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CONGRUENCE SUBGROUPS OF BRAID GROUPS AND CRYSTALLOGRAPHIC QUOTIENTS. PART I

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Abstract

This paper is the first of a two part series devoted to describing relations between congruence and crystallographic braid groups. We recall and introduce some elements belonging to congruence braid groups and we establish some (iso)-morphisms between crystallographic braid groups and corresponding quotients of congruence braid groups.

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1. Introduction

This paper delves into the relationship between two families of groups, subgroups and quotients of classical braid groups: *congruence subgroups of braid groups* and *crystallographic braid groups*, respectively introduced by Arnol'd [Arn68] and Tits [Tit66].

While both families are instances of more general groups with rich theoretical backgrounds, they have also garnered significant attention in recent (and less recent) literature on braid groups and relatives; see for instance [BM18, Nak21, Sty18] and also [ABGH20, BPS22, KM22] for congruence subgroups of braid groups and [A'C79,

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BM20, GGO17] as well as [GGOP21, BGM22, CdSJ023] for crystallographic braid groups. Let us provide an overview of the two general families to which these groups belong.

In the context of groups of matrices, a *congruence subgroup* of a matrix group with integer entries is a subgroup defined as the kernel of the mod m reduction of a linear group. The notion of congruence subgroups can be generalised for arithmetic subgroups of certain algebraic groups for which we can define appropriate reduction maps. A classical question about congruence subgroups is the *congruence subgroup problem*, first formulated in [BMS67]: in this seminal paper, Bass, Milnor and Serre prove that for $n \geq 3$, the group $\mathrm{SL}_n(\mathbb{Z})$ has the *congruence subgroup property*, meaning that every finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ contains a *principal congruence subgroup*. The literature devoted to this problem in several settings is vast (we refer to [Rag04] for a survey), linking the theory of arithmetic groups and geometric properties of related spaces.

In this spirit, we can define congruence subgroups of any group via a choice of representation into $\mathrm{GL}(n, \mathbb{Z})$. Let the braid group B_n be the mapping class group $\mathrm{Mod}(D_n)$ of the disc with n marked points D_n. We can define a symplectic representation and use it to define congruence subgroups of braid groups $B_n[m]$. We recall the details in Section 2, but let us give here an idea of the definitions of these groups. We start with the integral Burau representation of B_n, which is the representation $\rho\colon B_n \to \mathrm{GL}_n(\mathbb{Z})$ obtained by evaluating the (unreduced) Burau representation $B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ at $t = -1$. Describing the representation from a topological point of view, one can see that the integral Burau representation is symplectic, and can be regarded as a representation:

$$\rho\colon B_n \to \begin{cases} \mathrm{Sp}_{n-1}(\mathbb{Z}) & \text{for } n \text{ odd,} \\ (\mathrm{Sp}_n(\mathbb{Z}))_u & \text{for } n \text{ even,} \end{cases}$$

where $(\mathrm{Sp}_n(\mathbb{Z}))_u$ is the subgroup of $\mathrm{Sp}_n(\mathbb{Z})$ fixing a specific vector $u \in \mathbb{Z}^n$; see [GG16, Proposition 2.1] for a homological description of $(\mathrm{Sp}_n(\mathbb{Z}))_u$ in this context.

The *level m congruence subgroup*, $B_n[m]$, is the kernel of the mod m reduction of the integral Burau representation

$$\rho_m\colon B_n \to \begin{cases} \mathrm{Sp}_{n-1}(\mathbb{Z}/m\mathbb{Z}) & \text{for } n \text{ odd,} \\ (\mathrm{Sp}_n(\mathbb{Z}/m\mathbb{Z}))_u & \text{for } n \text{ even,} \end{cases} \tag{1-1}$$

for $m > 1$.

The second family of groups that we consider are *crystallographic groups*, appearing in the study of isometries of Euclidean spaces; see Section 3 for precise definitions and useful characterisations. In [GGO17], Gonçalves, Guaschi and Ocampo prove that certain quotients of the braid groups B_n are crystallographic, and use this result to study their torsion and other algebraic properties. The authors use this characterisation to prove that the group $B_n/[P_n, P_n]$ is crystallographic, where P_n denotes the pure braid group on n strands and $[P_n, P_n]$ its commutator subgroup. This quotient, that we refer

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to as the *crystallographic braid group*, was introduced by Tits in [Tit66] as *groupe de Coxeter étendu*; see [BGM22] for a short survey.

Congruence subgroups and crystallographic structures share a point of contact. It follows from Arnol'd's work [Arn68] that the pure braid group P_n can be characterised as the congruence subgroup $B_n[2]$. With this equivalence and the results of [GGO17] in mind, it is natural to ask: how are congruence subgroups of braid groups and crystallographic groups related? This question was also recently raised in [KNS24] for small Coxeter groups. In this paper, we propose to explore the interplay between congruence subgroups of braid groups and crystallographic groups, opening several questions that we will develop in a further work [BDOS24].

The paper is organised as follows. In Section 2, we provide some basic definitions and properties that are useful in this paper, such as the Burau representation, symplectic structures, the definition of congruence subgroups and the actions of half-twists on symplectic groups. Section 3 contains the main body of this work. In Section 3.1, we prove the following general result about crystallographic groups.

THEOREM 3.4. *Consider the short exact sequence* $1 \longrightarrow K \longrightarrow G \stackrel{p}{\longrightarrow} Q \longrightarrow 1$ *where* K *is a free abelian group of finite rank and* Q *is a finite group such that the representation* $\varphi\colon Q \to \mathrm{Aut}(K)$*, induced from the action by conjugacy, is not injective. Suppose that the group* $p^{-1}(\mathrm{Ker}(\varphi))$ *is torsion free. Then* G *is a crystallographic group with holonomy group* ${}^{Q}\!/_{\mathrm{Ker}(\varphi)}$.

This theorem plays an important role in this work, since the techniques used in [GGO17] do not apply directly in this paper. This is because the representation

$$\Theta_m\colon \rho_m(B_n) \to \mathrm{Aut}\Big({}^{B_n[m]}\!/_{[B_n[m], B_n[m]]}\Big),$$

induced from the action by conjugacy of B_n on $B_n[m]$, is injective if and only if $m = 2$ (see Proposition 3.5), where ρ_m is the homomorphism defined in (1-1). We apply Theorem 3.4 to get the following result, which is proved in Section 3.2, relating congruence subgroups and crystallographic groups.

THEOREM 3.6. *Let* $n \geq 3$ *be an odd integer and let* $m \geq 3$ *be a prime number. If the abelian group* ${}^{B_n[m]}\!/_{[B_n[m], B_n[m]]}$ *is torsion free, then the group* ${}^{B_n}\!/_{[B_n[m], B_n[m]]}$ *is crystallographic with dimension equal to rank* $\Big({}^{B_n[m]}\!/_{[B_n[m], B_n[m]]}\Big)$ *and holonomy group* ${}^{\rho_m(B_n)}\!/_{Z(\rho_m(B_n))}$.

