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Growth of Homology of Centre-by-metabelian Pro-p Groups

Dessislava H. Kochloukova and Aline G. S. Pinto

Abstract. For a centre-by-metabelian pro-p group G of type FP_{2m}, for some $m \geq 1$, we show that $\sup_{M \in \mathcal{A}} \operatorname{rk} H_i(M, \mathbb{Z}_p) < \infty$, for all $0 \leq i \leq m$, where $\mathcal{A}$ is the set of all subgroups of p-power index in G and, for a finitely generated abelian pro-p group V, $\operatorname{rk} V = \dim V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

1 Introduction

J. S. Wilson proved that the Golod–Shafarevich inequality holds for finitely presented soluble pro-p groups. Using this, he proved that, for finitely presented soluble pro-p group G with a normal pro-p subgroup H such that $G/H \simeq \mathbb{Z}_p$, the pro-p group H is finitely generated [24, Corollary A, (iii)].

In the context of pro-p groups, the properties of being finitely generated and finitely presented can be translated as the homological properties FP_1 and FP_2, respectively. A pro-p group G has homological type FP_m if $\mathbb{Z}_p$, considered as a trivial $\mathbb{Z}_p[[G]]$-module, has a projective (free) resolution of pro-p $\mathbb{Z}_p[[G]]$-modules, where the modules in dimension up to m are finitely generated or, equivalently, if the homology groups $H_i(G, \mathbb{F}_p)$ are finite for $i \leq m$. So G is finitely generated if and only if G is FP_1 and G is finitely presented if and only if G is FP_2. Thus, Wilson's result can be stated as: for soluble pro-p groups of type FP_2, every normal pro-p subgroup with quotient $\mathbb{Z}_p$ is FP_1.

Little is known for finitely presented soluble pro-p groups. C. Corob Cook [7] showed that every virtually torsion-free, soluble, pro-p group of type FP_∞ is of finite rank (for groups of finite rank see [8]). J. King [13] classified the finitely presented metabelian pro-p groups. This was later generalized by Kochloukova in [14], where all metabelian pro-p groups of type FP_m were classified in terms of King's invariant (Theorem 2.4). Using this classification of metabelian pro-p groups of type FP_m, Kochloukova and Pinto proved [16] that every finitely generated metabelian pro-p group embeds in a metabelian pro-p group of type FP_m. The case $m = 2$ was proved earlier by Remeslenikov [21], much before King's classification of finitely presented metabelian pro-p groups was established. The abstract case of the same embedding

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result for finitely generated metabelian discrete groups when $m = 2$ was proved by G. Baumslag [1] and for general m was proved by Kochloukova and da Silva [18]. In the case of Lie algebras, the embedding property of metabelian Lie algebras holds, too, and was established by J. Groves and Kochloukova in [10].

Groves proved that a finitely presented abstract centre-by-metabelian group is abelian-by-polycyclic [9]. In particular, it has the maximal condition on normal subgroups, thus the central part is finitely generated. Kochloukova and Pinto [17] showed that this holds for finitely presented, centre-by-metabelian pro-p groups, *i.e.*, the central part is a finitely generated, abelian, pro-p group. This, together with Kochloukova's classification of metabelian pro-p groups of type FP_m, gives classification of centre-by-metabelian pro-p groups of type FP_m. Our first result generalizes Wilson's result [24, Corollary A, (iii)] when G is a centre-by-metabelian pro-p group of homological type FP_{2m}.

Theorem A *Let G be a centre-by-metabelian pro-p group of type FP_{2m}, where $m \geq 1$ is an integer. If H is a normal pro-p subgroup of G such that $G/H \cong \mathbb{Z}_p$, then H is of type FP_m.*

In the following result, we give an example of a metabelian pro-p group of type FP_3 with a normal pro-p subgroup that is not FP_2 and the quotient is $\mathbb{Z}_p$. This justifies our hypothesis on G in Theorem A. The example was based on King's examples of a finitely generated metabelian pro-p group H that is not finitely presented [13].

Proposition B *Let $p > 2$ be a prime number. Let Q_0 be the free abelian pro-p group on the set $\{s, t\}$ and $k = \mathbb{F}_p$ or $k = \mathbb{Z}_p$. Let $A = k[[Q_0]]/(s + s^{-1} + t + t^{-1} - 4)$, and $Q = \overline{\langle s, t, y\rangle} = \mathbb{Z}_p^3$ is generated as an abelian pro-p group by s, t and y, where y acts on A (via conjugation) by multiplication with $(s + s^{-1})/2$. Then $G = A \rtimes Q$ is a pro-p group of type FP_3, with a pro-p normal subgroup $H = A \rtimes Q_0$ such that $G/H \cong \mathbb{Z}_p$ and H is not of type FP_2.*

M. R. Bridson and Kochloukova [4] generalized Wilson's result in the following direction. For a finitely generated pro-p group H, let $d(H)$ be the minimal number of generators of H. They showed [4, Proposition A] that for a finitely presented soluble pro-p group G, one has $\sup_{G/H \cong \mathbb{Z}_p} d(H) < \infty$. Using this, they then proved [4, Corollary D] that for a finitely presented nilpotent-by-abelian-by-finite pro-p group, one has $\sup_{M \in \mathcal{A}} \operatorname{rk} H_1(M, \mathbb{Z}_p) < \infty$, where $\mathcal{A}$ is the set of all pro-p subgroups of finite index in G and, for an abelian pro-p group B, $\operatorname{rk} B := \dim_{\mathbb{Q}_p} B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the torsion free rank of B. Observe that in a pro-p group, a subgroup of finite index always has a p-power index. They also gave an example of a finitely presented metabelian pro-p group where this fails when one changes the field of coefficients from $\mathbb{Q}_p$ to $\mathbb{F}_p$, *i.e.*, $\sup_{M \in \mathcal{A}} \dim_{\mathbb{F}_p} H_1(M, \mathbb{F}_p) = \infty$.

The next result generalizes [4, Corollary D] for centre-by-metabelian pro-p groups. Recall that, by a result of Kochloukova and Pinto, the central part of a finitely presented centre-by-metabelian pro-p group is finitely generated [17]. The same is known to hold for the category of Lie algebras by a result of Bryant and Groves [5].

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Theorem C *Let G be a centre-by-metabelian pro-p group of type FP_{2m}, where $m \geq 1$. Then $\sup_{M \in \mathcal{A}} \operatorname{rk} H_i(M, \mathbb{Z}_p) < \infty$, for all $0 \leq i \leq m$, where $\mathcal{A}$ is the set of all subgroups of p-power index in G.*

We also show that for $m = 2$, the condition in Theorem C that G is of type FP_{2m} is necessary.

Proposition D *Let $p > 2$ be a prime number. For the group G defined in Proposition B for $k = \mathbb{Z}_p$, we have that $\sup_{M \in \mathcal{A}} \operatorname{rk} H_2(M, \mathbb{Z}_p) = \infty$, where $\mathcal{A}$ is the set of all subgroups of p-power index in G.*

Finally, based on Theorems A and C we suggest the following conjecture.

Conjecture *There is a function $\rho\colon \{1,2,3,\cdots\} \times \{1,2,3,\cdots\} \to \{1,2,3,\cdots\}$ such that for every soluble pro-p group G of soluble class k and of homological type $\mathrm{FP}_{\rho(k,n)}$ and for every normal pro-p subgroup H of G such that $G/H \simeq \mathbb{Z}_p$, we have that H is of type FP_n.*

2 Preliminaries

2.1 Homological Finiteness Properties of Pro-p Groups

Recall that for a pro-p group G and $k = \mathbb{F}_p$ or $k = \mathbb{Z}_p$, the completed group algebra $k[[G]]$ is the inverse limit of $(k/p^i k)[G/U]$ over all $i \geq 1$ and open normal subgroups U of G. The completed group algebra $k[[G]]$ is a local ring whose unique maximal ideal is the kernel of the canonical map $k[[G]] \to \mathbb{F}_p$ that sends G to 1 and k to $k/pk \simeq \mathbb{F}_p$.

A pro-p group G is of homological type FP_n if there is a projective resolution (in the category of pro-p modules, thus all differentials should be continuous) of the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ $\mathcal{P}\colon \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to \mathbb{Z}_p \to 0$, where all P_i are finitely generated for $i \leq n$. It is worth mentioning that by [25, Lemma 7.2.2] any abstract homomorphism $\rho\colon V \to W$ of pro-p R-modules, where $R = k[[G]]$, k a pro-p ring, and V and W are finitely generated pro-p R-modules, is automatically continuous.

By [20, Theorem 1.6] a pro-p group G is of type FP_n if and only if the pro-p homology $H_i(G, \mathbb{F}_p)$ is finite for all $i \leq n$. Thus G is FP_1 if and only if G is finitely generated as a pro-p group (we say simply finitely generated). And both $H_1(G, \mathbb{F}_p)$ and $H_2(G, \mathbb{F}_p)$ are finite if and only if G is finitely presented as a pro-p group [22, §7.8], *i.e.*, $G \simeq F/\overline{\langle S^F \rangle}$, where F is a free pro-p group with a finite basis, S is a finite subset of F, and $\overline{\langle S^F \rangle}$ is the normal pro-p subgroup generated by S.

2.2 Metabelian Pro-p Groups

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$ and $\mathbb{F}[[T]]$ be the formal power series algebra with a group of units $\mathbb{F}[[T]]^\times$. Let Q be a (topologically) finitely generated abelian pro-p group and $T(Q)$ be the set $\operatorname{Hom}(Q, \mathbb{F}[[T]]^\times)$ of continuous homomorphisms from Q to $\mathbb{F}[[T]]^\times$. By the universal property of $\mathbb{Z}_p[[Q]]$, each $v \in T(Q)$ extends to a

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unique continuous algebra homomorphism from $\mathbb{Z}_p[[Q]]$ to $\mathbb{F}[[T]]$, which we denote by $\overline{v}$.

Definition 2.1 ([13, Definition A]) Let Q be a finitely generated abelian pro-p group and A a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module. King's invariant is defined as

$$\Delta(A) = \{v \in T(Q) \mid \operatorname{Ann}_{\mathbb{Z}_p[[Q]]}(A) \le \operatorname{Ker} \overline{v}\} \cup \{1\},$$

where $\operatorname{Ann}_{\mathbb{Z}_p[[Q]]}(A) = \{\lambda \in \mathbb{Z}_p[[Q]] \mid A\lambda = 0\}$ is the annihilator of A in $\mathbb{Z}_p[[Q]]$.

