T$, we have

$$N_e\eta\hat{\tau} = N_g\hat{\tau} \Leftrightarrow N_e\eta = N_g \Leftrightarrow (e, g) \in S \Leftrightarrow (e, g)\chi \in S' \Leftrightarrow (e\sigma, g\tau) \in S'$$
$$\Leftrightarrow N_{e\sigma}\eta' = N_{g\tau} \Leftrightarrow N_e\hat{\sigma}\eta' = N_g\hat{\tau}.$$

We apply some of the above results to certain special congruences on a regular semigroup.

THEOREM 3.4. *The following statements concern any regular semigroup. The intersection of a* B-*congruence and an* SG-*congruence is an* ISBG-*congruence. Conversely, every* ISBG-*congruence is the intersection of a* B-*congruence and an* SG-*congruence, but these need not be unique. Furthermore*

$$(\textit{min } \text{B-}\textit{congruence}) \cap (\textit{min } \text{SG-}\textit{congruence}) = \textit{min } \text{ISBG-}\textit{congruence},$$
$$(\textit{min } \text{B-}\textit{congruence}) \vee (\textit{min } \text{SG-}\textit{congruence}) = \textit{min } \text{S-}\textit{congruence},$$

and the congruences on the left hand side commute.

Proof. If β is a B-congruence and ξ is an SG-congruence on a regular semigroup S, then $S/\beta\cap\xi$ is a subdirect product of S/β and S/ξ and hence $\beta\cap\xi$ is an ISBG-congruence on S, by Theorem 3.2. Conversely, let ζ be an ISBG-congruence on S. Then, by Theorem 3.2, S/ζ is a regular semigroup subdirect product of a band B and a semilattice of groups T. This subdirect product induces, in the usual way, a B-congruence β and an SG-congruence ξ on S/ζ whose intersection is the equality relation. Denoting the ζ-class containing an element a by $\bar{a}$, and defining β' and ξ' on S by $a\,\beta'\,b \Leftrightarrow \bar{a}\,\beta\,\bar{b}$ and $a\,\xi'\,b \Leftrightarrow \bar{a}\,\xi\,\bar{b}$, we obtain $S/\beta \simeq (S/\zeta)/\beta$, $S/\xi' \simeq (S/\zeta)/\xi$ and $\beta'\cap\xi' = \zeta$. Hence ζ is the intersection of the B-congruence β' and the SG-congruence ξ'. Very simple examples show that β' and ξ' need not be uniquely determined by ζ. The statement concerning the intersection of minimum congruences is a consequence of the preceding statements.

As for the remaining assertions, we first note that in the expression for the sup, the left hand side is obviously contained in the right hand side. Suppose finally that $a\mathcal{N}b$ for $a, b \in S$, let ζ be the minimum ISBG-congruence on S, and let $a \to \hat{a}$ be the canonical homomorphism of S onto $\hat{S} = S/\zeta$. Then $\hat{a}\mathcal{N}\hat{b}$ and thus $\hat{a}\,\hat{\beta}\hat{\xi}\,\hat{b}$, by Theorem 3.1, where $\hat{\beta}$ and $\hat{\xi}$ denote the minimum B- and SG-congruences, respectively, on $\hat{S}$. Hence, for some $c \in S$, we have $\hat{a}\,\hat{\beta}\,\hat{c}$, $\hat{c}\,\hat{\xi}\,\hat{b}$. Denote by β and ξ the minimum B- and SG-congruences, respectively, on S. It follows easily that then also $a\,\beta\,c$, $c\,\xi\,b$ and thus $a\,\beta\xi\,b$. Hence $\mathcal{N} \subseteq \beta\xi$ and thus $\mathcal{N} \subseteq \beta\vee\xi$. One shows similarly that $\mathcal{N} \subseteq \xi\beta$, which finally implies that $\beta\xi = \xi\beta = \beta\vee\xi = \mathcal{N}$.

COROLLARY 3.5. *Let S be a regular semigroup. Then, for any* B-*congruence β and any* SG-*congruence ξ on S such that $\beta, \xi \subseteq \mathcal{N}$, we have $\beta\xi = \xi\beta = \beta \vee \xi = \mathcal{N}$.*

4. Subdirect products of a band and a group. For this case, we first give an abstract characterization, then a construction and the corresponding isomorphism theorem. In addition, we derive some consequences concerning congruences on a regular semigroup.

THEOREM 4.1. *A semigroup S is regular and a subdirect product of a band and a group if and only if S is a band of groups and E_S is a unitary subset.*

Proof. Necessity. Let B be a band and G a group, and suppose that S is a regular subsemigroup of $B \times G$ and their subdirect product. By Theorem 3.2 we know that S is a band of groups and that E_S is a subsemigroup of S. Let $(e, 1), (e, 1)(f, a) \in E_S$, where $(f, a) \in S$ and 1 is the identity of G. Then $(ef, a) \in E_S$, which implies that $a = 1$; so $(f, 1) \in E_S$ and E_S is unitary.