In Section 3.3, we show that there is an isomorphism between the crystallographic braid group ${}^{B_n}\!/_{[P_n, P_n]}$ and a quotient of congruence subgroups as described in the next result.

THEOREM 3.12. *Let* m *be a positive integer and let* $n \geq 3$*. Consider the map*

$$\overline{\xi}\colon {}^{B_n}\!/_{[P_n, P_n]} \to {}^{B_n[m]}\!/_{[P_n, P_n] \cap B_n[m]}$$

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defined by $\overline{\xi}(\sigma_i) = \sigma_i^m$ *for all* $1 \leq i \leq n-1$. *If m is odd, then* $\overline{\xi}$ *is an isomorphism. As a consequence, for* $n \geq 3$ *and m odd,* ${}^{B_n[m]}\!/\!_{[P_n, P_n] \cap B_n[m]}$ *is a crystallographic group of dimension* $n(n-1)/2$ *and holonomy group* S_n.

2. Congruence subgroups

Let S be a connected, orientable surface, possibly with marked points and boundary components. The *mapping class group* $\mathrm{Mod}(S)$ of S is the group of homotopy classes of homeomorphisms of S that preserve the orientation, fix the set of marked points setwise and fix the boundary pointwise.

2.1. Braid groups and examples. Let S be a surface as above. We introduce a particular element of $\mathrm{Mod}(S)$ that is used throughout the paper. Let A be an annulus. The homeomorphism depicted in Figure 1 is called a *twist map*.

Now, let $c \subset S$ be a simple closed curve. The regular neighbourhood $\mathcal{N}(c)$ of c is homeomorphic to an annulus A. Consider the homeomorphism f_c that acts as a twist map on $\mathcal{N}(c)$ and as the identity on $S \setminus \mathcal{N}(c)$. The homotopy class of f_c is called a *Dehn twist about* c, denoted by T_c [FM12, Section 3.1].

Braid groups can be defined in several equivalent ways, long known to be equivalent; see for instance [BB05, KT08]. In this work, it is convenient to define them in terms of mapping class groups. Let D_n be a disc with $n \in \mathbb{N}$ marked points in its interior. The *braid group* B_n is $\mathrm{Mod}(D_n)$. For a geometric insight into twists in the context of braid groups, let D_n lie on the xy-plane with its centre on the x-axis. Denote the punctures from left to right by $p_1, p_2, \ldots, p_n$: the arc connecting p_i and p_{i+1} is denoted by a_i (see Figure 4). Consider a_i to be the diameter of a circle c such that the points p_i and p_{i+1} lie on c. Interchanging the points p_i and p_{i+1} by rotating them half way along c in the clockwise direction gives a homeomorphism of D_n, and its homotopy class in $\mathrm{Mod}(D_n)$ is called a *half-twist*, denoted by σ_i. Note that all conjugates of σ_i are called half-twists. In terms of presented groups, half-twists correspond to the Artin generators from Artin's presentation for B_n [Art25]:

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

Let $c \in D_n$ be a curve surrounding the points p_i, p_{i+1}. This curve is homotopic to the circle described above. We note that if σ_i is a half-twist, then σ_i^2 is a Dehn twist about the curve c. This Dehn twist is generalised, for $1 \leq i < j \leq n$, as

$$A_{i,j} = (\sigma_{j-1}\sigma_{j-2} \ldots \sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2} \ldots \sigma_{i+1})^{-1}.$$

We recall that a generating set of P_n is given by $\{A_{i,j}\}_{1 \leq i < j \leq n}$. Geometrically, the element $A_{i,j}$ can be represented as a Dehn twist about a curve surrounding punctures p_i and p_j. For instance, in Figure 2, we describe $A_{2,5}$ as the Dehn twist about the curve that surrounds punctures p_2 and p_5.

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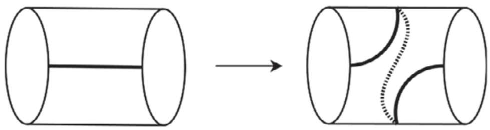

FIGURE 1. Twist map acts on an annulus.

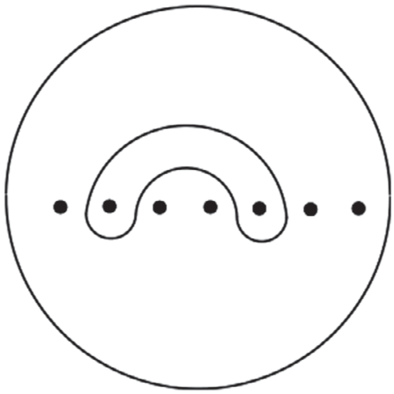

FIGURE 2. Dehn twist along the curve that surrounds the punctures p_2, p_5 is $A_{2,5}$.

We are interested in the action by conjugation of B_n on P_n. Recall from [MK99, Proposition 3.7, Ch. 3] that for all $1 \leq k \leq n-1$ and for all $1 \leq i < j \leq n$,

$$\sigma_k A_{i,j} \sigma_k^{-1} = \begin{cases} A_{i,j} & \text{if } k \neq i-1, i, j-1, j, \\ A_{i,j+1} & \text{if } j = k, \\ A_{i,j}^{-1} A_{i,j-1} A_{i,j} & \text{if } j = k+1 \text{ and } i < k, \\ A_{i,j} & \text{if } j = k+1 \text{ and } i = k, \\ A_{i+1,j} & \text{if } i = k < j-1, \\ A_{i,j}^{-1} A_{i-1,j} A_{i,j} & \text{if } i = k+1. \end{cases}$$

This action induces an action of $B_n/[P_n, P_n]$ on $P_n/[P_n, P_n]$; see [GGO17, Proposition 12]: let $\alpha \in B_n/[P_n, P_n]$ and let π be the permutation induced by α^{-1}, then $\alpha A_{i,j} \alpha^{-1} = A_{\pi(i),\pi(j)}$ in $P_n/[P_n, P_n]$.

Another important element of B_n that plays a crucial role in the paper is the Dehn twist (or a full twist) along a curve surrounding all marked points of D_n. We denote this element by Δ_n^2. In fact, Δ_n^2 generates the centre of B_n [Cho48]; in terms of half-twists,

$$\Delta_n^2 = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})^n.$$

2.2. Burau representation and symplectic structures. Braid groups naturally act on the homology of topological spaces obtained from the punctured disk. A construction arising in such a way is the Burau representation [Bur35]. One of the most famous representations of the braid group, originally introduced in terms of matrices assigned to the generators in Artin's presentation of B_n, the Burau representation is fundamental in low-dimensional topology. While this representation has been extensively studied, it still retains some mystery: a long standing candidate for proving the linearity of the braid group (later established independently in [Big01, Kra02]), the question of its faithfulness has remained open for quite some time. The Burau representation, faithful for $n \leq 3$ [MP69], eventually proved to be unfaithful for $n \geq 5$ (Moody [Moo91] proved unfaithfulness for $n \geq 9$, Long and Paton [LP93] for $n \geq 6$, and Bigelow [Big99] for $n = 5$). However, the case $n = 4$ remains open, with advances towards closing the problem being published recently [BB21, BT18, Dat22].