Kochloukova and Zalesskii [19] associated an invariant that is a subset of $T(Q)$ with any finitely generated pro-p group G . This invariant and the above invariant $\Delta(A)$ are quite hard to calculate in concrete examples.

We state below an important property of $\Delta(A)$. A similar result holds for the Bieri–Strebel invariant $\Sigma_A^c(Q)$ defined for a finitely generated $\mathbb{Z}Q$-module A [3].

Lemma 2.2 ([13, 2.3]) *Let B be a pro-p $\mathbb{Z}_p[[Q]]$-submodule of A. Then $\Delta(A) = \Delta(B) \cup \Delta(A/B)$.*

We say that A is *m-tame* over $\mathbb{Z}_p[[Q]]$ (or is m-tame as a pro-p $\mathbb{Z}_p[[Q]]$-module) if whenever $v_1, \dots, v_m \in \Delta(A)$ satisfy $v_1 \cdots v_m = 1$, then $v_1 = \cdots = v_m = 1$. From Lemma 2.2 we see that if A is m-tame and B is a pro-p $\mathbb{Z}_p[[Q]]$-submodule of A, then B is also m-tame.

King showed [13, Corollary G] that 2-tameness of A finitely characterizes presentation of any extension of A by Q. Using this he showed the following.

Proposition 2.3 ([13, Proposition H]) *Suppose that $p > 2$. Let Q_0 be a free abelian pro-p group on the set $\{s, t\}$ and let $A = \mathbb{F}_p[[Q_0]]/(s + s^{-1} + t + t^{-1} - 4)$. Then the split extension $A \rtimes Q_0$ is not of homological type* FP_2, i.e., *is not finitely presented.*

The classification of the metabelian pro-p groups of type FP_m is presented in the following theorem. The case $m = 2$ was done by King [11, Theorem C] and the case of a general natural number m was proved by Kochloukova [14].

Theorem 2.4 ([14, Theorem D]) *Suppose that $1 \to A \to G \to Q \to 1$ is an exact sequence of pro-p groups, where G is finitely generated, and A and Q are abelian, and consider A as a pro-p $\mathbb{Z}_p[[Q]]$-module via the action of Q induced by conjugation. Then the following are equivalent:*

(i) *G is of type FP_m over $\mathbb{Z}_p$;*

(ii) *the completed m-th exterior power $\widehat{\bigwedge}{}^m_{\mathbb{Z}_p}(A)$ of A is a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module via the diagonal Q-action;*

(iii) *the completed m-th tensor power $\widehat{\bigotimes}{}^m_{\mathbb{Z}_p} A$ of A is a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module via the diagonal Q-action;*

(iv) *the completed m-th symmetric tensor power $\widehat{S}^m_{\mathbb{Z}_p}(A)$ of A is a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module via the diagonal Q-action;*

(v) *A is m-tame over $\mathbb{Z}_p[[Q]]$.*

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Completed tensor powers, completed symmetric powers, and completed exterior powers of pro-p modules A over pro-p rings can be defined by the appropriate universal properties or can be constructed by taking inverse limits of the tensor, symmetric, and symmetric abstract powers of the finite p-quotients of A (for more properties on completed tensor product see [22, §5.5]). Note that by [6, Lemma 1.1] if k is a pro-p ring, V is a right pro-p k-module, and W is a left pro-p k-module, then there is a natural isomorphism $V \otimes_k W \simeq V \widehat{\otimes}_k W$ provided either V or W is a finitely presented pro-p k-module.

3 Pro-p Subgroups of G With $G/H \simeq \mathbb{Z}_p$

Theorem 3.1 *Let G be a metabelian pro-p group of type FP_{2m}, where $m \geq 1$ is an integer. If H is a normal pro-p subgroup of G such that $G/H \cong \mathbb{Z}_p$, then H is of type FP_m.*

Proof Let A be an abelian normal pro-p subgroup of G such that the quotient $Q = G/A$ is abelian. Set $A_0 = H \cap A$, $Q_0 = H/A_0$ and note that Q_0 is a pro-p subgroup of Q. Thus there is a short exact sequence $A_0 \to H \to Q_0$ of pro-p groups with A_0 and Q_0 abelian. Observe that A_0 is normal in G, hence A_0 is a pro-p $\mathbb{Z}_p[[Q]]$-submodule of A. Since $G/H \simeq \mathbb{Z}_p$, there are two cases:

- $[A:A_0] < \infty$ and $Q/Q_0 \simeq \mathbb{Z}_p$,
- $A/A_0 \simeq \mathbb{Z}_p$ and $[Q:Q_0] < \infty$.

In the first case, since G is of type FP_{2m}, by Theorem 2.4 we have that A is $2m$-tame over $\mathbb{Z}_p[[Q]]$. Hence, by Lemma 2.2, A_0 is $2m$-tame over $\mathbb{Z}_p[[Q]]$ and so, using again Theorem 2.4, $\widehat{\otimes}^{2m}_{\mathbb{Z}_p} A_0$ is a finitely generated $\mathbb{Z}_p[[Q]]$-module via the diagonal Q-action. Let $B = \widehat{\otimes}^{m}_{\mathbb{Z}_p} A_0$ and consider $\widetilde{G} = B \rtimes Q$, where Q acts diagonally on B. Since $B \widehat{\otimes}_{\mathbb{Z}_p} B$ is finitely generated as a $\mathbb{Z}_p[[Q]]$-module via the diagonal action, by Theorem 2.4 we obtain that $\widetilde{G}$ is of type FP_2. Now note that $\widetilde{H} := B \rtimes Q_0$ is a normal pro-p subgroup of $\widetilde{G}$ such that $\widetilde{G}/\widetilde{H} \cong \mathbb{Z}_p$. Then, by Wilson's result [24, Corollary A, (iii)], $\widetilde{H}$ is finitely generated and so $B = \widehat{\otimes}^{m}_{\mathbb{Z}_p} A_0$ is finitely generated as a $\mathbb{Z}_p[[Q_0]]$-module. By Theorem 2.4 this implies that A_0 is m-tame as a $\mathbb{Z}_p[[Q_0]]$-module and so, by Theorem 2.4 again, any extension of A_0 by Q_0 is of type FP_m. In particular, H is of type FP_m as required.

In the second case, let H_0 be the preimage of Q_0 in G, so there is a short exact sequence of groups $A \to H_0 \to Q_0$. Thus H_0 has finite index in G and so is of type FP_{2m}. Then by Theorem 2.4 we have that A is $2m$-tame as a $\mathbb{Z}_p[[Q_0]]$-module. Since A_0 is $\mathbb{Z}_p[[Q_0]]$-submodule of A, by Lemma 2.2, A_0 is also $2m$-tame as a$\mathbb{Z}_p[[Q_0]]$-module. Then by Theorem 2.4, H is FP_{2m}, hence is FP_m. ■

Proof of Theorem A Let $C \to G \to G/C$ be a central extension with G/C metabelian. By [17, Corollary 3.5], C is a finitely generated abelian pro-p group, hence $H \cap C$ is a finitely generated abelian pro-p group, hence of type FP_∞. Consider the short exact sequence of pro-p groups $C_0 \to H \to H/C_0$, where $C_0 = H \cap C$. Since C_0 is of type FP_∞ we have that H is of type FP_m if and only if H/C_0 is of type FP_m (the abstract case is proved in [2], the pro-p case is [12, Theorem 2]). Note that H/C_0

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is a normal subgroup of the metabelian pro-p group G/C. Furthermore, since C is a finitely generated abelian pro-p group and G is a pro-p group of type FP_{2m}, the quotient group G/C has type FP_{2m}. Finally $(G/C)/(H/C_0) \simeq G/HC$ is a quotient of $G/H \simeq \mathbb{Z}_p$, hence is either $\mathbb{Z}_p$ or finite. In the first case we can apply Theorem 3.1 to deduce that H/C_0 is of type FP_m. In the second case H/C_0 has finite index in G/C, hence has the same homological type as G/C, *i.e.*, H/C_0 is FP_{2m}, so is FP_m. ∎

Theorem 3.2 *Let $p > 2$ be a prime number. Let Q_0 be the free abelian pro-p group on the set $\{s,t\}$ and $k = \mathbb{F}_p$ or $k = \mathbb{Z}_p$. Let $A = k[[Q_0]]/(s + s^{-1} + t + t^{-1} - 4)$ and $Q = \overline{\langle s,t,y\rangle} \simeq \mathbb{Z}_p^3$ generated by s, t, and y, where y acts on A (via conjugation) by multiplication with $(s+s^{-1})/2$. Then the split extension $A \rtimes Q$ is of type* FP_3.

Proof Observe that $(s+s^{-1})/2$ is not an element of the unique maximal ideal of the local ring $k[[Q_0]]$, hence is invertible in $k[[Q_0]]$. Thus the pro-p group $A \rtimes Q$ is well defined.

If $k = \mathbb{Z}_p$ note that by Theorem 2.4 $A \rtimes Q$ is FP_3 if and only if $V = \widehat{\otimes}^3_{\mathbb{Z}_p} A$ is finitely generated as a $\mathbb{Z}_p[[Q]]$-module via the diagonal Q-action. Since $\mathbb{Z}_p[[Q]]$ is a local ring, V is finitely generated as a $\mathbb{Z}_p[[Q]]$-module if and only if V/pV is finitely generated as a $\mathbb{F}_p[[Q]]$-module (and these two conditions are equivalent to $V \widehat{\otimes}_{\mathbb{Z}_p[[Q]]} \mathbb{F}_p$ is finite). Finally since $V/pV \simeq \widehat{\otimes}^3_{\mathbb{Z}_p}(A/pA)$, we reduce to the case where $k = \mathbb{F}_p$. Thus henceforth we can assume that $k = \mathbb{F}_p$ and to prove the theorem it is enough to show that $(\widehat{\otimes}^3_{\mathbb{F}_p} A)\widehat{\otimes}_{\mathbb{F}_p[[Q]]}\mathbb{F}_p$ has finite dimension over $\mathbb{F}_p$.