Sufficiency. Using the notation of the proof of Theorem 3.2, we know that S is a subdirect product of E_S and $T = S/\xi$. Let $1_\alpha, 1_\alpha a \in E_T$, where $a \in T$. Then, for some $e, f \in E_S$, $(e, 1_\alpha)$, $(f, a) \in S\chi$ and thus $(ef, 1_\alpha a) \in E_{S\chi}$, since $1_\alpha a \in E_T$. Thus $(e, 1_\alpha), (e, 1_\alpha)(f, a) \in E_{S\chi}$, which, by the hypothesis on E_S, yields $(f, a) \in E_{S\chi}$, so that $a \in E_T$. Consequently E_T is a unitary subset of T and hence Proposition 2.6 is applicable to make T a subdirect product of $Y = T/\mathcal{N}$ and a group G. Let $\theta : t \to (t\gamma, t\delta)$ be the isomorphism of T into $Y \times G$ making T a subdirect product. Combining the two isomorphisms and using the notation of the proof of Theorem 3.2, we obtain

$$a \to (a^*, \bar{a}) \to (a^*, (\bar{a}\gamma, \bar{a}\delta)) \qquad (a \in S),$$

which makes S a subdirect product of E_S, Y, and G. Now define a function ψ by

$$\psi : a \to (a^*, \bar{a}\delta) \qquad (a \in S).$$

Then ψ is a homomorphism of S onto a subdirect product of E_S and G. Suppose that $(a^*, \bar{a}\delta) = (b^*, \bar{b}\delta)$. Then $a^* = b^*$, which implies that $a \mathcal{N} b$ and thus $\bar{a}\gamma \mathcal{N} \bar{b}\gamma$, which in turn yields $\bar{a}\gamma = \bar{b}\gamma$, since Y is a semilattice. Consequently $a\chi\theta = b\chi\theta$, which implies that $a = b$, proving that ψ is one-to-one.

Note that, if S is a regular semigroup subdirect product of a band B and a group G, the projections into B and G respectively induce the minimum B-congruence and the minimum G-congruence on S. In view of this, the above theorem is equivalent to [**5**, Theorem 2.5], which is formulated in terms of congruences. The proofs in [**5**] are different from ours. Several characterizations of the semigroups in Theorem 4.1 in terms of spined products were established by Yamada [**12**]. We give next a construction of all regular semigroup subdirect products of a given band B and a given group G which are subsemigroups of $B \times G$. To this end, it is convenient to introduce the following concept.

DEFINITION 4.2. Let Y be a semilattice and $\mathcal{S}$ be a family of nonempty subsets of a set X ordered by set inclusion. A function φ mapping Y into $\mathcal{S}$ is a *dual homomorphism* if $\alpha \leqq \beta$ always implies $\beta\varphi \subseteq \alpha\varphi$, and is *full* if $\bigcup_{\alpha \in Y} \alpha\varphi = X$.

For any group G, we denote by $\mathscr{L}(G)$ the lattice of all subgroups of G (where the order is set inclusion).

THEOREM 4.3. *Let B be a band and G be a group. Let $\eta : B/\mathscr{N} \to \mathscr{L}(G)$ be a full dual homomorphism and let*

$$S = \{(e, g) \in B \times G \mid g \in N_e \eta\}. \tag{3}$$

Then S is a regular semigroup subdirect product of B and G. Conversely, every regular semigroup subdirect product of B and G is essentially of this type.

Proof. Let S be as defined above. For $(e, g), (f, h) \in S$, we have $g \in N_e \eta$ and $h \in N_f \eta$. Since $N_{ef} \leqq N_e$, we obtain $g \in N_e \eta \subseteq N_{ef} \eta$ and analogously $h \in N_{ef} \eta$. But $N_{ef} \eta$ is a subgroup of G; so $gh \in N_{ef} \eta$, proving that $(ef, gh) \in S$. Consequently S is closed under multiplication. For $(e, g) \in S$ we have $g \in N_e \eta$; so also $g^{-1} \in N_e \eta$, which then shows that $(e, g^{-1}) \in S$, supplying an inverse of (e, g) in S. Since η is full, S is a subdirect product of B and G.

Conversely, let $S \subseteq B \times G$ be a subdirect product. For every $e \in B$, let

$$e\eta = \{g \in G \mid (e, g) \in S\}. \tag{4}$$

It follows easily that $e\eta$ is a subgroup of G. Further, if $e = efe$ and $g \in f\eta$, then

$$(e, g) = (e, 1)(f, g)(e, 1) \in S$$

and hence $g \in e\eta$. In particular, for $e \mathscr{N} f$ we obtain $e\eta = f\eta$, and we may consider η as a function from $B/\mathscr{N}$ into $\mathscr{L}(G)$. It also follows that η is a dual homomorphism, and its fullness is a consequence of the fact that the projection of S into G is onto. A comparison of (3) and (4) shows that S has the desired form.

We denote the semigroup S constructed above by $[B, \eta, G]$, and turn to the problem of isomorphism of two such semigroups. For a one-to-one correspondence φ of a set A onto a set B, we denote by $\bar{\varphi}$ the function defined for all subsets X of A by $X\bar{\varphi} = \{x\varphi \mid x \in X\}$, and we also denote by the same symbol the restriction of $\bar{\varphi}$ to any family of subsets of A.

THEOREM 4.4. *Let $S = [B, \eta, G]$ and $S' = [B', \eta', G']$. Further, let σ be an isomorphism of B onto B' and τ be an isomorphism of G onto G' such that the diagram*

$$\begin{tikzcd} B/\mathscr{N} \arrow[r, "\eta"] \arrow[d, "\hat{\sigma}"'] & \mathscr{L}(G) \arrow[d, "\bar{\tau}"] \\ B'/\mathscr{N} \arrow[r, "\eta'"] & \mathscr{L}(G') \end{tikzcd}$$

is commutative. Then the function χ defined by

$$(e, g)\chi = (e\sigma, g\tau) \qquad ((e, g) \in S)$$

is an isomorphism of S onto S'. Conversely, every isomorphism of S onto S' can be expressed in this way.