In this work, we are going to take the viewpoint of the Burau representation as a homological representation. Let $\pi = \pi_1(D_n, q)$ denote the fundamental group of D_n, where $q \in \partial D_n$. The function $\pi \to \mathbb{Z} \cong \langle t \rangle$ defines a covering space $\tilde{D}_n \to D_n$. Let Q be a set of all lifts of q. The action of t on $\tilde{D}_n$ induces a $\mathbb{Z}[t]$-module $\mathrm{H}_1(\tilde{D}_n, Q; \mathbb{Z}[t])$ of dimension n. Every mapping class in $\mathrm{Mod}(D_n)$ lifts to a unique mapping class in $\tilde{D}_n$. Hence, the (reducible) Burau representation is given by a map

$$\mathrm{Mod}(D_n) \to \mathrm{Aut}(\mathrm{H}_1(\tilde{D}_n, Q; \mathbb{Z}[t])).$$

This representation splits into a direct sum of an $(n-1)$- and a one-dimensional representation.

Fixing $t = -1$, the covering space becomes a two-fold branch cover $\Sigma \to D_n$, where Σ is homeomorphic to a surface of genus $g = (n-1)/2$ and one boundary component if n is odd, and $g = n/2 - 1$ and two boundary components if n is even [PV96]. As mentioned above, every mapping class in $\mathrm{Mod}(D_n)$ lifts to a unique mapping class in $\mathrm{Mod}(\Sigma)$ leading to an injection $\mathrm{Mod}(D_n) \to \mathrm{Mod}(\Sigma)$. Let $q \in \partial D_n$ be a point and Q be a set of all lifts of q. The reducible Burau representation at $t = -1$ [BM18, Section 2] (see also [BPS22]) is

$$\mathrm{Mod}(D_n) \to \mathrm{Mod}(\Sigma) \to \mathrm{Aut}(\mathrm{H}_1(\Sigma, Q; \mathbb{Z})).$$

For n odd, the module $\mathrm{H}_1(\Sigma, Q; \mathbb{Z})$ splits as $\mathrm{H}_1(\Sigma; \mathbb{Z}) \times \mathbb{Z}$ and the induced action of $\mathrm{Mod}(D_n)$ preserves a symplectic form on $\mathrm{H}_1(\Sigma; \mathbb{Z})$. Hence, the image of the latter representation is conjugate to $\mathrm{Sp}_{n-1}(\mathbb{Z})$ [GG16, Proposition 2.1]. When n is even, the module $\mathrm{H}_1(\Sigma, Q; \mathbb{Z})$ carries a symplectic structure. More precisely, if g is the genus of Σ, then let Σ' be a surface obtained by gluing a pair of pants in the boundary of Σ. Then Σ' is a surface genus $g + 1$ with one boundary component. We consider $\mathrm{H}_1(\Sigma, Q; \mathbb{Z})$ as a submodule of $\mathrm{H}_1(\Sigma'; \mathbb{Z})$. In Figure 3, we give a basis for each of the latter modules.

The representation obtained by the construction above is

$$\rho\colon B_n \to \begin{cases} \mathrm{Sp}_{n-1}(\mathbb{Z}) & \text{for } n \text{ odd}, \\ (\mathrm{Sp}_n(\mathbb{Z}))_u & \text{for } n \text{ even}, \end{cases} \tag{2-1}$$

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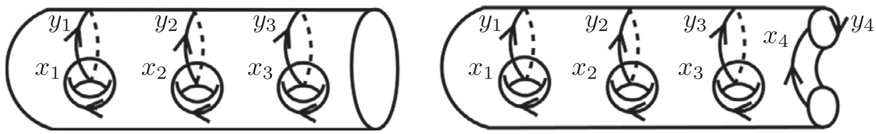

FIGURE 3. Generators for $H_1(\Sigma'; \mathbb{Z})$ on the left, and $H_1(\Sigma, Q; \mathbb{Z})$ on the right.

where, without loss of generality, we can choose $u = y_{2g+1}$. For the detailed construction, see [BM18, Section 2.1].

An analogue of the principal congruence subgroups for the braid groups B_n can be defined starting from the integral Burau representation. The *level m congruence subgroup* $B_n[m]$ is the kernel of the mod m reduction of the integral Burau representation

$$\rho_m \colon B_n \to \begin{cases} \mathrm{Sp}_{n-1}(\mathbb{Z}/m\mathbb{Z}) & \text{for } n \text{ odd,} \\ (\mathrm{Sp}_n(\mathbb{Z}/m\mathbb{Z}))_u & \text{for } n \text{ even,} \end{cases}$$

for $m > 1$.

In [Arn68], Arnol'd proved that the pure braid group P_n is isomorphic to the level 2 congruence subgroup $B_n[2]$ of the braid group B_n; see also [BM18, Section 2] for a sketch of the original argument. In [BM18], Brendle and Margalit go on to prove that $B_n[4]$ is isomorphic to the subgroup P_n^2, where P_n^2 is the subgroup of P_n generated by the squares of all elements.

A well-known family of elements in $B_n[m]$ are *braid Torelli elements*. Consider the symplectic representation (2-1). The kernel of this representation is denoted by $\mathcal{BI}_n$ and it is called *braid Torelli* group. Since the representation (1-1) is a mod m reduction of ρ, every element of $\mathcal{BI}_n$ is actually an element of $B_n[m]$. In particular, $\mathcal{BI}_n$ is generated by squares of Dehn twists about curves surrounding an odd number of marked points in D_n [BMP15]. In terms of half-twists, these elements are of the form

$$(\sigma_1 \cdots \sigma_k)^{2k+2},$$

where $k < n$ is even. This family of elements can be extended. If, for example, we denote by c a curve surrounding an odd number of marked points, then $T_c^2 \in \mathcal{BI}_n$. Other families of elements in $B_n[m]$, such as mod p involutions and centre maps, are described in [Sty18, Section 4].

2.3. Actions of half-twists on symplectic groups. Recall that $B_n \cong \mathrm{Mod}(D_n)$ and $\Sigma \to D_n$ is a two-fold branched cover. The image of the monomorphism $\mathrm{Mod}(D_n) \to \mathrm{Mod}(\Sigma)$ is called the *hyperelliptic mapping class group* denoted by $\mathrm{SMod}(\Sigma)$. Below, we explain how to lift elements of $\mathrm{Mod}(D_n)$ into $\mathrm{SMod}(\Sigma)$. Then we use these lifts to explain their action on $H_1(\Sigma, Q; \mathbb{Z})$.