Since Q_0 is a free abelian pro-p group on the set $\{s,t\}$, we have that $\mathbb{F}_p[[Q_0]]$ is isomorphic to the formal power series algebra $\mathbb{F}_p[[S,T]]$ over $\mathbb{F}_p$ in the commutative indeterminates S, T, where $S = s-1$ and $T = t-1$. Thus, $(A\widehat{\otimes}_{\mathbb{Z}_p}A\widehat{\otimes}_{\mathbb{Z}_p}A)\widehat{\otimes}_{\mathbb{F}_p[[Q]]}\mathbb{F}_p$ is isomorphic to $B = \mathbb{F}_p[[S_1,S_2,S_3,T_1,T_2,T_3]]/L$, where

$$L = (s_1s_2s_3 - 1, t_1t_2t_3 - 1, y_1y_2y_3 - 1, s_i + s_i^{-1} + t_i + t_i^{-1} - 4 \mid 1 \le i \le 3),$$

$S_i = s_i - 1$, $T_i = t_i - 1$, and $y_i = (s_i + s_i^{-1})/2$ for $1 \le i \le 3$. Thus to prove that B is finite, it is enough to show that the images of s_i, t_i in B are algebraic over $\mathbb{F}_p$, for $1 \le i \le 3$.

Define $\alpha_i := s_i + s_i^{-1}$ and $\beta_i := t_i + t_i^{-1}$, for $1 \le i \le 3$. So $y_i = \frac{\alpha_i}{2}$, $1 \le i \le 3$. Henceforth, for $a, b \in \mathbb{F}_p[[S_1,S_2,S_3,T_1,T_2,T_3]]$, we write $a \equiv b$ for $a - b \in L$, *i.e.*, the images of a and b in B are the same. Since $y_1y_2y_3 \equiv 1$, we get $\alpha_1\alpha_2\alpha_3 \equiv 8$. Moreover, $\alpha_i + \beta_i \equiv 4$ for $1 \le i \le 3$. Thus

$$(3.1)\quad 8 \equiv \alpha_1\alpha_2\alpha_3 \equiv \prod_{i=1}^{3}(s_i + s_i^{-1}) = s_1s_2s_3 + \frac{1}{s_1s_2s_3} + \frac{s_1s_2}{s_3} + \frac{s_1s_3}{s_2} + \frac{s_2s_3}{s_1} + \frac{s_1}{s_2s_3} + \frac{s_2}{s_1s_3} + \frac{s_3}{s_1s_2}.$$

Since $s_1s_2s_3 \equiv 1 \equiv s_1^{-1}s_2^{-1}s_3^{-1}$, $\frac{s_1s_2}{s_3} + \frac{s_1s_3}{s_2} + \frac{s_2s_3}{s_1} \equiv s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2$,

$$\frac{s_1}{s_2s_3} + \frac{s_2}{s_1s_3} + \frac{s_3}{s_1s_2} = \frac{s_1^2 + s_2^2 + s_3^2}{s_1s_2s_3} \equiv s_1^2 + s_2^2 + s_3^2,$$

and by (3.1), we get

$$s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2 + s_1^2 + s_2^2 + s_3^2 \equiv \frac{s_1s_2}{s_3} + \frac{s_1s_3}{s_2} + \frac{s_2s_3}{s_1} + \frac{s_1}{s_2s_3} + \frac{s_2}{s_1s_3} + \frac{s_3}{s_1s_2}$$
$$\equiv 8 - s_1s_2s_3 - \frac{1}{s_1s_2s_3} \equiv 8 - 1 - 1 = 6.$$

Then, since $s_1s_2s_3 \equiv 1$,

$$\sum_{i=1}^{3}\alpha_i^2 = \sum_{i=1}^{3}(s_i + \frac{1}{s_i})^2 = \sum_{i=1}^{3}s_i^2 + \sum_{i=1}^{3}\frac{1}{s_i^2} + 6 \tag{3.2}$$
$$= s_1^2 + s_2^2 + s_3^2 + \frac{s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2}{s_1^2s_2^2s_3^2} + 6$$
$$\equiv s_1^2 + s_2^2 + s_3^2 + s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2 + 6 \equiv 12.$$

Now since $\alpha_i + \beta_i \equiv 4$ and $\alpha_1\alpha_2\alpha_3 \equiv 8$, we have by (3.2)

$$\prod_{i=1}^{3}(4 - \beta_i) \equiv \alpha_1\alpha_2\alpha_3 \equiv 8 \quad \text{and} \quad \sum_{i=1}^{3}(4 - \beta_i)^2 \equiv \sum_{i=1}^{3}\alpha_i^2 \equiv 12.$$

Developing the left side in the above equations, we obtain

$$16(\beta_1 + \beta_2 + \beta_3) - 4(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3) + \beta_1\beta_2\beta_3 \equiv 56 \tag{3.3}$$

and

$$8(\beta_1 + \beta_2 + \beta_3) \equiv \beta_1^2 + \beta_2^2 + \beta_3^2 + 36. \tag{3.4}$$

Using that $t_1t_2t_3 \equiv 1$, we will rewrite equations (3.3) and (3.4) in terms of t_1, t_2, and t_3. For this, denote $a := \sum_{i=1}^{3} t_i$ and $b := t_1t_2 + t_1t_3 + t_2t_3 \equiv \sum_{i=1}^{3}\frac{1}{t_i}$. Note that since $t_1t_2t_3 \equiv 1$, $\sum_{i\neq j\neq k\neq i} t_i^2t_j^2t_k \equiv \sum_{i\neq j} t_it_j$. Thus,

$$\beta_1 + \beta_2 + \beta_3 = \sum_{i=1}^{3}(t_i + t_i^{-1}) \equiv a + b,$$

$$\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 = \sum_{i\neq j}\Big(t_i + \frac{1}{t_i}\Big)\Big(t_j + \frac{1}{t_j}\Big) = \sum_{i\neq j\neq k\neq i}\frac{(t_i^2+1)(t_j^2+1)}{t_it_jt_k}t_k$$
$$\equiv \sum(t_i^2t_j^2t_k + t_k + t_i^2t_k + t_j^2t_k)$$
$$\equiv \sum_{i\neq j}t_it_j + \sum_{k=1}^{3}t_k + (\sum_{i\neq j}t_it_j)(\sum_{i=1}^{3}t_k) - 3t_1t_2t_3$$
$$\equiv b + a + ba - 3,$$

$$\beta_1^2 + \beta_2^2 + \beta_3^2 = (\sum_{i=1}^{3}\beta_i)^2 - 2(\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)$$
$$\equiv (a+b)^2 - 2(b + a + ab - 3) = a^2 + b^2 - 2a - 2b + 6,$$

and

$$\beta_1\beta_2\beta_3 = \prod_{i=1}^{3}\Big(t_i + \frac{1}{t_i}\Big) \equiv \prod_{i=1}^{3}(t_i^2 + 1) \tag{3.5}$$
$$\equiv t_1^2t_2^2t_3^2 + 1 + t_1^2 + t_2^2 + t_3^2 + t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2.$$

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But

$$\sum_{i=1}^{3} t_i^2 = (\sum_{i=1}^{3} t_i)^2 - 2(t_1t_2 + t_1t_3 + t_2t_3) = a^2 - 2b \tag{3.6}$$

and so

$$\Big(\sum_{i=1}^{3} t_i^2\Big)\Big(\sum_{i=1}^{3} t_i^2\Big) = 2(t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2) + \sum_{i=1}^{3} t_i^4. \tag{3.7}$$

Note that, since t_i is a root of the polynomial $(x-t_1)(x-t_2)(x-t_3) = x^3 - ax^2 + bx - 1$, we have that $t_i^3 - at_i^2 + bt_i - 1 = 0$ and $t_i^4 - at_i^3 + bt_i^2 - t_i = 0$. Thus

$$\begin{aligned}\sum_{i=1}^{3} t_i^4 &= a\sum t_i^3 - b\sum t_i^2 + \sum t_i\\ &= a(a\sum t_i^2 - b\sum t_i + 3) - b\sum t_i^2 + \sum t_i\\ &= (a^2 - b)\sum t_i^2 + (1 - ab)\sum t_i + 3a\\ &\equiv (a^2 - b)(a^2 - 2b) - a^2b + 4a\end{aligned}$$

and so by (3.6) and (3.7)

$$\begin{aligned}2(t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2) &= (\sum t_i^2)^2 - \sum t_i^4\\ &\equiv (a^2 - 2b)(a^2 - 2b) - [(a^2 - b)(a^2 - 2b) - a^2b + 4a] = 2(b^2 - 2a).\end{aligned} \tag{3.8}$$

Therefore by (3.5), (3.6), and (3.8) $\beta_1\beta_2\beta_3 \equiv 2 + a^2 - 2b + b^2 - 2a$. Thus, in terms of a, b, equations (3.3) and (3.4) are, respectively,

$$a^2 + b^2 - 4ab + 10a + 10b \equiv 42 \quad \text{and} \quad a^2 + b^2 - 10a - 10b \equiv -42, \tag{3.9}$$

from which, by summing the above equations, we conclude that $2a^2 + 2b^2 - 4ab \equiv 0$, that is, $2(a-b)^2 \equiv 0$, so $a \equiv b$. Substituting in (3.9), we obtain $2a^2 - 20a + 42 \equiv 0$. Thus a and b are also algebraic over $\mathbb{F}_p$, from which we get t_1, t_2, t_3 algebraic over $\mathbb{F}_p$. This implies $\beta_i = t_i + t_i^{-1}$ and so $\alpha_i \equiv 4 - \beta_i$, for $i = 1, 2, 3$, are algebraic over $\mathbb{F}_p$. Since $\alpha_i = s_i + s_i^{-1}$ is algebraic over $\mathbb{F}_p$ and $(x - s_i)(x - s_i^{-1}) = x^2 - \alpha_i x + 1$, $1 \le i \le 3$, we also have that s_1, s_2, s_3 are algebraic over $\mathbb{F}_p$. ∎

The following corollary completes the proof of Proposition B.

Corollary 3.3 *Let Q_0, A, and Q be as in Theorem 3.2. Then $G = A \rtimes Q$ is a metabelian pro-p group of type* FP_3 *with a pro-p normal subgroup $H = A \rtimes Q_0$ such that $G/H \cong \mathbb{Z}_p$ and H is not of type* FP_2.