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

Proof. The proof is only slightly different from the proof of Theorem 3.3 and is omitted.

It follows from Theorem 4.3 that the projection $(e, 1) \to e$ furnishes an isomorphism of E_S onto B. We can use this in conjunction with Theorem 4.1 as follows. Let V be a variety of bands, call its elements *V-bands*, call a congruence σ on a semigroup S a *V-congruence* if $S/\sigma \in V$, and a band of groups a *V-band of groups* if $S/\mathscr{H} \in V$. From the above results, we easily deduce

COROLLARY 4.5. *The following statements concerning a regular semigroup S are equivalent.*

(i) *S is a subdirect product of a V-band and a group.*
(ii) *S is a subdirect product of V-bands and groups.*
(iii) *S is a V-band of groups and E_S is a unitary subset.*

In terms of congruences on a regular semigroup, we can draw the following conclusion, where UVG denotes a V-band of groups S for which E_S is a unitary subset of S.

COROLLARY 4.6. *The following statements concern any regular semigroup. The intersection of a* V*-congruence and a* G*-congruence is a* UVG*-congruence. Conversely, every* UVG*-congruence can be uniquely written as the intersection of a* V*-congruence and a* G*-congruence. In particular,*

$$(\textit{min V-congruence}) \cap (\textit{min G-congruence}) = \textit{min UVG-congruence}.$$

Proof. This follows from [5, Theorem 4.1] and the remarks above. It can also be proved, except for uniqueness, by modifying suitably the first part of the proof of Theorem 3.4.

Note that, for $V =$ the variety of all bands, Corollary 4.6 coincides with [5, Theorem 4.1] and, for $V =$ the variety of all semilattices, with [5, Theorem 4.2]. Simple examples show that a band congruence need not commute with a group congruence (but their sup is always the universal relation).

5. Subdirect products of a rectangular band and a semilattice of groups. We now perform a similar analysis for this case. An outstanding feature here is the profusion of special cases of semigroups and thus of types of congruences. For the next two theorems, Corollary 2.13 is of special interest.

THEOREM 5.1. *A semigroup S is regular and a subdirect product of a rectangular band and a semilattice of groups if and only if S is completely regular and E_S is a strongly normal band.*

Proof. Necessity. Let S be a regular semigroup subdirect product of a rectangular band M and a semilattice of groups T and assume that S is a subsemigroup of $M \times T$. It follows from Theorem 3.2 that S is completely regular, and, since both M and E_T are normal bands, the same holds for E_S. Suppose that $(e, 1_\alpha) < (f, 1_\beta)$, $(e, 1_\alpha) < (t, 1_\gamma)$, and $(f, 1_\beta)\mathscr{N}(t, 1_\gamma)$, where $(e, 1_\alpha)$, $(f, 1_\beta)$, $(t, 1_\gamma) \in E_S$. It follows easily that $1_\beta = 1_\gamma$ and that $(e, 1_\alpha) = (f, 1_\beta)(e, 1_\alpha) = (e, 1_\alpha)(f, 1_\beta)$ implies $e = fe = ef$, so that $e = f$; analogously $e = t$. Hence $(f, 1_\beta) = (t, 1_\gamma)$, showing that E_S is a strongly normal band.

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

Sufficiency. By [**11**, Theorem 4.3], we have that S is a band of groups. It follows from the proof of Theorem 3.2 that S is a subdirect product of E_S and S/ξ. Since E_S is strongly normal, Proposition 2.9 implies that E_S is a subdirect product of $E_S/\mathcal{N}$ and a rectangular band M. Thus S is a subdirect product of $E_S/\mathcal{N}$, M, and S/ξ. An argument quite similar to that in the sufficiency part of Theorem 4.1 shows that $E_S/\mathcal{N}$ can be omitted, which then proves that S is a subdirect product of M and S/ξ as required.

COROLLARY 5.2. *Semigroups in Theorem* 5.1 *can also be characterized as regular semigroup subdirect products of left zero semigroups, right zero semigroups, groups and groups with zero.*

Proof. This follows from Theorem 5.1 and an argument similar to that of a part of the proof of Theorem 3.2.

As in the preceding section, we can give an explicit construction of the semigroups under study. A different construction of all subdirect products of a rectangular band and an arbitrary semigroup is given in [**1**]. For any set X, let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X ordered under set inclusion.

A construction of these semigroups can be effected as follows.

THEOREM 5.3. *Let L be a left zero semigroup, R be a right zero semigroup, and T be a semilattice of groups. Let $\varphi : T/\mathcal{N} \to \mathcal{P}(L)$ and $\psi : T/\mathcal{N} \to \mathcal{P}(R)$ be full dual homomorphisms, and let*

$$S = \{(l, r, g) \in L \times R \times T \mid l \in N_g\varphi, r \in N_g\psi\}.$$

Then S is a regular semigroup subdirect product of L, R and T. Conversely, every regular semigroup subdirect product of L, R and T is essentially of this type.

Proof. Let S be as defined above. If $(l, r, g), (l', r', g') \in S$, then $l \in N_g\varphi \subseteq N_{gg'}\varphi$ and $r' \in N_{g'}\psi \subseteq N_{gg'}\psi$, so that $(l, r, g)(l', r', g') = (l, r', gg') \in S$. Regularity of S follows from the fact that N_g is a group for any $g \in T$. The fullness of both φ and ψ guarantees that S is a subdirect product of L, R and T.