Let Σ be a genus g surface as in Figure 4. The surface Σ is the 2-fold cover of the disc D_n. Each simple closed curve c_i is a lift of the arc a_i. Recall that σ_i is a half-twist along

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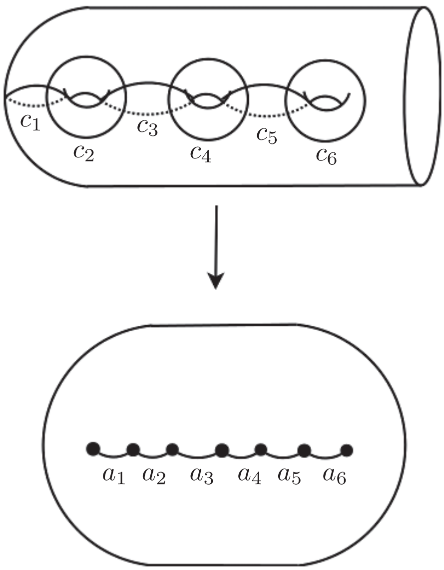

FIGURE 4. An example of a 2-fold cover of a marked disc. The simple closed curve c_i in the genus 3 surface becomes the arc a_i in the disc.

a_i. Then σ_i lifts to the Dehn twist T_{c_i}. This association describes the homomorphism $B_n \to \mathrm{SMod}(\Sigma)$ by $\sigma_i \mapsto T_{c_i}$.

Suppose that Σ is a genus $g \geq 1$ surface with one boundary component (similarly for two boundary components). Let T_c be a Dehn twist about a simple closed curve c and let $[c]$ be its homology class in $\mathrm{H}_1(\Sigma; \mathbb{Z})$. Denote by $t_{[c]}$ a transvection induced by T_c. The action of the transvection $t_{[c]}$ on a homology class u is defined by $t_{[c]}(u) = u + i(u, [c])[c]$, where $i(,)$ is a symplectic form. Therefore, the homomorphism $\rho_m \colon B_n \to \mathrm{Sp}_{n-1}(\mathbb{Z}/m\mathbb{Z})$ is defined by $\sigma_i \mapsto t_{[c_i]}$ (similarly for two boundary components). The next two lemmas describe the images of particular elements of B_n in the symplectic group over $\mathbb{Z}/m\mathbb{Z}$.

LEMMA 2.1. *For $m \geq 2$, we have that $\rho_m(\sigma_i^m) = 1$.*

PROOF. Since σ_i is mapped to the transvection $t_{[c_i]}$, we only need to compute the matrix form of $t_{[c_i]}$. It is easy to calculate the action of $t_{[c_i]}$ based on Figure 3. The result is conjugate to the following matrix:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I,$$

where I is the identity matrix of dimension $n-2$. The result follows by calculating the mth power of the latter matrix over $\mathbb{Z}/m\mathbb{Z}$. □

Lemma 2.1 leads to the question of whether $B_n[m]$ coincides with the group normally generated by σ_i^m. This is generally not the case (see [BDOS24] for further

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details): in fact, $B_n[m]$ is of finite index in B_n, while the group normally generated by σ_i^m is not (except pairs $(n,m) \in \{(3,3),(3,4),(3,5),(4,3),(5,3)\}$; see [Cox59]).

Recall that Δ_n^2 denotes the element $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ in B_n, generating the centre of B_n.

REMARK 2.2. The full twist Δ_n^2 has this notation since it is the square of the Garside element Δ_n, which is another crucial element in braid theory.

LEMMA 2.3. *If n is odd, then $\rho_m(\Delta_n^2)$ has order 2. If n is even, then $\rho_m(\Delta_n^2)$ has order m if $gcd(2,m) = 1$ or it has order $m/2$ if $gcd(2,m) = 2$.*

PROOF. Suppose that n is odd. The lift of $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ to Σ is the product of Dehn twists $(T_{c_1}T_{c_2}\cdots T_{c_{n-1}})^n$. Consider the basis $\{x_i, y_i\}$ depicted in Figure 3. Then the action of the product $(t_{[c_1]}t_{[c_2]}\cdots t_{[c_{n-1}]})^n$ reverses the orientation of x_i, y_i [Sty18]. Thus, it has order 2.

Suppose that n is even. The lift of $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ to Σ is the product $(T_{c_1}T_{c_2}\cdots T_{c_{n-1}})^n$. By the chain relation, the latter product is $T_{q_1}T_{q_2}$, where the curves q_1, q_2 are parallel to the boundary components of Σ [FM12, Proposition 4.12]. Since $[q_1] = [q_2] = y_{n-1}$, we have that $T_{q_1}T_{q_2}$ is mapped into the square transvection $t^2_{y_{n-1}}$. The transvection $t_{y_{n-1}}$ fixes all basis elements $\{x_i, y_i\}$ except x_{n-1}. Hence,

$$t^2_{y_{n-1}}(x_{n-1}) = x_{n-1} + 2y_{n-1}. \qquad \square$$

3. Crystallographic structures and congruence subgroups of the braid groups

We recall the definition of a crystallographic group.

DEFINITION 3.1. A group G is said to be a *crystallographic group* if it is a discrete and uniform subgroup of $\mathbb{R}^N \rtimes \mathrm{O}(N,\mathbb{R}) \subseteq \mathrm{Aff}(\mathbb{R}^N)$.

In [GGO17], there is a characterisation of crystallographic groups that is convenient in our context; see also [Dek96, Section 2.1].

LEMMA 3.2 [GGO17, Lemma 8]. *A group G is crystallographic if and only if there is an integer N and a short exact sequence*

$$1 \longrightarrow \mathbb{Z}^N \longrightarrow G \xrightarrow{\ \zeta\ } \Phi \longrightarrow 1$$

such that

(1) *Φ is finite;*
(2) *the integral representation $\Theta\colon \Phi \to \mathrm{Aut}(\mathbb{Z}^N)$, induced by conjugation on $\mathbb{Z}^N$ and defined by $\Theta(\phi)(x) = \pi x \pi^{-1}$, where $x \in \mathbb{Z}^N$, $\phi \in \Theta$ and $\pi \in G$ is such that $\zeta(\pi) = \phi$ is faithful.*

3.1. A general result on crystallographic groups.

In this subsection, we prove two results that are general and that are applied to the study of crystallographic structures on quotients of the braid group by commutator subgroups of congruence subgroups.

