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FREE RESOLUTIONS FOR CERTAIN CLASSES OF GROUPS

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1. Introduction

In a previous paper **[1]** we constructed a free resolution for a class of groups which include Fuchsian groups with compact orbit spaces **[2, 3]**, infinite polyhedral groups, plane crystallographic groups p^2, p^3, p^4 and p^6 and Dyck's groups **[4]**, and used this resolution for computation of the integral homology and cohomology of these groups. Lyndon **[5]** determined the cohomology of groups with a single defining relation. The plane crystallographic groups p^1 and pg and Artin's braid group B_3 are among these groups. In this paper we have constructed free resolutions for certain classes of groups—resolutions which are particularly suitable for direct computation of the homology and the cohomology of these groups for any coefficient module. These classes of groups include the plane crystallographic groups pm, cm and pgg. We have computed the integral homology and cohomology from each of the free resolutions obtained.

Let G be a group given by $G=F/R$, where F is a free group generated freely by $x_1, x_2, \ldots, x_m$ and R is the normal closure of $r_1, r_2, \ldots, r_n \in F$. Lyndon **[5, 6]** showed that a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ can be started as follows:

$$\ldots\ldots \to Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \tag{1}$$

where Y_0 is a right $\mathbb{Z}G$-module free on $\alpha_1, \alpha_2, \ldots, \alpha_m$,

Y_1 .... $\beta_1, \beta_2, \ldots, \beta_n$,

and the homomorphisms are given by

$$\varepsilon(g) = 1 \in \mathbb{Z}, \text{ for each } g \in G, \text{ extended by linearity,}$$

$$d_0(\alpha_i) = h_i - 1, \qquad i = 1, 2, \ldots, m,$$

$$d_1(\beta_j) = \sum_{i=1}^{m} \alpha_i \pi\left(\frac{\partial r_j}{\partial x_i}\right), \qquad j = 1, 2, \ldots, n,$$

each extended by $\mathbb{Z}G$-linearity, where $\pi: \mathbb{Z}F \to \mathbb{Z}G$ is the homomorphism induced by the canonical homomorphism $F \to \frac{F}{R} = G$, each $h_i = \pi(x_i)$, and each $\frac{\partial r_j}{\partial x_i}$ is defined by

$$r_j - 1 = \sum_{i=1}^{m} (x_i - 1) \frac{\partial r_j}{\partial x_i}, \text{ (see [7]).}$$

Each of the free resolutions we have constructed is an extension of (1).

We shall use two results, viz. Propositions 1 and 2, of [**1**] in this paper and they are all that we need for our proofs. For the sake of completeness, we state these two results together as Proposition 1 below.

Let H be a subgroup of a group G and $\bar{H}$ a set of generators of H. For any subset S of $\mathbb{Z}G$, let $\mathrm{Ann}_R S$ denote the right annhilator of S.

Define

$$s(H) = \sum_{h \in H} h, \quad \text{if } H \text{ is finite,}$$

$$= 0, \quad \text{otherwise.}$$

Proposition 1. ([**1**])

$$\text{If } H \text{ is finite}, \mathrm{Ann}_R\{s(H)\} = \sum_{h \in \bar{H}} (h-1) \cdot \mathbb{Z}G. \tag{P1}$$

$$\bigcap_{h \in \bar{H}} \mathrm{Ann}_R \{h-1\} = s(H) \cdot \mathbb{Z}G. \tag{P2}$$

In this paper we have used the symbol s_j to denote the element $1 + h_j + \dots + h_j^{m_j - 1}$ of $\mathbb{Z}G$, where h_j is an element of order m_j in a group G.

2.

Let G be a group given by

generators: $h_1, h_2, \dots, h_k, h_{k+1}$;

relations: $h_1^{m_1} = \dots = h_k^{m_k} = e$,

$$h_1 h_{k+1} = h_{k+1} h_1, \dots, h_l h_{k+1} = h_{k+1} h_l (l \leqq k)$$

If $k = l = 2$, and $m_1 = m_2 = 2$, $G \cong pm$ ([**4**], p. 136).