Proof By Theorem 3.2 we have that $G = A \rtimes Q$ is of type FP_3 and by Proposition 2.3 $H = A \rtimes Q_0$ is not of type FP_2. ∎

Proof of Proposition D Recall that $p > 2$. We show first that it suffices to show that

$$\sup_{k\ge1} \mathrm{rk}(\widehat{\bigwedge}^2 A)\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p = \infty. \tag{3.10}$$

Indeed, consider the Lyndon–Hochschild–Serre spectral sequence

$$E_{i,j}^2 = H_i(Q^{p^k}, H_j(A, \mathbb{Z}_p)) \Rightarrow H_{i+j}(A \rtimes Q^{p^k}, \mathbb{Z}_p).$$

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Note that $E^2_{i,j}$ depends on k, but we do not put an extra index k to $E^2_{i,j}$ in order not to confuse the notation.

By definition

$$E^2_{0,2} = (\widehat{\wedge}^2 A)\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p,$$
$$E^2_{3,0} = H_3(Q^{p^k}, \mathbb{Z}_p) \simeq \widehat{\wedge}^3 \mathbb{Z}_p^3,$$
$$E^2_{2,1} = H_2(Q^{p^k}, A).$$

Note that since $A \rtimes Q$ is FP_3, it is FP_2, and by [4, Proposition A + Proposition B]

$$\sup_{k\geq 1} \dim_{\mathbb{Q}_p}(A \otimes_{\mathbb{Z}_p[[Q^{p^k}]]} \mathbb{Q}_p) < \infty.$$

Then, by Theorem 5.5 from the next section, $\sup_{k\geq1} \dim_{\mathbb{Q}_p} H_i(Q^{p^k}, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$, for all i. In particular, this holds for $i = 2$, and so $\sup_{k\geq1} \mathrm{rk}(E^2_{2,1}) < \infty$. Observe that by the description of $E^2_{3,0}$ above we have that $\sup_{k\geq1} \mathrm{rk}(E^2_{3,0}) < \infty$. Then, since all possible nonzero differentials that finish at $E^*_{0,2}$ should start at $E^2_{2,1}$ or $E^3_{3,0}$, we deduce that $\sup_{k\geq1} \mathrm{rk}(E^\infty_{0,2}) = \infty$ if and only if (3.10) holds. Suppose now $\sup_{k\geq1} \mathrm{rk}(E^\infty_{0,2}) = \infty$. Since $E^\infty_{0,2}$ is a subgroup of $H_2(A \rtimes Q^{p^k}, \mathbb{Z}_p)$, one has

$$\sup_{k\geq1} \mathrm{rk}(H_2(A \rtimes Q^{p^k}, \mathbb{Z}_p)) \geq \sup_{k\geq1} \mathrm{rk}(E^\infty_{0,2}) = \infty,$$

as required.

To prove (3.10)) consider the decomposition of $\mathbb{Z}_p[[Q]]$-modules $\widehat{\otimes}^2 A = V_1 \oplus V_2$, where the completed tensor product is over $\mathbb{Z}_p$, Q acts diagonally, and for $\theta\colon \widehat{\otimes}^2 A \to \widehat{\otimes}^2 A$ given by $\theta(a_1\widehat{\otimes}a_2) = a_2\widehat{\otimes}a_1$, we set

$$V_1 = \{v - \theta(v) \mid v \in \widehat{\otimes}^2 A\} \quad \text{and} \quad V_2 = \{v + \theta(v) \mid v \in \widehat{\otimes}^2 A\}.$$

Then V_1 is isomorphic to $\widehat{\wedge}^2 A$ via the canonical map $\widehat{\otimes}^2 A \to \widehat{\wedge}^2 A$, where the completed exterior product is over $\mathbb{Z}_p$.

Consider the epimorphism of pro-p rings

$$\rho\colon \widehat{\otimes}^2 A = \mathbb{Z}_p[[S_1, S_2, T_1, T_2]]/(s_1 + s_1^{-1} + t_1 + t_1^{-1} - 4, s_2 + s_2^{-1} + t_2 + t_2^{-1} - 4) \longrightarrow A$$

sending s_2 and s_1^{-1} to s^{-1} and sending t_2 and t_1^{-1} to t^{-1}. Note that the diagonal action of the generators of Q on $\widehat{\otimes}^2 A$ is given by multiplication by s_1s_2, t_1t_2, and $\frac{s_1+s_1^{-1}}{2}\,\frac{s_2+s_2^{-1}}{2}$. Then the map ρ induces an epimorphism of pro-p groups

$$\rho_k\colon (\widehat{\otimes}^2 A)\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p \longrightarrow A/((s + s^{-1})^{2p^k} - 2^{2p^k}) = W_k.$$

Note that

$$(\widehat{\otimes}^2 A)\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p = (V_1\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p) \oplus (V_2\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p),$$

and $s_1^i - s_2^i \in V_1 \subseteq \widehat{\otimes}^2 A$. Then $\alpha_i = \overline{s^i - s^{-i}} = \rho_k((s_1^i - s_2^i)\widehat{\otimes}1) \in \rho_k(V_1\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]}\mathbb{Z}_p)$, where $\overline{s^i - s^{-i}}$ denotes the image of $s^i - s^{-i}$ in A. Note that $\{\alpha_i\}_{1\leq i\leq 2p^k-1}$ generates a

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free $\mathbb{Z}_p$-submodule of W_k of rank $2p^k - 1$. Hence,

$$\operatorname{rk}(\widehat{\bigwedge}^2 A \widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]} \mathbb{Z}_p) \geq \operatorname{rk}(V_1 \widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^k}]]} \mathbb{Z}_p) \geq 2p^k - 1. \qquad \blacksquare$$

4 Some Homology Groups Classified up to Torsion

Recall that $H_i(G, V)$ denotes the pro-p homology of the pro-p group G with coefficients in a pro-p $\mathbb{Z}_p[[G]]$-module V. By definition, pro-p modules over pro-p rings are compact, hence $V \neq \mathbb{Q}_p$. Even in the case when A is an abelian pro-p group that is not torsion-free and $i \geq 3$, the homology $H_i(A, \mathbb{Z}_p)$ has a complicated structure, though there is a natural embedding $\widehat{\bigwedge}^i_{\mathbb{Z}_p} A \to H_i(A, \mathbb{Z}_p)$, as shown by King [12, Theorem B]. The following lemma shows that $H_i(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ can be easily described. As pointed out at the beginning of the paragraph, we cannot resolve the problem by moving $\mathbb{Q}_p$ inside the homology functor. Recall that $\widehat{\bigwedge}^i_{\mathbb{Z}_p} A$ denotes the completed exterior power of A over $\mathbb{Z}_p$.

Lemma 4.1 *Let A be an abelian pro-p group.*

(i) *$H_i(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (\widehat{\bigwedge}^i_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for all $i \geq 1$;*

(ii) *if Q is a finitely generated pro-p abelian group and A a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module, we have*

$$H_i(Q, H_j(A, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H_i(Q, \widehat{\bigwedge}^j_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

for $i \geq 0$, $j \geq 1$.

Proof (i) If B is an abelian, torsion-free, pro-p group, then $H_i(B, \mathbb{Z}_p) \cong \widehat{\bigwedge}^i_{\mathbb{Z}_p} B$ for all $i \geq 1$ [12, Theorem B]. So the lemma follows trivially for torsion-free abelian pro-p groups.

Now let $\operatorname{tor} A$ be the torsion pro-p subgroup of A. From the short exact sequence of abelian pro-p groups $\operatorname{tor} A \hookrightarrow A \xrightarrow{\alpha} M$, where $M = A/\operatorname{tor} A$, we obtain the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{i,j} = H_i(M, H_j(\operatorname{tor} A, \mathbb{Z}_p)) \Longrightarrow H_{i+j}(A, \mathbb{Z}_p).$$

Since, for $j \neq 0$, $H_j(\operatorname{tor} A, \mathbb{Z}_p)$ is torsion, we have $E^2_{i,j}$ is also torsion for $j \neq 0$. Also, the spectral sequence says that $H_n(A, \mathbb{Z}_p)$ has a filtration with factors isomorphic to each $E^\infty_{i,j}$ such that $i + j = n$, *i.e.*, there is a filtration of abelian pro-p groups

$$\Delta_{-1} = 0 \subseteq \Delta_0 \subseteq \cdots \subseteq \Delta_i \subseteq \Delta_{i+1} \subseteq \cdots \subseteq \Delta_n = H_n(A, \mathbb{Z}_p),$$

such that $\Delta_i/\Delta_{i-1} \simeq E^\infty_{i,n-i}$ for every $0 \leq i \leq n$. Moreover, $E^\infty_{i,j}$ is a subquotient of $E^2_{i,j}$, so it is also torsion for $j \neq 0$, so $E^\infty_{i,j} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ for $j \neq 0$. Therefore, since $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an exact functor, we obtain a filtration of $\mathbb{Q}_p$-vector spaces

$$V_{-1} = 0 \subseteq V_0 \subseteq \cdots \subseteq V_i \subseteq V_{i+1} \subseteq \cdots \subseteq V_n = H_n(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where $V_i = \Delta_i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Using again that $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an exact functor, we get that $V_i/V_{i-1} \simeq (\Delta_i/\Delta_{i-1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq E^\infty_{i,n-i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, for $0 \leq i \leq n$ and hence $V_i/V_{i-1} = 0$ for

$0 \le i \le n-1$ and so $0 = V_{-1} = V_0 = \cdots = V_{n-1}$. Then

$$H_n(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_n = V_n / V_{n-1} \cong E^{\infty}_{n,0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Moreover, from the differential map $d^r_{n,0}\colon E^r_{n,0} \to E^r_{n-r,r-1}$, since $E^r_{n-r,r-1}$ is torsion, we have that $E^r_{n,0}/E^{r+1}_{n,0} = E^r_{n,0}/\operatorname{Ker}(d^r_{n,0}) \simeq \operatorname{Im}(d^r_{n,0}) \subseteq E^r_{n-r,r-1}$ is torsion for all $r \ge 2$. So $E^2_{n,0}/E^{\infty}_{n,0}$ is torsion and, since $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is exact, we obtain

$$E^{\infty}_{n,0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong E^2_{n,0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Thus

$$\begin{aligned} H_n(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &\cong E^2_{n,0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &= H_n(M, H_0(\operatorname{tor} A, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_n(M, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \end{aligned}$$

and, since M is torsion-free, $H_n(M, \mathbb{Z}_p) \cong \widehat{\bigwedge}^n_{\mathbb{Z}_p} M$.