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THEOREM 3.3. *Let $\phi\colon G \to F$ be a surjective homomorphism with F a finite group. Let K denote the kernel of ϕ. Suppose that there is a nontrivial element of the centre of G that does not belong to K. Then the representation $\eta\colon F \to \mathrm{Aut}\big({}^{K}\!/_{[K,K]}\big)$, induced from the action by conjugacy of ${}^{G}\!/_{[K,K]}$ on ${}^{K}\!/_{[K,K]}$, is not injective.*

PROOF. Since $[K, K]$ is characteristic in K and K is normal in G, then $[K, K]$ is normal in G. Hence, we may consider the action by conjugacy of ${}^{G}\!/_{[K,K]}$ on ${}^{K}\!/_{[K,K]}$. This induces a representation $\eta\colon F \to \mathrm{Aut}\big({}^{K}\!/_{[K,K]}\big)$. Let $z \in Z(G)$ be a nontrivial element in the centre of G such that $z \notin K$. We note that $\bar{z}$ does not belong to ${}^{K}\!/_{[K,K]}$. Furthermore, since $z \in Z(G)$,

$$\bar{z}\bar{k}\bar{z}^{-1} = \bar{k}, \quad \text{for every element } \bar{k} \in {}^{K}\!/_{[K,K]}. \tag{3-1}$$

Let $\bar{\phi}(\bar{z}) = t$, where $\bar{\phi}\colon {}^{G}\!/_{[K,K]} \to F$. Notice that t is a nontrivial element in F. So, we conclude that η is not injective since $\eta(t)$ is the identity homomorphism (see (3-1)). □

In the following result, we consider the case where the holonomy representation defined in Lemma 3.2 is not injective and give conditions for the middle group to be a crystallographic group.

THEOREM 3.4. *Consider the short exact sequence $1 \longrightarrow K \longrightarrow G \overset{p}{\longrightarrow} Q \longrightarrow 1$ where K is a free abelian group of finite rank and Q is a finite group such that the representation $\varphi\colon Q \to \mathrm{Aut}(K)$, induced from the action by conjugacy, is not injective. Suppose that the group $p^{-1}(\mathrm{Ker}(\varphi))$ is torsion free. Then G is a crystallographic group with holonomy group ${}^{Q}\!/_{\mathrm{Ker}(\varphi)}$.*

PROOF. First, we note that $p^{-1}(\mathrm{Ker}(\varphi))$ is a Bieberbach group, since it is finitely generated, torsion free and virtually abelian; see [Dek96, Theorem 3.1.3(4)].

Now, we prove that $p^{-1}(\mathrm{Ker}(\varphi))$ is free abelian. Since $p^{-1}(\mathrm{Ker}(\varphi))$ is a Bieberbach group, it fits in a short exact sequence $1 \to A \to p^{-1}(\mathrm{Ker}(\varphi)) \to F \to 1$ where F is a finite group and A is a free abelian group containing K as a normal subgroup of finite index. Suppose now that F is not the trivial group. Let $x \in p^{-1}(\mathrm{Ker}(\varphi))$ be an element that is mapped onto a nontrivial element in F. We know that the induced map $F \to \mathrm{Aut}(A)$ is injective, so conjugation by x induces a nontrivial automorphism of A. However, since K is of finite index in the free abelian group A, this implies that conjugation by x also induces a nontrivial automorphism of K. But this is not possible since $x \in p^{-1}(\mathrm{Ker}(\varphi))$.

Hence, $p^{-1}(\mathrm{Ker}(\varphi))$ is free abelian and we obtain the sequence

$$1 \longrightarrow p^{-1}(\mathrm{Ker}(\varphi)) \longrightarrow G \overset{\bar{p}}{\longrightarrow} {}^{Q}\!/_{\mathrm{Ker}(\varphi)} \longrightarrow 1$$

such that the middle group is a crystallographic group. □

3.2. Crystallographic structures and congruence subgroups of braid groups. In this subsection, we study a quotient of B_n, namely, ${}^{B_n}\!/_{[B_n[m], B_n[m]]}$. Since

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$B_n/[B_n[2], B_n[2]]$ is crystallographic [GGO17, Proposition 1], being isomorphic to the crystallographic braid group $B_n/[P_n, P_n]$, it is reasonable to ask whether $B_n/[B_n[m], B_n[m]]$ is crystallographic for any positive integer m. Here we give conditions for this statement to hold.

The following short exact sequence

$$1 \longrightarrow B_n[m] \longrightarrow B_n \xrightarrow{\rho_m} \rho_m(B_n) \longrightarrow 1$$

induces a short exact sequence on the quotients

$$1 \longrightarrow B_n[m]/[B_n[m], B_n[m]] \longrightarrow B_n/[B_n[m], B_n[m]] \xrightarrow{\overline{\pi}} \rho_m(B_n) \longrightarrow 1. \tag{3-2}$$

The action by conjugacy of $B_n/[B_n[m], B_n[m]]$ on $B_n[m]/[B_n[m], B_n[m]]$ induces a homomorphism

$$\Theta_m\colon \rho_m(B_n) \to \mathrm{Aut}\left(B_n[m]/[B_n[m], B_n[m]]\right). \tag{3-3}$$

As a consequence of Theorem 3.3, we have the following result.

PROPOSITION 3.5. *The representation* $\Theta_m\colon \rho_m(B_n) \to \mathrm{Aut}\left(B_n[m]/[B_n[m], B_n[m]]\right)$*, induced from the action by conjugacy of* B_n *on* $B_n[m]$*, is injective if and only if* $m = 2$*.*

PROOF. For $m = 2$, the abelian group $B_n[2]/[B_n[2], B_n[2]]$ has finite rank and is torsion free. Furthermore, Θ_2 in injective; see [GGO17, Proof of Proposition 1].

Let $m \geq 3$. Recall that the element $\Delta_n^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ represents the full twist on $\mathrm{Mod}(D_n) \cong B_n$, which generates the centre of B_n. From Lemma 2.3, for any n, the element $\rho_m(\Delta_n^2)$ is nontrivial and of finite order. Thus, $\Delta_n^2 \notin B_n[m]$. Therefore, the induced element in $B_n/[B_n[m], B_n[m]]$ does not belong to $B_n[m]/[B_n[m], B_n[m]]$. From Theorem 3.3, the homomorphism $\Theta_m\colon \rho_m(B_n) \to \mathrm{Aut}\left(B_n[m]/[B_n[m], B_n[m]]\right)$ is not injective. □

Since the representation Θ_m is not injective for $m \geq 3$, we cannot apply Lemma 3.2 in this case. However, we may give general conditions such that the group $B_n/[B_n[m], B_n[m]]$ is crystallographic. We have the following result about crystallographic structures and quotients of braid groups by commutators of congruence subgroups.

THEOREM 3.6. *Let* $n \geq 3$ *be an odd integer and let* $m \geq 3$ *be a prime number. If the abelian group* $B_n[m]/[B_n[m], B_n[m]]$ *is torsion free, then the group* $B_n/[B_n[m], B_n[m]]$ *is crystallographic with dimension equal to rank*$\left(B_n[m]/[B_n[m], B_n[m]]\right)$ *and holonomy group* $\rho_m(B_n)/Z(\rho_m(B_n))$*.*

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PROOF. From Theorem 3.4, if $\overline{\rho_m}^{-1}(\mathrm{Ker}(\Theta_m))$ is torsion free, where ρ_m and Θ_m are the homomorphisms defined in (3-2) and (3-3), respectively, then the group ${B_n}/{[B_n[m], B_n[m]]}$ is crystallographic.