Conversely, let $S \subseteq L \times R \times T$ be a regular semigroup subdirect product of L, R and T. For every $g \in T$, let

$$g\varphi = \{l \in L \mid (l, r, g) \in S \quad \text{for some} \quad r \in R\},$$

$$g\psi = \{r \in R \mid (l, r, g) \in S \quad \text{for some} \quad l \in L\}.$$

The projection of S into $L \times R$ is a subdirect product of L and R, and hence must coincide with $M = L \times R$ by Proposition 2.12. Now let $g \in T$ and let 1_α be the identity of the maximal subgroup of T containing g. If $l \in g\varphi$, then $(l, r, g) \in S$ for some $r \in R$ and hence $(l, r, 1_\alpha) \in S$, so that $l \in 1_\alpha\varphi$. Conversely, let $l \in 1_\alpha\varphi$. Then $(l, r, 1_\alpha), (l', r', g) \in S$ for some $l' \in L, r, r' \in R$. Consequently

$$(l, r, g) = (l, r, 1_\alpha)(l', r', g)(l, r, 1_\alpha) \in S,$$

showing that $l \in g\varphi$. Thus $g\varphi = 1_\alpha\varphi$ and we may consider φ as a function mapping $T/\mathcal{N}$ into $\mathcal{P}(L)$. Let $g, h \in T$ and assume that $N_g \geqq N_h$. If $l \in N_g\varphi$, then $(l, r, g), (l', r', h) \in S$ for some

$l' \in L$, $r, r' \in R$ and hence

$$(l, r, ghg) = (l, r, g)(l', r', h)(l, r, g) \in S,$$

so that $l \in N_h \varphi$, since $N_{ghg} = N_h$. Thus $N_g \varphi \subseteq N_h \varphi$, which says that φ is a dual homomorphism; its fullness follows from the fact that the projection of S into L is onto.

A similar argument is applicable to ψ. If $(l, r, g) \in S$, then $l \in N_g \varphi$ and $r \in N_g \psi$. Conversely, if $l \in N_g \varphi$ and $r \in N_g \psi$, then there exist $l' \in L$ and $r' \in R$ such that $(l, r', 1_\alpha)$, $(l', r, g) \in S$, where 1_α has the same meaning as above; thus $(l, r, g) = (l, r', 1_\alpha)(l', r, g) \in S$ as required.

For the semigroup S constructed above we use the notation $(L, \varphi; T; \psi, R)$ and now turn to isomorphisms of such semigroups.

THEOREM 5.4. *Let $S = (L, \varphi; T; \psi, R)$ and $S' = (L', \varphi'; T'; \psi', R')$. Further, let $\lambda : L \to L'$ and $\rho : R \to R'$ be bijections and θ be an isomorphism of T onto T' such that the diagram*

$$\begin{tikzcd}
\mathscr{P}(L) \arrow[d, "\bar{\lambda}"'] & T/\mathscr{N} \arrow[l, "\varphi"'] \arrow[r, "\psi"] \arrow[d, "\hat{\theta}"] & \mathscr{P}(R) \arrow[d, "\bar{\rho}"] \\
\mathscr{P}(L') & T'/\mathscr{N} \arrow[l, "\varphi'"'] \arrow[r, "\psi'"] & \mathscr{P}(R')
\end{tikzcd}$$

is commutative. Then the function χ defined by

$$(l, r, g)\chi = (l\lambda, r\rho, g\theta) \qquad ((l, r, g) \in S)$$

is an isomorphism of S onto S'. Conversely, every isomorphism of S onto S' can be expressed in this way.

Proof. For χ given as above, one shows, analogously as in the proof of Theorem 3.3, that χ is an isomorphism of S onto S'.

Conversely, let χ be an isomorphism of S onto S'. Then $M = L \times R$ is a rectangular band and T is a semilattice $Y = T/\mathscr{N}$ of groups G_α with identity 1_α; the corresponding meanings are attached to M', Y', $G'_{\alpha'}$, $1'_{\alpha'}$. We define the functions δ and θ by

$$(e, g)\chi = ((e, g)\delta, (e, g)\theta) \qquad (e \in M, g \in T); \tag{5}$$

hence $\delta : S \to M'$ and $\theta : S \to T'$ are homomorphisms. Noting that

$$E_S = \{(e, 1_\alpha) \in M \times T \,|\, e = (l, r), l \in N_{1_\alpha} \varphi, r \in N_{1_\alpha} \psi\},$$

we infer that $(e, 1_\alpha)\theta = 1'_{(e,1_\alpha)\pi}$ for some function $\pi : E_S \to Y'$. If $(e, 1_\alpha), (f, 1_\alpha) \in E_S$, then $(e, 1_\alpha)\mathscr{N}(f, 1_\alpha)$, so that $1'_{(e,1_\alpha)\pi} \mathscr{N} 1'_{(f,1_\alpha)\pi}$ and thus $1'_{(e,1_\alpha)\pi} = 1'_{(f,1_\alpha)\pi}$. Abusing the notation slightly, we may write $(e, 1_\alpha)\pi = \alpha\pi$, where now $\pi : Y \to Y'$ and $(e, 1_\alpha)\theta = 1'_{\alpha\pi}$. It then follows from (5) that

$$(e, 1_\alpha)\chi = ((e, 1_\alpha)\delta, 1'_{\alpha\pi}) \qquad ((e, 1_\alpha) \in E_S). \tag{6}$$