We claim that

$$(\widehat{\bigwedge}^n_{\mathbb{Z}_p} M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong (\widehat{\bigwedge}^n_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{4.1}$$

Consider the commutative diagram

$$\begin{array}{ccccc} \ker\gamma & \hookrightarrow & \widehat{\bigwedge}^n_{\mathbb{Z}_p} A & \xrightarrow{\gamma} & \widehat{\bigwedge}^n_{\mathbb{Z}_p} M \\ \uparrow & & \uparrow & & \uparrow \\ \ker\beta & \hookrightarrow & \widehat{\bigotimes}^n_{\mathbb{Z}_p} A & \xrightarrow{\beta} & \widehat{\bigotimes}^n_{\mathbb{Z}_p} M \end{array}$$

where the vertical maps are the canonical maps from completed tensor powers to completed exterior powers, $\beta = \widehat{\otimes}^n \alpha$ and recall that $\alpha\colon A \to M$ is the canonical projection. To prove (4.1) it is sufficient to show that $\ker\beta$ is torsion, since this implies that $\ker\gamma$ is torsion, hence $\ker\gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ and so (4.1) holds.

We show that $\ker\beta$ is torsion by induction on n. Consider the canonical epimorphisms

$$\varphi = \widehat{\otimes}^{n-1}\alpha\colon \widehat{\otimes}^{n-1}_{\mathbb{Z}_p} A \longrightarrow \widehat{\otimes}^{n-1}_{\mathbb{Z}_p} M,$$
$$\alpha\widehat{\otimes}\varphi = \widehat{\otimes}^n \alpha\colon \widehat{\otimes}^n_{\mathbb{Z}_p} A \longrightarrow \widehat{\otimes}^n_{\mathbb{Z}_p} M.$$

Since $\alpha\widehat{\otimes}\varphi$ is the composition $(1\widehat{\otimes}\varphi) \circ (\alpha\widehat{\otimes}1)$, we have that $\ker(\alpha\widehat{\otimes}\varphi)$ is the image of $(\ker\alpha\widehat{\otimes}_{\mathbb{Z}_p}(\widehat{\otimes}^{n-1}_{\mathbb{Z}_p} A)) \oplus (A\widehat{\otimes}_{\mathbb{Z}_p} \ker\varphi)$ in $A\widehat{\otimes}_{\mathbb{Z}_p}(\widehat{\otimes}^{n-1}_{\mathbb{Z}_p} A) = \widehat{\otimes}^n_{\mathbb{Z}_p} A$. By inductive hypothesis, $\ker\varphi$ is torsion and by construction $\ker\alpha = \operatorname{tor}(A)$ is torsion. So $(\ker\alpha\widehat{\otimes}_{\mathbb{Z}_p}(\widehat{\otimes}^{n-1}_{\mathbb{Z}_p} A)) \oplus (A\widehat{\otimes}_{\mathbb{Z}_p} \ker\varphi)$ is torsion. Thus $\ker(\alpha\widehat{\otimes}\varphi)$ is torsion. This finishes the induction step and so the proof of the claim.

(ii) The case $i = 0$ follows from (i), so we can assume from now on that $i \ge 1$.

Consider the spectral sequence $E^2_{i,j} = H_i(\operatorname{tor} A, H_j(A/\operatorname{tor} A, \mathbb{Z}_p))$ associated with the short exact sequence $0 \to \operatorname{tor}(A) \to A \to A/\operatorname{tor}(A) \to 0$. By the proof of (i) $H_j(A, \mathbb{Z}_p)$ has a filtration $0 = \Delta_{-1} \subseteq \Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_j = H_j(A, \mathbb{Z}_p)$ such that $\Delta_i/\Delta_{i-1} = E^{\infty}_{i,j-i}$, $E^{\infty}_{i,j-i}$ is torsion for $i \ne j$ and $E^{\infty}_{j,0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (\widehat{\bigwedge}^j_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Thus

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Δ_{j-1} is torsion and by the long exact sequence in homology applied for the short exact sequence $0 \to \Delta_{j-1} \to H_j(A, \mathbb{Z}_p) \to E^{\infty}_{j,0} \to 0$

$$\cdots \longrightarrow H_i(Q, \Delta_{j-1}) \xrightarrow{\alpha_i} H_i(Q, H_j(A, \mathbb{Z}_p)) \xrightarrow{\beta_i} H_i(Q, E^{\infty}_{j,0}) \xrightarrow{\partial_i} H_{i-1}(Q, \Delta_{j-1}) \longrightarrow \cdots$$

Then there is an exact complex

$$0 \longrightarrow H_i(Q, \Delta_{j-1})/\ker \alpha_i \longrightarrow H_i(Q, H_j(A, \mathbb{Z}_p)) \xrightarrow{\beta_i} H_i(Q, E^{\infty}_{j,0}) \longrightarrow \operatorname{im} \partial_i \to 0.$$

Since Δ_{j-1} is torsion, both $H_i(Q, \Delta_{j-1})$ and $H_{i-1}(Q, \Delta_{j-1})$ are torsion, hence

$$H_i(Q, \Delta_{j-1})/\ker \alpha_i$$

and $\operatorname{im} \partial_i$ are torsion. Hence the map β_i induces an isomorphism

$$H_i(Q, H_j(A, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H_i(Q, E^{\infty}_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{4.2}$$

On other hand $E^{\infty}_{j,0}$ is a subquotient of $E^2_{j,0}$ and

$$E^2_{j,0} \simeq H_j(A/\operatorname{tor}(A), \mathbb{Z}_p) \simeq \widehat{\bigwedge}^j_{\mathbb{Z}_p}(A/\operatorname{tor}(A)).$$

Furthermore, by the proof of Lemma 4.1 (i) $E^2_{j,0}/E^{\infty}_{j,0}$ is torsion. Note that the short exact sequence $0 \to E^{\infty}_{j,0} \to E^2_{j,0} \to E^2_{j,0}/E^{\infty}_{j,0} \to 0$ gives a long exact sequence in homology

$$\cdots \longrightarrow H_i(Q, E^{\infty}_{j,0}) \longrightarrow H_i(Q, E^2_{j,0}) \longrightarrow H_i(Q, E^2_{j,0}/E^{\infty}_{j,0}) \longrightarrow H_{i-1}(Q, E^{\infty}_{j,0}) \longrightarrow \cdots$$

Applying the exact functor $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, we obtain another long exact sequence

$$\cdots \longrightarrow H_i(Q, E^{\infty}_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p lra H_i(Q, E^2_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow H_i(Q, E^2_{j,0}/E^{\infty}_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow H_{i-1}(Q, E^{\infty}_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \cdots$$

Since $E^2_{j,0}/E^{\infty}_{j,0}$ is torsion, $H_i(Q, E^2_{j,0}/E^{\infty}_{j,0})$ is torsion, too; hence in the above long exact sequence $H_i(Q, E^2_{j,0}/E^{\infty}_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$ for every $i \geq 0$. Thus there is an isomorphism

$$H_i(Q, E^2_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H_i(Q, E^{\infty}_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{4.3}$$

By the proof of Lemma 4.1 (i) the epimorphism $\gamma\colon \widehat{\bigwedge}^j_{\mathbb{Z}_p} A \to \widehat{\bigwedge}^j_{\mathbb{Z}_p}(A/\operatorname{tor}(A))$ and the canonical isomorphism $\delta\colon \widehat{\bigwedge}^j_{\mathbb{Z}_p} A/\operatorname{tor}(A) \to H_j(A/\operatorname{tor}(A), \mathbb{Z}_p) = E^2_{j,0}$ induce isomorphisms

$$(\widehat{\bigwedge}^j_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (\widehat{\bigwedge}^j_{\mathbb{Z}_p} A/\operatorname{tor}(A)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq E^2_{j,0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Thus, the map $\mu = \delta\gamma\colon \widehat{\bigwedge}^j_{\mathbb{Z}_p} A \to E^2_{j,0}$ has torsion kernel and torsion co-kernel. Thus μ induces an isomorphism

$$H_i(Q, \widehat{\bigwedge}^j_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H_i(Q, E^2_{j,0}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \text{ for } i \geq 0, j \geq 1. \tag{4.4}$$

Finally (4.2), (4.3), and (4.4) complete the proof. ∎

Lemma 4.2 *Let G be a pro-p group, G_0 a pro-p open, normal, subgroup in G, and V a pro-p $\mathbb{Z}_p[[G]]$-module. Then $H_n(G,V)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \simeq H_0(G/G_0, H_n(G_0,V))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.*

Proof Consider the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{i,n-i} = H_i(G/G_0, H_{n-i}(G_0,V))$$

converging to $H_n(G,V)$. Note that every open subgroup in a pro-p group has a p-power index. In particular, G/G_0 is a finite p-group. Thus $H_i(G/G_0,-)$ is torsion for every $i>0$, hence $E^2_{i,n-i}$ is torsion for $i>0$. Then $E^\infty_{i,n-i}$ is torsion for every $i>0$. By the convergence of the spectral sequence, $H_n(G,V)$ has a filtration with quotients $E^\infty_{i,n-i}$ for $0\le i\le n$. Since $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is an exact functor, $H_n(G,V)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ has filtration with quotients $E^\infty_{i,n-i}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, but $E^\infty_{i,n-i}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = 0$ for $i>0$. Hence

$$H_n(G,V)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \simeq E^\infty_{0,n}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p. \tag{4.5}$$

Note that all differentials that start at $E^s_{0,n}$ finish in the second quadrant, hence are zero. And all differentials that end at $E^s_{0,n}$ start at $E^s_{s,n+1-s}$, and $E^s_{s,n+1-s}$ is torsion, hence $E^{s+1}_{0,n} = E^s_{0,n}/\operatorname{im}(d^s_{s,n+1-s})$ and $E^s_{0,n}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \simeq E^{s+1}_{0,n}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ for every $s\ge 2$. Thus,

$$H_0(G/G_0, H_n(G_0,V))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = E^2_{0,n}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \simeq E^\infty_{0,n}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p. \tag{4.6}$$

Finally (4.5) and (4.6) complete the proof. ∎

5 Growth of Homology: Proof of Theorem C in the Metabelian Case

We observe that a version of Theorem C works for a discrete, finitely presented, centre-by-metabelian group G, since by a result of Groves [9] the central part of G is finitely generated and for discrete metabelian groups Theorem C holds [15].