We note that the $\mathrm{Ker}(\Theta_m)$ is isomorphic to $Z(\rho_m(B_n))$ the centre of $\rho_m(B_n)$, since the full twist Δ_n^2 generates the centre of B_n and $\rho_m(\Delta_n^2)$ belongs to the normal subgroup $\mathrm{Ker}(\Theta_m)$ of the symplectic group $\rho_m(B_n)$. Recall that, under the assumptions of the statement, $Z(\rho_m(B_n))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We consider now the following short exact sequence:

$$1 \longrightarrow {B_n[m]}/{[B_n[m], B_n[m]]} \longrightarrow \overline{\rho_m}^{-1}(\mathrm{Ker}(\Theta_m)) \xrightarrow{\overline{\rho_m}} Z(\rho_m(B_n)) \longrightarrow 1$$

such that the kernel is torsion free (by hypothesis), the class of the element $\Delta_n^2 \in B_n$ is a nontrivial element of $\overline{\rho_m}^{-1}(\mathrm{Ker}(\Theta_m))$ and $1 \neq \overline{\Delta_n^4} \in {B_n[m]}/{[B_n[m], B_n[m]]}$.

Applying a standard method to give presentations for group extensions [Joh90, Ch. 10] and using the fact that the full twist generates the centre of B_n, we conclude that the middle group $\overline{\rho_m}^{-1}(\mathrm{Ker}(\Theta_m))$ is free abelian and its rank corresponds to the rank of the free abelian group ${B_n[m]}/{[B_n[m], B_n[m]]}$. □

REMARK 3.7. As far as we know, it is still an open problem whether ${B_n[m]}/{[B_n[m], B_n[m]]}$ is torsion free for any n and m except for a few cases. It is well known that the group ${B_n[2]}/{[B_n[2], B_n[2]]}$ is free abelian of rank $\binom{n}{2}$. Also, the groups ${B_3[3]}/{[B_3[3], B_3[3]]}$ and ${B_3[4]}/{[B_3[4], B_3[4]]}$ are torsion free of ranks 4 and 6, respectively; see [BDOS24].

3.3. Symmetric quotients of congruence subgroups of braid groups. From the definition of congruence subgroups, we get an inclusion $\iota\colon B_n[m] \to B_n$ that induces a homomorphism $\bar{\iota}\colon {B_n[m]}/{[P_n, P_n] \cap B_n[m]} \to {B_n}/{[P_n, P_n]}$. In general, $\bar{\iota}$ is not an isomorphism. In the following result, we study it in more detail.

THEOREM 3.8. *Let m be an odd positive integer and let $n \geq 3$. The homomorphism induced from the inclusion $\iota\colon B_n[m] \to B_n$,*

$$\bar{\iota}\colon {B_n[m]}/{[P_n, P_n] \cap B_n[m]} \to {B_n}/{[P_n, P_n]},$$

is injective. Furthermore, the group $\bar{\iota}\left({B_n[m]}/{[P_n, P_n] \cap B_n[m]}\right)$ is a normal proper subgroup of ${B_n}/{[P_n, P_n]}$ such that the quotient is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^{n(n-1)/2}$.

REMARK 3.9. For $n = 2$, the quotient groups of Theorem 3.8 are isomorphic.

Before delving into the proof, we state two technical lemmas that are needed.

LEMMA 3.10. *Let N, H, G groups be such that $H \leq G$ and N is a normal subgroup of G. Then the inclusion homomorphism $\iota\colon H \hookrightarrow G$ induces an injective homomorphism*

$$\kappa\colon {H}/{N \cap H} \to {G}/{N}.$$

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LEMMA 3.11. *Consider the following commutative diagrams of (vertical and horizontal) short exact sequences of groups in which every square is commutative:*

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & U & \xrightarrow{\beta_1} & V & \xrightarrow{\pi_1} & W & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \zeta} & & \\ 1 & \longrightarrow & X & \xrightarrow{\beta_2} & Y & \xrightarrow{\pi_2} & Z & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & & & \\ & & R & \xrightarrow{\mu} & S & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & A & \xrightarrow{\iota_1} & B & \xrightarrow{\rho_1} & C & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \\ 1 & \longrightarrow & D & \xrightarrow{\iota_2} & E & \xrightarrow{\rho_2} & F & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \\ & & & & G & \xrightarrow{\phi} & H & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array}$$

(1) (a) *If β_i is an inclusion, for $i = 1, 2$, and ζ is an isomorphism, then μ is an isomorphism.*

(b) *If α_i is an inclusion, for $i = 1, 2$, and μ is an isomorphism, then ζ is an isomorphism.*

(2) *Suppose that, for $i = 1, 2$, the homomorphisms ι_i and ψ_i are inclusions. Then η is an isomorphism if and only if ϕ is.*

Lemma 3.10 is a simple consequence of the standard isomorphism theorem between $H/N \cap H$ and the subgroup NH/N of G/N, while Lemma~3.11 can be easily proven using diagram chasing.

PROOF OF THEOREM 3.8. From [ABGH20, Theorem 3.1 and its proof], we have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B_n[2m] & \longrightarrow & B_n[m] & \xrightarrow{\tau_m} & S_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \iota} & & \| & & \\ 1 & \longrightarrow & P_n & \longrightarrow & B_n & \xrightarrow{\tau} & S_n & \longrightarrow & 1 \end{array}$$

where τ is the natural surjective homomorphism that sends each braid generator σ_i to the transposition $(i,\ i+1)$, τ_m is the restriction of τ to the subgroup $B_n[m]$, ι is the natural inclusion from the definition of congruence subgroups and ψ is the restriction of ι to the subgroup $B_n[2m]$.

Now, we consider the following diagram induced from the commutative square on the left, where the vertical arrows on this square are inclusion homomorphisms and $\psi|$

is the restriction of ψ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [P_n,P_n]\cap B_n[2m] & \longrightarrow & B_n[2m] & \longrightarrow & {}^{B_n[2m]}\!\big/_{[P_n,P_n]\cap B_n[2m]} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi|} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \overline{\psi}} & & \\ 1 & \longrightarrow & [P_n,P_n] & \longrightarrow & P_n & \longrightarrow & {}^{P_n}\!\big/_{[P_n,P_n]} & \longrightarrow & 1 \end{array}$$

From Lemma 3.10, the third arrow $\overline{\psi}$ on the right is also injective. Since ${}^{P_n}\!\big/_{[P_n,P_n]}$ is a free abelian group of rank $n(n-1)/2$, then ${}^{B_n[2m]}\!\big/_{[P_n,P_n]\cap B_n[2m]}$ is a free abelian group of finite rank, at most $n(n-1)/2$. From [ABGH20, Corollary 2.4], the element $A_{i,j}^m$ belongs to $B_n[2m]$ for all $1 \le i < j \le n$, where $\{A_{i,j} \mid 1 \le i < j \le n\}$ is the set of Artin generators of P_n. Since ${}^{P_n}\!\big/_{[P_n,P_n]}$ is generated by the set of cosets $\{\overline{A_{i,j}} \mid 1 \le i < j \le n\}$, it follows that $\{\overline{A_{i,j}^m} \mid 1 \le i < j \le n\}$ is a basis of ${}^{B_n[2m]}\!\big/_{[P_n,P_n]\cap B_n[2m]}$, so it has rank $n(n-1)/2$. Furthermore, from the above,

$$\frac{{}^{P_n}\!\big/_{[P_n,P_n]}}{\overline{\psi}\Big({}^{B_n[2m]}\!\big/_{[P_n,P_n]\cap B_n[2m]}\Big)} \cong (\mathbb{Z}/m\mathbb{Z})^{n(n-1)/2}. \tag{3-4}$$