Definition 1.

$$\ldots\xrightarrow{\sigma} Y\xrightarrow{\tau} Y\xrightarrow{\sigma} Y\xrightarrow{\tau} Y\xrightarrow{\sigma} Y\xrightarrow{d_1} Y_0\xrightarrow{d_0}\to G\xrightarrow{\varepsilon}\mathbb{Z}\to 0, \tag{2}$$

is a sequence of right $\mathbb{Z}G$-modules and $\mathbb{Z}G$-homomorphisms defined as follows:

Y_0 is a right G-module free on $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$;

Y.................................$\beta_1, \beta_2, \ldots, \beta_{k+l}$;

and the homomorphisms are given by

$\varepsilon(g)=1\in\mathbb{Z}$, for each $g\in G$, and extended by linearity;

$d_0(\alpha_i)=h_i-1, \qquad i=1,2,\ldots,k, \quad k+1;$

$d_1(\beta_j)=\alpha_j s_j, \qquad j=1,2,\ldots,k;$

$d_1(\beta_{k+j'})=\alpha_{j'}(h_{k+1}-1)-\alpha_{k+1}(h_{j'}-1), \qquad j'=1,2,\ldots,l;$

$\sigma(\beta_j)=\beta_j(h_j-1), \qquad j=1,2,\ldots,k;$

$\sigma(\beta_{k+j'})=\beta_{k+j'}s_{j'}-\beta_{j'}(h_{k+1}-1), \qquad j'=1,2,\ldots,l;$

$\tau(\beta_j)=\beta_j s_j, \qquad j=1,2,\ldots,k;$

$\tau(\beta_{k+j'})=\beta_{k+j'}(h_{j'}-1)+\beta_{j'}(h_{k+1}-1), \qquad j'=1,2,\ldots,l;$

and each extended by $\mathbb{Z}G$-linearity.

Theorem 1. *The sequence* (2) *is a free* $\mathbb{Z}G$*-resolution of* $\mathbb{Z}$.

Proof. Since (2) is an extension of (1), it will be sufficient to verify the exactness of (2) only at the fifth, the sixth and the seventh terms from the right.

The verification of the fact that the image of each homomorphism is contained in the kernel of the following homomorphism is straightforward.

We only have to show that (i) $\operatorname{Ker} d_1\subseteq\operatorname{Im}\sigma$, (ii) $\operatorname{Ker}\sigma\subseteq\operatorname{Im}\tau$, and (iii) $\operatorname{Ker}\tau\subseteq\operatorname{Im}\sigma$.

First, let $\gamma=\sum_{s=1}^{k+l}\beta_s\gamma_s\in\operatorname{Ker} d_1$. Then, since Y_0 is free on $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$, it follows that

$$\begin{aligned}
s_1\gamma_1+(h_{k+1}-1)\gamma_{k+1} &= 0\\
\vdots\qquad\qquad & \quad\vdots\\
s_l\gamma_l+(h_{k+1}-1)\gamma_{k+1} &= 0\\
s_{l+1}\gamma_{l+1} &= 0\\
\vdots\qquad\qquad & \quad\vdots\\
s_k\gamma_k &= 0\\
(1-h_1)\gamma_{k+1}+\ldots+(1-h_l)\gamma_{k+l} &=
\end{aligned} \tag{3}$$

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For each $j=1,2,\ldots,l$, multiplication of the j-th equation by h_j-1 gives $(h_j-1)(h_{k+1}-1)\gamma_{k+j}=0$, i.e., $(h_{k+1}-1)(h_j-1)\gamma_{k+j}=0$, so that $\gamma_{k+j}=s_j\gamma'_{k+j}$, for some $\gamma'_{k+j}\in\mathbb{Z}G$, by (P2). Substitution of this value in the same equation yields $\gamma_j=(h_j-1)\gamma'_j-(h_{k+1}-1)\gamma'_{k+j}$, for some $\gamma'_j\in\mathbb{Z}G$, by (P1). Also, for each $j'=l+1, l+2,\ldots,k$, $\gamma_{j'}=(h_{j'}-1)\gamma'_{j'}$, for some $\gamma'_{j'}\in\mathbb{Z}G$. Hence $\gamma=\sigma(\sum_{s=1}^{k+l}\beta_s\gamma'_s)$, so that (i) holds.

The verifications of (ii) and (iii) are similar and easier, and so are left out.