Lemma 5.1 *Let Q be a finitely generated abelian pro-p group and B a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module such that $B\widehat{\otimes}_{\mathbb{Z}_p}B$ is a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module via the diagonal Q-action. Then $\sup_{M\in\mathcal{A}}\dim_{\mathbb{Q}_p} B\otimes_{\mathbb{Z}_p[[M]]}\mathbb{Q}_p<\infty$, where $\mathcal{A}$ is the set of all subgroups of p-power index in $G = B\rtimes Q$ and we view B as a $\mathbb{Z}_p[[G]]$-module via the canonical epimorphism $G\to Q$.*

Proof By Theorem 2.4, since $B\widehat{\otimes}B$ is a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module via the diagonal action, $G = B\rtimes Q$ is finitely presented. By [4, Proposition A, Theorem C] $\sup_{M\in\mathcal{A}}\operatorname{rk} H_1(M,\mathbb{Z}_p)<\infty$ and $\sup_{M\in\mathcal{A}}\dim_{\mathbb{Q}_p} B\otimes_{\mathbb{Z}_p[[M]]}\mathbb{Q}_p<\infty$. ∎

In the following lemma, $\widehat{\operatorname{Tor}}_j^A$ denotes the derived functor of $\widehat{\otimes}_A$ in the category of pro-p A-modules [22, §6.1].

Lemma 5.2 *Let Q be the abelian pro-p group $\mathbb{Z}_p^n = \overline{\langle q_1,\dots,q_n\rangle}$, $A = \mathbb{Z}_p[[Q]]/I$ a pro-p ring, and for every positive integer m, denote by A_m the closed ideal of A generated by the image of $\{q_1^{p^m}-1,\dots,q_n^{p^m}-1\}$ in A. Suppose that*

(i) $\sup_{m\ge1}\dim_{\mathbb{Q}_p}(A/A_m)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p<\infty$;

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(ii) *for* $m \geq 1$, V_m *is a finitely generated, right pro-*p A/A_m*-module and* W_m *is a finitely generated, left pro-*p A/A_m*-module such that*

$$a = \sup_{m\geq 1} \dim_{\mathbb{Q}_p}(V_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) < \infty \quad \text{and} \quad b = \sup_{m\geq 1} \dim_{\mathbb{Q}_p}(W_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) < \infty.$$

Then for every $j \geq 0$,

$$\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_m, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty \tag{5.1}$$

and

$$\bigcup_{m\geq 1} \mathcal{C}_m \text{ is finite,} \tag{5.2}$$

where $\mathcal{C}_m$ *is the set of isomorphism classes of abstract simple* $(A/A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$*-modules.*

Proof 1. We first show (5.2). Observe that A/A_m is a quotient of $S_m = \mathbb{Z}_p[Q/Q^{p^m}]$ and S_m is a finitely generated $\mathbb{Z}_p$-module. Thus $R_m = (A/A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\mathbb{Q}_p$-algebra that is finite-dimensional over $\mathbb{Q}_p$.

Let D be an abstract simple R_m-module. Then D is a simple quotient of R_m, *i.e.*, a finite field extension of $\mathbb{Q}_p$, thus a local field. Note that D is generated by the image $\overline{Q}$ of Q in D and $\mathbb{Q}_p$ and $\overline{Q}$ is a finite abelian p-group. Any finite subgroup in the multiplicative group of a field is cyclic, hence $\overline{Q} = \langle \alpha \rangle$ and α is a primitive p^s-root of 1 for some $s \leq m$. Then the minimal polynomial of α over $\mathbb{Q}_p$ is $(x^{p^s} - 1)/(x^{p^{s-1}} - 1)$ and so $\dim_{\mathbb{Q}_p} D = p^s - p^{s-1} \leq \sup_{m\geq 1} \dim_{\mathbb{Q}_p}(A/A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$. Then there exists $s_0 \geq 0$ such that $s \leq s_0$ for every D, and D is a simple quotient of $D_0 = \mathbb{Q}_p[x]/(x^{p^{s_0}} - 1)$. Finally since D_0 is finite-dimensional over $\mathbb{Q}_p$, we deduce that D_0 is an Artinian $\mathbb{Q}_p$-algebra, hence has only finitely many maximal ideals. This completes the proof of (5.2).

2. Consider filtrations of pro-p A/A_m-modules

$$0 = F_{0,m} \subset F_{1,m} \subset \cdots \subset F_{t-1,m} \subset F_{t,m} = V_m,$$
$$0 = E_{0,m} \subset E_{1,m} \subset \cdots \subset E_{t'-1,m} \subset E_{t',m} = W_m,$$

of V_m and W_m, respectively, such that the quotients $V_{s,m} := F_{s,m}/F_{s-1,m}$ and $W_{s',m} := E_{s',m}/E_{s'-1,m}$ are non-trivial and each one is either finite or, after tensoring with $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, is a simple abstract $R_m = (A/A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module. This is possible since we can assume that $V_{s,m}$ and $W_{s',m}$ are simple A/A_m-modules. This implies that $V_{s,m}$ and $W_{s',m}$ are both cyclic A/A_m-modules, so can be considered as ring quotients of A/A_m and by the simplicity condition both $V_{s,m}$ and $W_{s',m}$ are fields. Furthermore, since A/A_m is a finitely generated $\mathbb{Z}_p$-module, we deduce that $V_{s,m}$ and $W_{s',m}$ are finitely generated $\mathbb{Z}_p$-modules, *i.e.*, finitely generated abelian pro-p groups. If $V_{s,m}$ (resp. $W_{s',m}$) is an infinite field, then this infinite field contains $\mathbb{Q}_p$, hence $V_{s,m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq V_{s,m}$ (resp. $W_{s',m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq W_{s',m}$).

Now we will need to combine abstract and pro-p Tor functors. Recall that $\widehat{\mathrm{Tor}}_j^A$ denotes the pro-p Tor functor (as mentioned before Lemma 5.2) and Tor_j^A denotes the abstract Tor functor, *i.e.*, the derived functor of the abstract tensor product $\otimes_A$ in the category of abstract A-modules.

Note that the short exact sequence $0 \to F_{s-1,m} \to F_{s,m} \to V_{s,m} \to 0$ gives rise to a long exact sequence

$$\ldots \longrightarrow \widehat{\mathrm{Tor}}_j^A(F_{s-1,m}, W_m) \longrightarrow \widehat{\mathrm{Tor}}_j^A(F_{s,m}, W_m) \longrightarrow \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \longrightarrow \cdots.$$

Then, after applying the exact functor $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, we obtain a long exact sequence

$$\cdots \longrightarrow \widehat{\mathrm{Tor}}_j^A(F_{s-1,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \widehat{\mathrm{Tor}}_j^A(F_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \cdots,$$

hence for every $j \geq 0$,

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(F_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(F_{s-1,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p + \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Then by induction on s we obtain

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(F_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \sum_{1\leq j\leq s} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{j,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In particular, for $s = t$, we get that

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_m, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \sum_{1\leq s\leq t} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{5.3}$$

Note that if $V_{s,m}$ is finite, then $\widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m)$ is torsion, hence

$$\widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0.$$

Then from (5.3) we obtain

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_m, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq a_0 \cdot \max_{1\leq s\leq t} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \tag{5.4}$$

where a_0 is the number of the factors $V_{s,m}$ such that $V_{s,m}$ is infinite. Thus

$$a_0 \leq \dim_{\mathbb{Q}_p}(V_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \leq a.$$

In a similar way, using the long exact sequence in $\widehat{\mathrm{Tor}}_*^A$, we can show that

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq b_0 \cdot \max_{1\leq s'\leq t'} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_{s',m}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \tag{5.5}$$

where b_0 is the number of the factors $W_{s',m}$ such that $W_{s',m}$ is infinite. Thus

$$b_0 \leq \dim_{\mathbb{Q}_p}(W_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \leq b.$$

By (5.4) and (5.5) we obtain that

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_m, W_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq ab \cdot \max_{1\leq s\leq t} \max_{1\leq s'\leq t'} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_{s',m}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

We claim that

$$\widehat{\mathrm{Tor}}_j^A(V_{s,m}, W_{s',m}) \simeq \mathrm{Tor}_j^A(V_{s,m}, W_{s',m}). \tag{5.6}$$

Indeed, since $V_{s,m}$ and $W_{s',m}$ are finitely generated $\mathbb{Z}_p$-modules and A is a Noetherian ring (both as a pro-p and an abstract ring), we deduce that there are free resolutions of $V_{s,m}$ and $W_{s',m}$ as abstract A-modules with all free modules finitely generated. All finitely generated free A-modules are free pro-p A-modules and all differentials in the above free abstract resolutions are automatically continuous since the modules are finitely generated [25, Lemma 7.2.2]. Then using these resolutions to compute both $\widehat{\mathrm{Tor}}_*^A$ and Tor_*^A and the fact that for every $k \geq 1$, we have $A^k \widehat{\otimes}_A - \simeq A^k \otimes_A -$ and $-\widehat{\otimes}_A A^k \simeq - \otimes_A A^k$, imply (5.6).

Since $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an exact functor, we deduce that

$$\mathrm{Tor}_j^A(V_{s,m}, W_{s',m}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Tor}_j^A(V_{s,m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, W_{s',m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

Finally, since $V_{s,m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $W_{s',m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are abstract simple R_m-modules, by (5.2) there are only finitely many possibilities for $V_{s,m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $W_{s',m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then

$$\sup_{s,s',m} \dim_{\mathbb{Q}_p} \mathrm{Tor}_j^A(V_{s,m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, W_{s',m} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) < \infty.$$

This completes the proof. ∎

Recall that for a pro-p group Q and a pro-p $\mathbb{Z}_p[[Q]]$-module A,

$$H_i(Q, A) = \widehat{\mathrm{Tor}}_i^{\mathbb{Z}_p[[Q]]}(A, \mathbb{Z}_p).$$

By definition Q^{p^j} is the pro-p subgroup of Q generated by $\{q^{p^j} \mid q \in Q\}$.