Considering [BPS22, Proposition 3.1], Arnol'd's result $B_n[2] = P_n$ and with some set theoretical equivalences, we can see that

$$\begin{aligned} [P_n,P_n]\cap B_n[m] &= ([P_n,P_n]\cap P_n)\cap B_n[m] \\ &= [P_n,P_n]\cap (P_n \cap B_n[m]) \\ &= [P_n,P_n]\cap B_n[2m]. \end{aligned}$$

The following diagram is induced from the commutative square on the left, where the vertical arrows on this square are inclusion homomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [P_n,P_n]\cap B_n[m] & \longrightarrow & B_n[m] & \longrightarrow & {}^{B_n[m]}\!\big/_{[P_n,P_n]\cap B_n[m]} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \overline{\iota}} & & \\ 1 & \longrightarrow & [P_n,P_n] & \longrightarrow & B_n & \longrightarrow & {}^{B_n}\!\big/_{[P_n,P_n]} & \longrightarrow & 1 \end{array}$$

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From Lemma 3.10, the third arrow $\bar{\iota}$ on the right is also injective. With this information and (3-4), we construct the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & {}^{B_n[2m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]} & \hookrightarrow & {}^{B_n[m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]} & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \bar{\iota}} & & \| & & \\
1 & \longrightarrow & {}^{P_n}\!/\!{}_{[P_n,P_n]} & \hookrightarrow & {}^{B_n}\!/\!{}_{[P_n,P_n]} & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & (\mathbb{Z}/m\mathbb{Z})^{n(n-1)/2} & \xrightarrow{\ \mu\ } & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

From Lemma 3.11 item (1), the homomorphism μ is an isomorphism and we get the result. □

Let $n \geq 3$. Recall from [ABGH20, Lemma 2.3] that the element σ_i^m belongs to $B_n[m]$ for all $1 \leq i \leq n-1$, where $\{\sigma_i \mid 1 \leq i \leq n-1\}$ is the set of Artin generators of B_n. Although the set map $\xi \colon B_n \to B_n[m]$ defined by $\xi(\sigma_i) = \sigma_i^m$, for all $1 \leq i \leq n-1$, is not a homomorphism, when m is odd, it induces an isomorphism on the quotient groups $\bar{\xi} \colon {}^{B_n}\!/\!{}_{[P_n,P_n]} \to {}^{B_n[m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]}$, as we show in the next result.

THEOREM 3.12. *Let m be a positive integer and let $n \geq 3$. Consider the map*

$$\bar{\xi} \colon {}^{B_n}\!/\!{}_{[P_n,P_n]} \to {}^{B_n[m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]}$$

defined by $\bar{\xi}(\sigma_i) = \sigma_i^m$ for all $1 \leq i \leq n-1$. If m is odd, then $\bar{\xi}$ is an isomorphism. As a consequence, for $n \geq 3$ and m odd, ${}^{B_n[m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]}$ is a crystallographic group of dimension $n(n-1)/2$ and holonomy group S_n.

PROOF. Suppose that $n \geq 3$ and m is an odd positive integer and consider the map

$$\bar{\xi} \colon {}^{B_n}\!/\!{}_{[P_n,P_n]} \to {}^{B_n[m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]}$$

defined by $\bar{\xi}(\sigma_i) = \sigma_i^m$ for all $1 \leq i \leq n-1$. To show that $\bar{\xi}$ is a homomorphism, it is enough to verify that Artin's relations are preserved by $\bar{\xi}$.

Let $1 \leq i, j \leq n$ such that $|i-j| \geq 2$. From Artin's relation $\sigma_i\sigma_j = \sigma_j\sigma_i$, we obtain $\sigma_i^m\sigma_j^m = \sigma_j^m\sigma_i^m$ in $B_n[m]$, which is then preserved by $\bar{\xi}$.

Let $1 \leq i \leq n-2$. The equality $\sigma_i^m\sigma_{i+1}^m\sigma_i^m\sigma_{i+1}^{-m}\sigma_i^{-m}\sigma_{i+1}^{-m} = 1$ is valid in ${}^{B_n[m]}\!/\!{}_{[P_n,P_n]\cap B_n[m]}$. In fact, suppose that $m = 2k+1$, then from the action of

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conjugation in $B_n/[P_n, P_n]$ described in Section 2.1,

$$\begin{aligned}
\sigma_i^m \sigma_{i+1}^m \sigma_i^m \sigma_{i+1}^{-m} \sigma_i^{-m} \sigma_{i+1}^{-m} &= A_{i,i+1}^k \sigma_i A_{i+1,i+2}^k \sigma_{i+1} A_{i,i+1}^k \sigma_i \sigma_{i+1}^{-1} A_{i+1,i+2}^{-k} \sigma_i^{-1} A_{i,i+1}^{-k} \sigma_{i+1}^{-1} A_{i+1,i+2}^{-k} \\
&= A_{i,i+1}^k A_{i,i+2}^k A_{i+1,i+2}^k A_{i,i+1}^{-k} A_{i,i+2}^{-k} A_{i+1,i+2}^{-k} \\
&= 1 \in B_n/[P_n, P_n].
\end{aligned}$$

From Theorem 3.8, the homomorphism

$$\bar{\iota}\colon B_n[m]/[P_n, P_n] \cap B_n[m] \to B_n/[P_n, P_n]$$

is injective, and then $\sigma_i^m \sigma_{i+1}^m \sigma_i^m \sigma_{i+1}^{-m} \sigma_i^{-m} \sigma_{i+1}^{-m} = 1$ in $B_n[m]/[P_n, P_n] \cap B_n[m]$.

Now, consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_n/[P_n, P_n] & \longrightarrow & B_n/[P_n, P_n] & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \bar{\xi}|} & & \big\downarrow{\scriptstyle \bar{\xi}} & & \big\| & & \\
1 & \longrightarrow & B_n[2m]/[P_n, P_n] \cap B_n[m] & \longrightarrow & B_n[m]/[P_n, P_n] \cap B_n[m] & \longrightarrow & S_n & \longrightarrow & 1
\end{array}$$

As seen in the proof of Theorem 3.8, the free abelian groups $B_n[2m]/[P_n, P_n] \cap B_n[2m]$ and $P_n/[P_n, P_n]$ of rank $n(n-1)/2$ have bases $\{\overline{A_{i,j}^m} \mid 1 \le i < j \le n\}$ and $\{\overline{A_{i,j}} \mid 1 \le i < j \le n\}$, respectively. Since

$$\bar{\xi}|\colon P_n/[P_n, P_n] \to B_n[2m]/[P_n, P_n] \cap B_n[2m]$$

is a homomorphism such that $\bar{\xi}|(A_{i,j}) = A_{i,j}^m$, for all $1 \le i < j \le n$, it is an isomorphism. Therefore, from the five lemma, $\bar{\xi}$ is an isomorphism.