Integral Homology and Cohomology of G.

$$H_0(G,\mathbb{Z})\cong\mathbb{Z},$$

$$H_1(G,\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}_{m_1}\oplus\ldots\oplus\mathbb{Z}m_k,$$

$$H_{2r}(G,\mathbb{Z})\cong\mathbb{Z}_{m_1}\oplus\ldots\oplus\mathbb{Z}m_l,$$

$$H_{2r+1}(G,\mathbb{Z})\cong\mathbb{Z}_{m_1}\oplus\ldots\oplus\mathbb{Z}m_k,$$

for each $r\geqq 1$.

$$H^0(G,\mathbb{Z})\cong\mathbb{Z},$$

$$H^1(G,\mathbb{Z})\cong\mathbb{Z},$$

$$H^{2r}(G,\mathbb{Z})\cong\mathbb{Z}m_1\oplus\ldots\oplus\mathbb{Z}m_k,$$

$$H^{2r+1}(G,\mathbb{Z})=\mathbb{Z}m_1\oplus\ldots\oplus\mathbb{Z}m_l,$$

for each $r\geqq 1$.

3.

We next consider a group G given by generators: h_1, h_2; relations: $h_1^{m_1}=e$, $h_1h_2^{m_2}=h_2^{m_2}h_1$.

If $m_1=m_2=2$, then $G\cong cm$ ([**4**], p. 136).

Definition 2.

$$\ldots\xrightarrow{\sigma}Y\xrightarrow{\tau}Y\xrightarrow{\sigma}Y\xrightarrow{\tau}Y\xrightarrow{\sigma}Y\xrightarrow{d_2}Y\xrightarrow{d_1}Y\xrightarrow{d_0}\mathbb{Z}G\xrightarrow{\varepsilon}\mathbb{Z}\to 0 \tag{4}$$

is a sequence of right $\mathbb{Z}G$-modules and $\mathbb{Z}G$-homomorphisms defined as follows:

Y is a right $\mathbb{Z}G$-module free on α_1, α_2; and the homomorphisms are given by

$\varepsilon(g)=1\in\mathbb{Z}$, for each $g\in G$, and extended by linearity;

$d_0(\alpha_i)=h_i-1, \qquad i=1,2;$

$$d_1(\alpha_1)=\alpha_1 s_1,\ d_1(\alpha_2)=\alpha_1(h_2^{m_2}-1)+\alpha_2 s_2(1-h_1);$$

$$d_2(\alpha_1)=\alpha_1(h_1-1),\ d_2(\alpha_2)=-\alpha_1(h_2^{m_2}-1)+\alpha_2 s_1;$$

$$\sigma(\alpha_1)=\alpha_1 s_1,\ \sigma(\alpha_2)=\alpha_1(h_2^{m_2}-1)+\alpha_2(h_1-1);$$

$$\tau(\alpha_1)=\alpha_1(h_1-1),\ \tau(\alpha_2)=-\alpha_1(h_2^{m_2}-1)+\alpha_2 s_2;$$

and each extended by $\mathbb{Z}G$-linearity.

Theorem 2. *The sequence* (4) *is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.*

Proof. It is easy to see that (4) extends (1).

As in Theorem 1, we shall only verify that $\operatorname{Ker} d_1 \subseteq \operatorname{Im} d_2$.
Let $\gamma=\alpha_1\gamma_1+\alpha_2\gamma_2 \in \operatorname{Ker} d_1$. Then,

$$\begin{aligned} s_1\gamma_1+(h_2^{m_2}-1)\gamma_2 &= 0, \\ s_2(1-h_1)\gamma_2 \qquad &= 0. \end{aligned} \tag{5}$$

Multiplication of the second equation by h_2-1 and subsequent application of P2 gives

$$\gamma_2=s_1\gamma'_2,$$

for some $\gamma'_2 \in \mathbb{Z}G$.

Substitution of this value in the first equation yields

$$\gamma_1=(h_1-1)\gamma'_1-(h_2^{m_2}-1)\gamma'_2,$$

for some $\gamma'_1 \in \mathbb{Z}G$, by P1.

Hence $\gamma=d_2(\alpha_1\gamma'_1+\alpha_2\gamma'_2)$. Thus, $\operatorname{Ker} d_1 \subseteq \operatorname{Im} d_2$.