Lemma 5.3 *Let $Q = \mathbb{Z}_p^n$ and $0 \to A_1 \to A \to A_2 \to 0$ be a short exact sequence of pro-p $\mathbb{Z}_p[[Q]]$-modules such that $\sup_{m\geq 1} \dim_{\mathbb{Q}_p} H_i(Q^{p^m}, A_j) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$ for $j = 1, 2$. Then $\sup_{m\geq 1} \dim_{\mathbb{Q}_p} H_i(Q^{p^m}, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$.*

Proof By the long exact sequence in homology

$$\cdots \longrightarrow H_i(Q^{p^m}, A_1) \longrightarrow H_i(Q^{p^m}, A) \longrightarrow H_i(Q^{p^m}, A_2) \longrightarrow \cdots$$

we get that

$$\dim_{\mathbb{Q}_p} H_i(Q^{p^m}, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \dim_{\mathbb{Q}_p} H_i(Q^{p^m}, A_1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p + \dim_{\mathbb{Q}_p} H_i(Q^{p^m}, A_2) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$ ∎

Let $Q = \mathbb{Z}_p^n = \overline{\langle q_1, \dots, q_n \rangle}$ and so $Q^{p^m} = \overline{\langle q_1^{p^m}, \dots, q_n^{p^m} \rangle}$. There is a pro-$p$ version of the Koszul complex in the abstract case [23, Corollary 4.5.5]. It is obtained from the abstract version after applying the functor $\mathbb{Z}_p[[Q]] \otimes_{\mathbb{Z}[Q_0]}$, where $Q_0 = \mathbb{Z}^n$ is an abstract group with pro-p completion Q. Thus the pro-p version of the Koszul complex is

$$\mathcal{P}_m: \quad \cdots \longrightarrow P_{k,m} \xrightarrow{\partial_{k,m}} P_{k-1,m} \longrightarrow \cdots \longrightarrow P_{1,m} \xrightarrow{\partial_{1,m}} P_{0,m} \xrightarrow{\partial_{0,m}} \mathbb{Z}_p \longrightarrow 0,$$

where $P_{0,m} = \mathbb{Z}_p[[Q^{p^m}]]$, $P_{k,m} = \bigoplus_{1\leq i_1 < \cdots < i_k \leq n} \mathbb{Z}_p[[Q^{p^m}]] e_{i_1} \cdots e_{i_k}$ for $k \geq 1$ and $\partial_{0,m}$ is the augmentation map. The differential $\partial_{k,m}: P_{k,m} \to P_{k-1,m}$, where $k \geq 1$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, is given by

$$\partial_{k,m}(e_{i_1} \cdots e_{i_k}) = \sum_{1\leq j\leq k} (-1)^j (q_{i_j}^{p^m} - 1) e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_k},$$

where the hat in $e_{i_1}\cdots\widehat{e_{i_j}}\cdots e_{i_k}$ means that the term e_{i_j} is erased in the product. Let A be a right pro-p $\mathbb{Z}_p[[Q^{p^m}]]$-module. Applying the functor $(A\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^m}]]}-)$ to the complex $\mathcal{P}_m$, we obtain the complex

$$\mathcal{S}_m := A\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^m}]]}\mathcal{P}_m: \quad \cdots \longrightarrow S_{k,m} \xrightarrow{\widehat{\partial}_{k,m}} S_{k-1,m} \longrightarrow \cdots$$
$$\cdots \longrightarrow S_{0,m} \xrightarrow{\widehat{\partial}_{0,m}} A\widehat{\otimes}_{\mathbb{Z}_p[[Q^{p^m}]]}\mathbb{Z}_p \longrightarrow 0,$$

where $\widehat{\partial}_{k,m} := \mathrm{id}_A\widehat{\otimes}\partial_{k,m}$, $S_{0,m} = A$, and $S_{k,m} = \bigoplus_{1\le i_1<\cdots<i_k\le n} Ae_{i_1}\cdots e_{i_k}$ for $k \ge 1$. Note that since all modules $P_{i,m}$ in $\mathcal{P}_m$ are finitely presented pro-p $\mathbb{Z}_p[[Q]]$-modules, we have that $A\widehat{\otimes}_{\mathbb{Z}_p[[Q]]}P_{i,m} \simeq A\otimes_{\mathbb{Z}_p[[Q]]}P_{i,m}$.

Lemma 5.4 *Let $Q = \mathbb{Z}_p^n = \overline{\langle q_1,\dots,q_n\rangle}$, $A = \mathbb{Z}_p[[Q]]/I$ for some ideal I in $\mathbb{Z}_p[[Q]]$, and let A_m be the ideal of A generated by $q_1^{p^m}-1,\dots,q_n^{p^m}-1$. Assume that*

$$\sup_{m\ge1}\dim_{\mathbb{Q}_p} A/A_m\otimes_{\mathbb{Z}_p}\mathbb{Q}_p < \infty.$$

Then for every $i \ge 0$ and $j \ge 0$

$$\sup_{m\ge1}\dim_{\mathbb{Q}_p}\widehat{\mathrm{Tor}}_j^A(A/A_m,\mathrm{Ker}(\widehat{\partial}_{i,m}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p < \infty \tag{5.7}$$

and

$$\sup_{m\ge1}\dim_{\mathbb{Q}_p} H_i(\mathcal{S}_m)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p < \infty. \tag{5.8}$$

Proof Since $\mathbb{Z}_p[[Q]]$ is an abstract Noetherian ring, every abstract ideal in $\mathbb{Z}_p[[Q]]$ is finitely generated and so is automatically closed. In particular, A_m is a closed ideal in A.

1. We show first that (5.8) follows from (5.7). Observe that

$$H_i(\mathcal{S}_m) = \widehat{\mathrm{Tor}}_i^{\mathbb{Z}_p[[Q^{p^m}]]}(A,\mathbb{Z}_p)$$

is an A-module, where Q^{p^m} acts trivially, so is an A/A_m-module. Note that $H_i(\mathcal{S}_m) = \mathrm{Ker}(\widehat{\partial}_{i,m})/\mathrm{Im}(\widehat{\partial}_{i+1,m})$, hence $A_m\,\mathrm{Ker}(\widehat{\partial}_{i,m}) \subseteq \mathrm{Im}(\widehat{\partial}_{i+1,m})$ and so there is a surjective map $\widehat{\mathrm{Tor}}_0^A(A/A_m,\mathrm{Ker}(\widehat{\partial}_{i,m})) = \mathrm{Ker}(\widehat{\partial}_{i,m})/A_m\,\mathrm{Ker}(\widehat{\partial}_{i,m}) \to H_i(\mathcal{S}_m)$. This induces a surjective map $\widehat{\mathrm{Tor}}_0^A(A/A_m,\mathrm{Ker}(\widehat{\partial}_{i,m}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to H_i(\mathcal{S}_m)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, hence

$$\sup_{m\ge1}\dim_{\mathbb{Q}_p} H_i(\mathcal{S}_m)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \le \sup_{m\ge1}\dim_{\mathbb{Q}_p}\widehat{\mathrm{Tor}}_0^A(A/A_m,\mathrm{Ker}(\widehat{\partial}_{i,m}))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

2. To prove (5.7) we first consider the case $i = 0$; but then $\mathrm{Ker}(\widehat{\partial}_{0,m}) = A_m$. The following exact sequence is a part of the long exact sequence in pro-p homology

$$0 = \widehat{\mathrm{Tor}}_{j+1}^A(A/A_m,A) \to \widehat{\mathrm{Tor}}_{j+1}^A(A/A_m,A/A_m) \to \widehat{\mathrm{Tor}}_j^A(A/A_m,A_m)$$
$$\to \widehat{\mathrm{Tor}}_j^A(A/A_m,A) = 0 \quad \text{for } j \ge 1.$$

Thus for $j \ge 1$

$$\widehat{\mathrm{Tor}}_{j+1}^A(A/A_m,A/A_m) \simeq \widehat{\mathrm{Tor}}_j^A(A/A_m,A_m) = \widehat{\mathrm{Tor}}_j^A(A/A_m,\mathrm{Ker}(\widehat{\partial}_{0,m})). \tag{5.9}$$

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By our assumption $\sup_{m\geq 1} \dim_{\mathbb{Q}_p} A/A_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$. Then by (5.1) from Lemma 5.2 $\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_{j+1}^A(A/A_m, A/A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$. Hence by (5.9)

$$\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(A/A_m, A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty \quad \text{for } j \geq 1.$$

Finally for $j = 0$, observe that A_m/A_m^2 is an A/A_m-module generated by the images of $q_1^{p^m} - 1, \dots, q_n^{p^m} - 1$, hence

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_0^A(A/A_m, A_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = (A_m/A_m^2) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq n \,.\, \dim_{\mathbb{Q}_p}(A/A_m \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

This completes the proof when $i = 0$.

By induction on i we can assume that (5.7) holds for $i-1$, *i.e.*,

$$\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(A/A_m, Ker(\widehat{\partial}_{i-1,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty, \tag{5.10}$$

and by case I we have that

$$\sup_{m\geq 1} \dim_{\mathbb{Q}_p} H_{i-1}(\mathcal{S}_m) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty. \tag{5.11}$$

Consider the short exact sequence $0 \to \mathrm{Ker}(\widehat{\partial}_{i,m}) \to S_{i,m} \to \mathrm{Im}(\widehat{\partial}_{i,m}) \to 0$ of pro-p A-modules, where $S_{i,m}$ is a module of the Koszul complex $\mathcal{S}_m$, so by definition is a finitely generated free pro-p A-module. Then for $j \geq 2$, there is a long exact sequence in homology

$$\cdots \to 0 = \widehat{\mathrm{Tor}}_j^A(A/A_m, S_{i,m}) \longrightarrow \widehat{\mathrm{Tor}}_j^A(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \longrightarrow \widehat{\mathrm{Tor}}_{j-1}^A(A/A_m, \mathrm{Ker}(\widehat{\partial}_{i,m})) \longrightarrow \widehat{\mathrm{Tor}}_{j-1}^A(A/A_m, S_{i,m})) = 0 \longrightarrow \cdots.$$

Similarly, there is an exact sequence

$$0 = \widehat{\mathrm{Tor}}_1^A(A/A_m, S_{i,m}) \longrightarrow \widehat{\mathrm{Tor}}_1^A(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \longrightarrow \widehat{\mathrm{Tor}}_0^A(A/A_m, Ker(\widehat{\partial}_{i,m})) \longrightarrow \widehat{\mathrm{Tor}}_0^A(A/A_m, S_{i,m})) \longrightarrow \widehat{\mathrm{Tor}}_0^A(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \longrightarrow 0.$$

Then for $j \geq 2$,

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_{j-1}^A(A/A_m, \mathrm{Ker}(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{5.12}$$

and

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_0^A(A/A_m, Ker(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_1^A(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p + \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_0^A(A/A_m, S_{i,m}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{5.13}$$