For $g \in G_\alpha$ and $(e, 1_\alpha) \in S$, we have $(e, g) \in S$ and $(e, g)\mathscr{N}(e, 1_\alpha)$, which implies that $(e, g)\theta \mathscr{N}(e, 1_\alpha)\theta$. In view of (5) and (6) we must have $(e, g)\theta \in G'_{\alpha\pi}$. If also $(f, 1_\alpha) \in S$, then

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

for the same reason $(f, g)\theta \in G'_{\alpha\pi}$ and $(f, 1_\alpha)\theta = 1'_{\alpha\pi}$, so that

$$(e, g)\theta = (e, g)\theta 1'_{\alpha\pi} = (e, g)\theta(f, 1_\alpha)\theta = (ef, g)\theta = (e, 1_\alpha)\theta(f, g)\theta = 1'_{\alpha\pi}(f, g)\theta = (f, g)\theta,$$

and we may write $(e, g)\theta = g\theta$ and consider θ as a homomorphism of T into T'. With the same notation, $(e, g)(e, 1_\alpha) = (e, 1_\alpha)(e, g)$ and, applying δ to this, we have

$$(e, g)\delta(e, 1_\alpha)\delta = (e, 1_\alpha)\delta(e, g)\delta,$$

which in a rectangular band yields $(e, g)\delta = (e, 1_\alpha)\delta$. Since $g \in G_\alpha$ is arbitrary, it follows that we may write $(e, g)\delta = e\delta$ and consider δ as a homomorphism of M into M'. Consequently (5) becomes

$$(e, g)\chi = (e\delta, g\theta) \qquad (e \in M, g \in T). \tag{7}$$

For every $e' \in M'$ there exists $g' \in T'$ such that $(e', g') \in S'$ and conversely. On the other hand, if $(e', g') \in S'$, there exists $(e, g) \in S$ such that $(e, g)\chi = (e', g')$, which shows that both δ and θ are onto functions.

Let $e\delta = f\delta$; then $(e, 1_\alpha), (f, 1_\beta) \in S$ for some $\alpha, \beta \in Y$. Hence

$$(e, 1_\alpha)(f, 1_\beta)(e, 1_\alpha) = (efe, 1_{\alpha\beta\alpha}) = (e, 1_{\alpha\beta}) \in S$$

and analogously $(f, 1_{\alpha\beta}) \in S$. But then $(e, 1_{\alpha\beta})\chi = (f, 1_{\alpha\beta})\chi$, by (7), and thus $e = f$.

Suppose next that $g\theta = h\theta$. Then $g \in G_\alpha$, $h \in G_\beta$ for some $\alpha, \beta \in Y$, and $(e, g), (f, h) \in S$ for some $e, f \in M$, and thus $(e, 1_\alpha), (f, 1_\beta) \in S$. Since $1_\alpha \mathcal{N} g$ and $1_\beta \mathcal{N} h$, we have $1_\alpha \theta \mathcal{N} g\theta$ and $1_\beta \theta \mathcal{N} h\theta$, which then implies that $1_\alpha \theta = 1_\beta \theta$. It follows that

$$\begin{aligned}(e, 1_\alpha)\chi = (e\delta, 1_\alpha \theta) &= ((e\delta)(f\delta)(e\delta), (1_\alpha \theta)(1_\beta \theta)(1_\alpha \theta)) \\ &= (e, 1_\alpha)\chi(f, 1_\beta)\chi(e, 1_\alpha)\chi = (efe, 1_{\alpha\beta\alpha})\chi = (e, 1_{\alpha\beta})\chi;\end{aligned}$$

hence $1_\alpha = 1_{\alpha\beta}$ and thus $\alpha = \alpha\beta$. A similar argument shows that $\beta = \alpha\beta$ and hence $\alpha = \beta$. But then $(ef, g), (ef, h) \in S$ and

$$(ef, g)\chi = ((ef)\delta, g\theta) = ((ef)\delta, h\theta) = (ef, h)\chi;$$

therefore $g = h$.

According to Proposition 2.14, we can write

$$(l, r)\delta = (l\lambda, r\rho) \qquad ((l, r) \in M)$$

for some bijections $\lambda : L \to L'$ and $\rho : R \to R'$. For any $(l, r) \in M$ and $g \in T$, we obtain

$$\begin{aligned}l \in N_g \varphi, r \in N_g \psi &\Leftrightarrow ((l, r), g) \in S \Leftrightarrow ((l, r)\delta, g\theta) \in S' \Leftrightarrow ((l\lambda, r\rho), g\theta) \in S' \\ &\Leftrightarrow l\lambda \in N_{g\theta}\varphi', r\rho \in N_{g\theta}\psi' \Leftrightarrow l\lambda \in N_g \hat{\theta}\varphi', r\rho \in N_g \hat{\theta}\psi',\end{aligned}$$

which proves the commutativity of the diagram.

We may now consider the congruences on a regular semigroup essentially encountered in Theorem 5.1. Note that CRISN relates to "completely regular, idempotents strongly normal", and M-congruence stands for a "matrix congruence" (or "rectangular band congruence").