Integral Homology and Cohomology of G.

$$H_0(G,\mathbb{Z})\cong\mathbb{Z},$$

$$H_1(G,\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}m_1,$$

$$H_l(G,\mathbb{Z})\cong\mathbb{Z}m_1, \quad \text{for each} \quad l\geqq 2.$$

$$H^0(G,\mathbb{Z})\cong\mathbb{Z},$$

$$H^1(G,\mathbb{Z})\cong\mathbb{Z},$$

$$H^l(G,\mathbb{Z})\cong\mathbb{Z}m_1, \quad \text{for each} \quad l\geqq 2.$$

4.

We finally consider a group G given by generators: h_1, h_2; relations: $(h_1h_2)^2 = (h_1^{-1}h_2)^2 = e$. Then $G \cong pgg$ ([4], p. 136).

Definition 3.

$$\ldots \xrightarrow{\sigma} Y \xrightarrow{\tau} Y \xrightarrow{\sigma} Y \xrightarrow{\tau} Y \xrightarrow{\sigma} Y \xrightarrow{d_1} Y \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \tag{6}$$

is a sequence of right $\mathbb{Z}G$-modules and $\mathbb{Z}G$-homomorphisms defined as follows:

Y is a right $\mathbb{Z}G$-module free on α_1, α_2, and the homomorphisms are given by

$\varepsilon(g) = 1 \in \mathbb{Z}$, for each $g \in G$, and extended by linearity;

$d_0(\alpha_i) = h_i - 1, \qquad i = 1, 2;$

$d_1(\alpha_1) = \alpha_1 h_2(h_1h_2 + 1) + \alpha_2(h_1h_2 + 1)$, $d_1(\alpha_2) = -\alpha_1(h_1^{-1}h_2 + 1) + \alpha_2(h_1^{-1}h_2 + 1)$;

$\sigma(\alpha_1) = \alpha_1(h_1h_2 - 1)$, $\sigma(\alpha_2) = \alpha_2(h_1^{-1}h_2 - 1)$;

$\tau(\alpha_1) = \alpha_1(h_1h_2 + 1)$, $\tau(\alpha_2) = \alpha_2(h_1^{-1}h_2 + 1)$;

and each extended by $\mathbb{Z}G$-linearity.

Theorem 3. *The sequence* (6) *is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.*

Proof. As before we shall only verify that $\operatorname{Ker} d_1 \subseteq \operatorname{Im} \sigma$. Let $\gamma = \alpha_1\gamma_1 + \alpha_2\gamma_2 \in \operatorname{Ker} d_1$. Then

$$\begin{aligned} h_2(h_1h_2 + 1)\gamma_1 - (h_1^{-1}h_2 + 1)\gamma_2 &= 0, \\ (h_1h_2 + 1)\gamma_1 + (h_1^{-1}h_2 + 1)\gamma_2 &= 0. \end{aligned} \tag{7}$$

Adding the equations, we have $(h_2 + 1)(h_1h_2 + 1)\gamma_1 = 0$, so that $\gamma_1 = (h_1h_2 - 1)\gamma'_1$, for some $\gamma'_1 \in \mathbb{Z}G$, by (P1) and the fact that h_2 has infinite order. Substitution in the second equation gives $(h_1^{-1}h_2 + 1)\gamma_2 = 0$, and so, $\gamma_2 = (h_1^{-1}h_2 - 1)\gamma'_2$, for some $\gamma'_2 \in \mathbb{Z}G$. Thus, $\gamma = \sigma(\alpha_1\gamma'_1 + \alpha_2\gamma'_2)$, and so $\operatorname{Ker} d_1 \subseteq \operatorname{Im} \sigma$.

Integral Homology and Cohomology of G.

$$\begin{aligned} &H_0(G, \mathbb{Z}) \cong \mathbb{Z}, \\ &H_1(G, \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4, \\ &H_{2l}(G, \mathbb{Z}) \cong 0, \\ &H_{2l+1}(G, \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \end{aligned}$$

for each $l \geqq 1$.

$$H^0(G, \mathbb{Z}) \cong \mathbb{Z},$$

$$H^1(G, \mathbb{Z}) \cong 0,$$

$$H^2(G, \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4,$$

$$H^{2l-1}(G, \mathbb{Z}) \cong 0,$$

$$H^{2l}(G, \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

for each $l \geqq 2$.

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