Note that $\widehat{\mathrm{Tor}}_0^A(A/A_m, S_{i,m}) = (A/A_m)\widehat{\otimes}_A S_{i,m} = (A/A_m)^{\binom{n}{i}}$. This together with (5.13) and (5.12) imply that to complete the proof it remains to show that

$$\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}_j^A(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty \quad \text{for } j \geq 1.$$

By Lemma 5.2 and (5.11)

$$\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}^{A}_{j+1}(A/A_m, H_{i-1}(\mathcal{S}_m)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty. \tag{5.14}$$

Finally using the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\widehat{\partial}_{i,m}) \longrightarrow \mathrm{Ker}(\widehat{\partial}_{i-1,m}) \longrightarrow H_{i-1}(\mathcal{S}_m) \longrightarrow 0$$

of pro-p A-modules, we have a long exact sequence in homology

$$\cdots \longrightarrow \widehat{\mathrm{Tor}}^{A}_{j+1}(A/A_m, H_{i-1}(\mathcal{S}_m)) \longrightarrow \widehat{\mathrm{Tor}}^{A}_{j}(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \longrightarrow \widehat{\mathrm{Tor}}^{A}_{j}(A/A_m, \mathrm{Ker}(\widehat{\partial}_{i-1,m})) \longrightarrow \cdots,$$

and hence we get

$$\dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}^{A}_{j}(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}^{A}_{j+1}(A/A_m, H_{i-1}(\mathcal{S}_m)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p + \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}^{A}_{j}(A/A_m, \mathrm{Ker}(\widehat{\partial}_{i-1,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Hence by (5.10) and (5.14), $\sup_{m\geq 1} \dim_{\mathbb{Q}_p} \widehat{\mathrm{Tor}}^{A}_{j}(A/A_m, \mathrm{Im}(\widehat{\partial}_{i,m})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$. ∎

Theorem 5.5 *Let Q be a finitely generated abelian pro-p group and A a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module. If $\sup_{t\geq 1} \dim_{\mathbb{Q}_p} A \otimes_{\mathbb{Z}_p[[Q^{p^t}]]} \mathbb{Q}_p < \infty$, then*

$$\sup_{t\geq 1} \dim_{\mathbb{Q}_p} H_i(Q^{p^t}, A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty, \quad \textit{for all } i.$$

Proof By going down to a subgroup of finite index in Q, we can assume that $Q = \mathbb{Z}_p^n = \overline{\langle q_1, \dots, q_n \rangle}$. Using induction on the number of generators of A as a $\mathbb{Z}_q[[Q]]$-module and Lemma 5.3, we can reduce to the case when A is a cyclic $\mathbb{Z}_p[[Q]]$-module, *i.e.*, $A = \mathbb{Z}_p[[Q]]/I$ for some ideal in $\mathbb{Z}_p[[Q]]$ (since $\mathbb{Z}_p[[Q]]$ is an abstract Noetherian ring, every abstract ideal in $\mathbb{Z}_p[[Q]]$ is closed). Then we can apply (5.8). ∎

Theorem 5.6 *Let G be a metabelian pro-p group of type* FP_{2m}. *Then*

$$\sup_{M\in\mathcal{A}} \mathrm{rk}\ H_i(M, \mathbb{Z}_p) < \infty, \quad \textit{for all } 0 \leq i \leq m,$$

where $\mathcal{A}$ is the set of all subgroups of p-power index in G.

Proof Let A be a pro-p abelian subgroup of G such that $G/A \cong Q$ is abelian. Let $G_1 \in \mathcal{A}$, Q_1 be the image of G_1 in Q and $A_1 = A \cap G_1$, so $G_1/A_1 \cong Q_1$.

The Lyndon–Hochschild–Serre spectral sequence in pro-p homology

$$E^2_{r,s} = H_r(Q_1, H_s(A_1, \mathbb{Z}_p)) \Longrightarrow H_{r+s}(G_1, \mathbb{Z}_p)$$

implies that

$$\dim_{\mathbb{Q}_p} H_j(G_1, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \sum_{r=0}^{j} \dim_{\mathbb{Q}_p} E^{\infty}_{r,j-r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq \sum_{r=0}^{j} \dim_{\mathbb{Q}_p} E^2_{r,j-r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

By $[A : A_1] < \infty$ and Lemma 4.1 we obtain

$$E^2_{r,s} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_r(Q_1, H_s(A, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{5.15}$$

Since $[Q:Q_1] < \infty$, there is $t > 0$ such that $Q^{p^t} := \overline{\langle q^{p^t} \mid q \in Q \rangle} \subset Q_1$ and, by Lemma 4.2 for every pro-p $\mathbb{Z}_p[[Q_1]]$-module L,

$$H_r(Q_1, L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_0(Q_1/Q^{p^t}, H_r(Q^{p^t}, L)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Hence $\dim_{\mathbb{Q}_p} H_r(Q_1, L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \le \dim_{\mathbb{Q}_p} H_r(Q^{p^t}, L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, so applying for $L = H_{j-r}(A, \mathbb{Z}_p)$, we get

(5.16) $\dim_{\mathbb{Q}_p} H_r(Q_1, H_{j-r}(A, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \le \dim_{\mathbb{Q}_p} H_r(Q^{p^t}, H_{j-r}(A, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$

Then by (5.15), (5.16), and Lemma 4.1

$$\sup_{[Q:Q_1]<\infty} \dim_{\mathbb{Q}_p} E^2_{r,j-r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \le \sup_{t\ge1} \dim_{\mathbb{Q}_p} H_r(Q^{p^t}, \widehat{\bigwedge}{}^{j-r}_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Thus, to show that $\sup_{G_1 \in \mathcal{A}} \dim_{\mathbb{Q}_p} H_j(G_1, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$, for all $0 \le j \le m$, it is sufficient to prove that $\sup_{t\ge1} \dim_{\mathbb{Q}_p} H_r(Q^{p^t}, \widehat{\bigwedge}{}^{k}_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty$, for all $0 \le r, k \le m$. Now, since G is a metabelian pro-p group of type FP_{2m}, by Theorem 2.4, $\widehat{\bigwedge}{}^{k}_{\mathbb{Z}_p} A$ is finitely generated as a pro-p $\mathbb{Z}_p[[Q]]$-module for all $k \le 2m$. So applying Theorem 5.5 with $B = \widehat{\bigwedge}{}^{k}_{\mathbb{Z}_p} A$, we see it is enough to show that

$$\sup_{t\ge1} \dim_{\mathbb{Q}_p} H_0(Q^{p^t}, \widehat{\bigwedge}{}^{k}_{\mathbb{Z}_p} A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p < \infty, \quad \text{for all } 0 \le k \le m.$$

But this follows from Lemma 5.1. ∎

6 Proof of Theorem C: the General Case

Let $C = Z(G)$; thus G/C is metabelian. Let $M \in \mathcal{A}$; consider the short exact sequence of pro-p groups $C \cap M \hookrightarrow M \to M/(C \cap M)$ and the associated Lyndon–Hochschild–Serre spectral sequence

$$E^2_{i,j} = H_i(M/(C \cap M), H_j(C \cap M, \mathbb{Z}_p)) \Longrightarrow H_{i+j}(M, \mathbb{Z}_p).$$

Since G is of type FP_{2m}, $m \ge 1$, by [17, Corollary 3.5], C and so $C \cap M$ are finitely generated abelian pro-p groups. Also, since C is central, $M/(C \cap M)$ acts trivially (via conjugation) on $C \cap M$. This implies that

$$E^2_{i,j} = H_i(M/(C \cap M), H_j(C \cap M, \mathbb{Z}_p)) = H_i(M/(C \cap M), \mathbb{Z}_p) \widehat{\otimes}_{\mathbb{Z}_p} H_j(C \cap M, \mathbb{Z}_p).$$

Moreover, since $[G:M] < \infty$, G and M are pro-p groups of the same homological type [12, Theorem 2]. So, by [17, Theorem 3.6], $M/(C \cap M)$ is of type FP_{2m}. Thus $H_i(M/(C \cap M), \mathbb{Z}_p)$ is finitely generated as a $\mathbb{Z}_p$-module for $0 \le i \le 2m$. Also, since $C \cap M$ is a finitely generated abelian pro-p group, $H_j(C \cap M, \mathbb{Z}_p)$ is finitely generated as $\mathbb{Z}_p$-module for all j, hence is finitely presented as $\mathbb{Z}_p$-module. Then by [6, Lemma 1.1]

$$H_i(M/M \cap C, \mathbb{Z}_p) \widehat{\otimes}_{\mathbb{Z}_p} H_j(C \cap M, \mathbb{Z}_p) \cong H_i(M/M \cap C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} H_j(C \cap M, \mathbb{Z}_p),$$

and so

$$\begin{aligned} E^2_{i,j} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &\cong (H_i(M/M \cap C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} H_j(C \cap M, \mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\cong (H_i(M/M \cap C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} (H_j(C \cap M, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p). \end{aligned}$$

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Thus, for $0 \le i \le 2m$ and any j,

$$\operatorname{rk}(E^2_{i,j}) = \operatorname{rk} H_i(M/M \cap C, \mathbb{Z}_p) \cdot \operatorname{rk} H_j(C \cap M, \mathbb{Z}_p) \le \operatorname{rk} H_i(M/M \cap C, \mathbb{Z}_p) \cdot \binom{\operatorname{rk} C}{j}, \tag{6.1}$$

Finally, by Theorem 5.6, since $M/(C \cap M)$ has a p-power index in the metabelian pro-p group G/C of type FP_{2m}, $\sup_{M \in \mathcal{A}} \operatorname{rk}\ H_i(M/M \cap C, \mathbb{Z}_p) < \infty$, for $0 \le i \le m$. Therefore, from the spectral sequence convergence and (6.1), we obtain

$$\sup_{M \in \mathcal{A}} \operatorname{rk} H_i(M, \mathbb{Z}_p) \le \sum_{\alpha+\beta=i} \sup_{M \in \mathcal{A}} \operatorname{rk} E^2_{\alpha,\beta} < \infty, \quad \text{for } 0 \le i \le m.$$

∎

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Department of Mathematics, State University of Campinas (UNICAMP), 13083-859 Campinas-SP, Brazil
Email: desi@ime.unicamp.br

Department of Mathematics, University of Brasília, 70297-400 Brasília, Brazil
Email: pinto.aline@gmail.com