THEOREM 5.5. *The following statements concern any regular semigroup. The intersection of an* M-*congruence and an* SG-*congruence is a* CRISN-*congruence. Conversely, every* CRISN-*congruence can be written uniquely as the intersection of an* M-*congruence and an* SG-*congruence. In particular,*

$$(\textit{min}\ \text{M-}\textit{congruence})\cap(\textit{min}\ \text{SG-}\textit{congruence}) = \textit{min}\ \text{CRISN-}\textit{congruence}.$$

Proof. The proof of the first two statements, except for uniqueness, follows the same pattern as the first part of the proof of Theorem 3.4 by referring to Theorem 5.1 instead of Theorem 3.2 and may safely be omitted. The last statement follows easily from the preceding two. It thus remains to establish uniqueness. We let μ and μ' be M-congruences and ξ and ξ' be SG-congruences on a regular semigroup S and suppose that $\mu \cap \xi = \mu' \cap \xi'$.

Let $a\,\mu\, b$ and let a' be an inverse of a. Then $aa'\,\mu\, ba'$ and thus $(aa')(ba')\,\mu\,(ba')(aa')$. Further, $aa' \in E_S$ and hence $(aa')(ba')\,\xi\,(ba')(aa')$, since the idempotents of S/ξ are in the centre of S/ξ. Consequently $(aa')(ba')\,\mu \cap \xi\,(ba')(aa')$, which by the hypothesis yields $(aa')(ba')\,\mu' \cap \xi'\,(ba')(aa')$. It follows that $(aa')(ba')\,\mu'\,(ba')(aa')$ and hence $aa'ba'a\,\mu'\,ba'a$. Since μ is an M-congruence, it follows that $a\,\mu'\,ba$. A similar argument can be used to show that also $b\,\mu\, ba$, which then implies that $a\,\mu'\,b$, proving $\mu \subseteq \mu'$. By symmetry, we conclude that $\mu = \mu'$.

Finally let $a\,\xi\, b$ and let a' be an inverse of a. Then $a\,\mu\, aa'ba'a$, since μ is an M-congruence and $a\,\xi\, aa'ba'a$, since $a\,\xi\, b$. Thus $a\,\mu \cap \xi\, aa'ba'a$ and the hypothesis implies that $a\,\mu' \cap \xi'\, aa'ba'a$, so that $a\,\xi'\, aa'ba'a$. For any $x \in S$, let $\bar{x}$ be the ξ'-class containing x. Then $aa'\,\mathscr{N}\, a$ implies $\overline{aa'}\,\mathscr{N}\,\bar{a}$ and thus $\overline{aa'}$ is the identity of the maximal subgroup G of S/ξ' containing $\bar{a}$; similarly $\overline{a'a}$ has the same property. Letting $e = \overline{aa'} = \overline{a'a}$, we have $a = e\bar{b}e$. Analogously $b = f\bar{a}f$, where f is the identity of the maximal subgroup of S/ξ' containing $\bar{b}$. Hence

$$\bar{a} = e\bar{b}e = ef\bar{a}fe = fe\bar{a}ef = f\bar{a}f = \bar{b}$$

and thus $a\,\xi'\, b$, proving that $\xi \subseteq \xi'$. By symmetry, we also have $\xi' \subseteq \xi$ and therefore $\xi = \xi'$.

For the sake of completeness, we state the following result. Note that L refers to a left zero semigroup, R to a right zero semigroup.

THEOREM 5.6 [**10**, Theorem 1]. *The following statements are valid in any semigroup. The intersection of an* L-*congruence and an* R-*congruence is an* M-*congruence. Conversely, every* M-*congruence can be written uniquely as the intersection of an* L-*congruence and an* R-*congruence. In particular*

$$(\textit{min}\ \text{L-}\textit{congruence})\cap(\textit{min}\ \text{R-}\textit{congruence}) = \textit{min}\ \text{M-}\textit{congruence}.$$

COROLLARY 5.7. *The following statements concern any regular semigroup. The intersection of an* L-*congruence, an* R-*congruence, and an* SG-*congruence is a* CRISN-*congruence. Conversely, every* CRISN-*congruence can be written uniquely as the intersection of an* L-*congruence, an* R-*congruence and an* SG-*congruence.*

Proof. Combine Theorems 5.5 and 5.6.

Note that the sup of any M-congruence and any SG-congruence is the universal relation. We can now deduce easily the results corresponding to the ones above for regular semigroup subdirect products of (i) a left (resp. right) zero semigroup and a semilattice of groups (ii) a

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

rectangular band and a semilattice. For (i), we make the following substitutions in the above statements:

In Theorem 5.1: rectangular band $\rightarrow$ left zero semigroup,

E_S strongly normal $\rightarrow$ E_S left strongly normal;

in Theorem 5.6: M, CRISN $\rightarrow$ L, CRILSN;

and in Corollary 5.2, Theorem 5.3 and Theorem 5.4 omit all references to right zero semigroups.
A similar modification can be carried through for (ii) and need not be stated explicitly.

6. Subdirect products of a rectangular group and a semilattice. This case, and its subcases, finally round up our study of subdirect products. The similarity with the preceding cases allows us to abbreviate or omit several proofs. In connection with the next two theorems, it is useful to recall Corollary 2.13.

THEOREM 6.1. *A semigroup S is regular and a subdirect product of a rectangular group and a semilattice if and only if S is completely regular and E_S is a unitary strongly normal subband of S.*

Proof. Necessity. Let $S \subseteq M \times Y \times G$ be a subdirect product with M a rectangular band, Y a semilattice, and G a group, and let T be the projection of S in $Y \times G$. Hence T is a regular semigroup subdirect product of a semilattice and a group, which by Proposition 2.6 implies that T is a semilattice of groups and that E_T is a unitary subset of T. Consequently S is a subdirect product of the rectangular band M and the semilattice of groups T and thus, by Theorem 5.1, we have that S is completely regular and E_S is a strongly normal band. Let Q be an isomorphic copy of S in $M \times T$, and let $(e, 1_\alpha), (e, 1_\alpha)(f, g) \in E_Q$. Then $1_\alpha, 1_\alpha g \in E_T$, which implies that $g \in E_T$, so that $(f, g) \in E_Q$. It then follows that E_S is a unitary subset of S.

Sufficiency. First Theorem 5.1 implies that S is a subdirect product of a rectangular band M and a semilattice of groups T. Assume that $S \subseteq M \times T$ and let $1_\alpha, 1_\alpha g \in E_T$. Then, for some $e, f \in M$, we have $(e, 1_\alpha), (f, g) \in S$ and thus $(e, 1_\alpha), (e, 1_\alpha)(f, g) \in E_S$, which by the hypothesis yields $(f, g) \in E_S$ and thus $g \in E_T$. This shows that E_T is unitary and hence, applying Proposition 2.6, we conclude that T is a subdirect product of a semilattice Y and a group G. By transitivity of subdirect products, we have that S is a subdirect product of M, Y and G as required.

A construction of these semigroups is given in

THEOREM 6.2. *Let L be a left zero semigroup, R be a right zero semigroup, Y be a semilattice and G be a group. Let $\lambda : Y \rightarrow \mathscr{P}(L)$, $\rho : Y \rightarrow \mathscr{P}(R)$, $\eta : Y \rightarrow \mathscr{L}(G)$ be full dual homomorphisms and let*

$$S = \{(l, r, \alpha, g) \in L \times R \times Y \times G \mid l \in \alpha\lambda, r \in \alpha\rho, g \in \alpha\eta\}.$$

Then S is a regular semigroup subdirect product of L, R, Y and G. Conversely, every regular semigroup subdirect product of L, R, Y and G is essentially of this type.

Proof. One may use Theorems 4.3 and 5.3 to deduce this theorem, or prove it directly by modifying slightly the proofs of these theorems. Hence the proof may be omitted.

We denote the semigroup S constructed above by $(Y; L, \lambda; R, \rho; G, \eta)$ and establish next when two such semigroups are isomorphic.

THEOREM 6.3. *Let* $S = (Y; L, \lambda; R, \rho; G, \eta)$ *and* $S' = (Y'; L', \lambda'; R', \rho'; G', \eta')$. *Further, let* $\varphi : L \to L'$, $\psi : R \to R'$, $\sigma : Y \to Y'$, $\tau : G \to G'$ *be onto isomorphisms such that the diagrams*

$$\begin{tikzcd} \mathscr{P}(L) \arrow[d, "\bar{\varphi}"'] & Y \arrow[l, "\lambda"'] \arrow[r, "\rho"] \arrow[d, "\sigma"'] & \mathscr{P}(R) \arrow[d, "\bar{\psi}"] \\ \mathscr{P}(L') & Y' \arrow[l, "\lambda'"'] \arrow[r, "\rho'"] & \mathscr{P}(R') \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta"] \arrow[d, "\sigma"'] & \mathscr{L}(G) \arrow[d, "\bar{\tau}"] \\ Y \arrow[r, "\eta'"] & \mathscr{L}(G) \end{tikzcd}$$

are commutative. Then the function χ *defined by*

$$(l, r, \alpha, g)\chi = (l\varphi, r\psi, \alpha\sigma, g\tau) \qquad ((l, r, \alpha, g) \in S)$$

is an isomorphism of S *onto* S'. *Conversely, every isomorphism of* S *onto* S' *can be expressed in this way.*

Proof. The proof of the direct part follows the same pattern as the corresponding part of the proof of Theorem 3.3. For the converse, we let B be the projection of S in $L \times R \times Y$ and B' be the projection of S' in $L' \times R' \times Y'$. Then S is a subdirect product of B and G, and S' is a subdirect product of B' and G', and we are able to apply Theorems 4.3 and 4.4. This yields isomorphisms σ' of B onto B' and τ of G onto G' as in Theorem 4.4. Furthermore, B is a subdirect product of L, R and Y, and B' is a subdirect product of L', R' and Y'; so we may apply Theorems 5.3 and 5.4. This yields the remaining isomorphisms φ, ψ, σ. Note that the mapping $N_{(l,r,\alpha)} \to \alpha$ $((l, r, \alpha) \in B)$ is an isomorphism of $S/\mathcal{N}$ onto Y. Combining all these statements gives the desired result; we omit the details.

We are now in a position to consider the congruences on a regular semigroup that are induced by the subdirect product figuring in Theorem 6.1. The abbreviation CRUSN stands for "completely regular, idempotents form a unitary strongly normal subband".

THEOREM 6.4. *The following statements concern any regular semigroup. The intersection of an* L-*congruence, an* R-*congruence, an* S-*congruence and a* G-*congruence is a* CRUSN-*congruence. Conversely, every* CRUSN-*congruence can be written uniquely as the intersection of an* L-*congruence, an* R-*congruence, an* S-*congruence and a* G-*congruence. In particular*

$$(\textit{min } \text{L-}\textit{congruence}) \cap (\textit{min } \text{R-}\textit{congruence}) \cap (\textit{min } \text{S-}\textit{congruence}) \cap (\textit{min } \text{G-}\textit{congruence}) = \textit{min } \text{CRUSN-}\textit{congruence}.$$

Proof. The proofs of all the statements except for the uniqueness part follow the pattern of parts of the proof of Theorem 3.4 by referring to Theorem 6.1 instead of Theorem 3.2 and need not be carried out here. To prove the uniqueness part, we let λ and λ' be L-congruences, ρ and ρ' be R-congruences, η and η' be S-congruences, σ and σ' be G-congruences on a regular

https://doi.org/10.1017/S0017089500001701 Published online by Cambridge University Press

semigroup S, such that $\lambda\cap\rho\cap\eta\cap\sigma = \lambda'\cap\rho'\cap\eta'\cap\sigma'$. Writing $(\lambda\cap\rho\cap\eta)\cap\sigma = (\lambda'\cap\rho'\cap\eta')\cap\sigma'$, we infer from Corollary 4.6 that $\lambda\cap\rho\cap\eta = \lambda'\cap\rho'\cap\eta'$ and $\sigma = \sigma'$. Next, writing $(\lambda\cap\rho)\cap\eta = (\lambda'\cap\rho')\cap\eta'$, we deduce from Theorem 5.5 that $\lambda\cap\rho = \lambda'\cap\rho'$ and $\eta = \eta'$. Finally, $\lambda\cap\rho = \lambda'\cap\rho'$, by Theorem 5.6, yields $\lambda = \lambda'$, $\rho = \rho'$ as required.

In the same way as in the preceding section, we are now able to derive the statements corresponding to those in this section and concerning regular semigroup subdirect products of (i) a left (resp. right) group and a semilattice, (ii) a left (resp. right) zero semigroup and a semilattice. For (i) we make the following changes:

In Theorem 6.1, omit R and take E_S a unitary left strongly normal band;
in Theorem 6.4, omit R and write CRULSN instead of CRUSN;

and in Theorems 6.2 and 6.3 omit all reference to right zero semigroups. The changes to be made for case (ii) need not be stated explicitly.

Also note that S in Theorem 6.1 is a subdirect product of subdirectly irreducible left zero semigroups, right zero semigroups, semilattices and groups. Recall that the semigroups of each of the first three classes contain exactly two elements.

From some of the considerations so far and a brief look at the direct product of the form $L \times R \times G$, we easily deduce

PROPOSITION 6.5. *Let* λ, ρ, μ, σ *and* η *be, respectively, an* L-, R-, M-, G-, *and* S-*congruence on an arbitrary semigroup. Then*

$$\lambda\rho = \rho\lambda = \lambda\sigma = \sigma\lambda = \rho\sigma = \sigma\rho = \mu\sigma = \sigma\mu = \lambda \vee \eta = \rho \vee \eta = \mu \vee \eta = \sigma \vee \eta = \mathscr{U},$$

but η *need not commute with any of* λ, ρ, μ *or* σ.

ACKNOWLEDGEMENT. The author is indebted to Professor A. H. Clifford and the referee for several useful suggestions concerning the presentation of this paper.

Added in proof. The first displayed statement in Theorem 3.4 has been established by B. M. Schein in the paper entitled " A note on radicals in regular semigroups ", *Semigroup Forum*, **3** (1971), 84–85.

REFERENCES

1. J. L. Chrislock and T. Tamura, Notes on subdirect products of semigroups and rectangular bands, *Proc. Amer. Math. Soc.* **20** (1969), 511–514.

2. A. H. Clifford, Bands of semigroups, *Proc. Amer. Math. Soc.* **5** (1954), 499–504.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Amer. Math. Soc., Math. Surveys No. 7, Vol. I (Providence, R.I., 1961).

4. P. H. H. Fantham, On the classification of a certain type of semigroup, *Proc. London Math. Soc.* **10** (1960), 409–427.

5. J. M. Howie and G. Lallement, Certain fundamental congruences on a regular semigroup, *Proc. Glasgow Math. Assoc.* **7** (1966), 145–156.

6. N. Kimura, The structure of idempotent semigroups (I), *Pacific J. Math.* **8** (1958), 257–275.

7. M. Petrich, Sur certaines classes de demi-groupes I, *Acad. Roy. Belg. Bull. Cl. Sci.* **49** (1963), 785–798.

8. M. Petrich, The maximal semilattice decomposition of a semigroup, *Math. Zeit.* **85** (1964), 68–82.

9. M. Petrich, The structure of a class of semigroups which are unions of groups, *Notices Amer. Math. Soc.* **12** (1965), Abstr. No. 619–151.

10. M. Petrich, The maximal matrix decomposition of a semigroup, *Portugal. Math.* **25** (1966), 15–33.

11. M. Petrich, Regular semigroups satisfying certain conditions on idempotents and ideals, *Trans. Amer. Math. Soc.* **170** (1972), 245–267.

12. M. Yamada, Strictly inverse semigroups, *Bull. Shimane Univ.* **13** (1964), 128–138.

13. M. Yamada, Regular semigroups whose idempotents satisfy permutation identities, *Pacific J. Math.* **21** (1967), 371–392.

14. M. Yamada and N. Kimura, Note on idempotent semigroups II, *Proc. Japan Acad.* **34** (1958), 110–112.

PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK
PENNSYLVANIA

D