The last part follows from the corresponding result on the crystallographic braid group $B_n/[P_n, P_n]$; see [GGO17, Proposition 1]. □

A group G is called *co-Hopfian* if it is not isomorphic to any of its proper subgroups, or equivalently, if every injective homomorphism $\phi\colon G \to G$ is surjective. It is known that the braid group B_n is not co-Hopfian. However, for $n \ge 4$, the quotient by its centre is co-Hopfian; see [BM06].

COROLLARY 3.13. *Let $n \ge 3$. The crystallographic braid group $B_n/[P_n, P_n]$ is not co-Hopfian.*

PROOF. It follows from Theorems 3.8 and 3.12. □

REMARK 3.14. We note that in this paper, we do not use Lemma 3.11 item (2). However, it will be useful in [BDOS24].

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References

[ABGH20] J. Appel, W. Bloomquist, K. Gravel and A. Holden, 'On quotients of congruence subgroups of braid groups', Preprint, 2020, arXiv:2011.13876.

[A'C79] N. A'Campo, 'Tresses, monodromie et le groupe symplectique', *Comment. Math. Helv.* **54**(2) (1979), 318–327.

[Arn68] V. I. Arnol'd, 'A remark on the branching of hyperelliptic integrals as functions of the parameters', *Funkcional. Anal. i Priložen.* **2**(3) (1968), 1–3.

[Art25] E. Artin, 'Theorie der Zöpfe', *Abh. Math. Semin. Univ. Hambg.* **4** (1925), 47–72.

[BB05] J. S. Birman and T. E. Brendle, 'Braids: a survey', in: *Handbook of Knot Theory* (Elsevier B. V., Amsterdam, 2005), 19–103.

[BB21] V. Bharathram and J. Birman, 'On the Burau representation of B4', *Involve* **14**(1) (2021), 143–154.

[BDOS24] P. Bellingeri, C. Damiani, O. Ocampo and C. Stylianakis, 'Congruence subgroups of braid groups and crystallographic quotients. Part II', Preprint, 2024, arXiv:2404.05804.

[BGM22] P. Bellingeri, J. Guaschi and S. Makri, 'Unrestricted virtual braids and crystallographic braid groups', *Bol. Soc. Mat. Mex. (3)* **28**(3) (2022), Article no. 63, 16 pages.

[Big99] S. J. Bigelow, 'The Burau representation is not faithful for $n = 5$', *Geom. Topol.* **3** (1999), 397–404.

[Big01] S. J. Bigelow, 'Braid groups are linear', *J. Amer. Math. Soc.* **14**(2) (2001), 471–486.

[BM06] R. W. Bell and D. Margalit, 'Braid groups and the co-Hopfian property', *J. Algebra* **303**(1) (2006), 275–294.

[BM18] T. E. Brendle and D. Margalit, 'The level four braid group', *J. reine angew. Math.* **735** (2018), 249–264.

[BM20] V. Beck and I. Marin, 'Torsion subgroups of quasi-abelianized braid groups', *J. Algebra* **558** (2020), 3–23.

[BMP15] T. E. Brendle, D. Margalit and A. Putman, 'Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at $t = -1$', *Invent. Math.* **200**(1) (2015), 263–310.

[BMS67] H. Bass, J. Milnor and J.-P. Serre, 'Solution of the congruence subgroup problem for sl_n $(n \geq 3)$ and sp_{2n} $(n \geq 2)$', *Publ. Math. Inst. Hautes Études Sci.* **33** (1967), 59–137.

[BPS22] W. Bloomquist, P. Patzt and N. Scherich, 'Quotients of braid groups by their congruence subgroups,' *Proc. Amer. Math. Soc.*, to appear.

[BT18] A. Beridze and P. Traczyk, 'Burau representation for $n = 4$', *J. Knot Theory Ramifications* **27**(3) (2018), Article no. 1840002, 6 pages.

[Bur35] W. Burau, 'Über zopfgruppen und gleichsinnig verdrillte verkettungen', *Abh. Math. Semin. Univ. Hambg.* **11** (1935), 179–186.

[CdSJO23] P. C. Cerqueira dos Santos Júnior and O. Ocampo, 'Virtual braid groups, virtual twin groups and crystallographic groups', *J. Algebra* **632** (2023), 567–601.

[Cho48] W.-L. Chow, 'On the algebraical braid group', *Ann. of Math. (2)* **49** (1948), 654–658.

[Cox59] H. S. M. Coxeter, 'Factor groups of the braid group', in: *Proceedings of the Fourth Canadian Mathematical Congress (Banff, 1957)* (University of Toronto Press, 1959), 95–122.

[Dat22] A. Datta, 'A strong characterization of the entries of the Burau matrices of 4-braids: The Burau representation of the braid group B_4 is faithful almost everywhere', Preprint, 2022, arXiv:2209.10826.

https://doi.org/10.1017/S1446788724000089 Published online by Cambridge University Press

[Dek96] K. Dekimpe, *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Mathematics, 1639 (Springer-Verlag, Berlin, 1996).

[FM12] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series, 49 (Princeton University Press, Princeton, NJ, 2012).

[GG16] J.-M. Gambaudo and É. Ghys, 'Braids and signatures', *Bull. Soc. Math. France* **133**(4) (2005), 541–579.

[GGO17] D. L. Gonçalves, J. Guaschi and O. Ocampo, 'A quotient of the Artin braid groups related to crystallographic groups', *J. Algebra* **474** (2017), 393–423.

[GGOP21] D. L. Gonçalves, J. Guaschi, O. Ocampo and C. Pereiro, 'Crystallographic groups and flat manifolds from surface braid groups', *Topology Appl.* **293** (2021), Article no. 107560, 16 pages.

[Joh90] D. L. Johnson, *Presentations of Groups*, London Mathematical Society Student Texts, 15 (Cambridge University Press, Cambridge, 1990).

[KM22] K. Kordek and D. Margalit, 'Representation stability in the level 4 braid group', *Math. Z.* **302** (2022), 433–482.

[KNS24] P. Kumar, T. K. Naik and M. Singh, 'Congruence subgroups and crystallographic quotients of small Coxeter groups', *Forum Math.* **36**(1) (2024), 193–213.

[Kra02] D. Krammer, 'Braid groups are linear', *Ann. of Math. (2)* **155**(1) (2002), 131–156.

[KT08] C. Kassel and V. Turaev, *Braid Groups*, Graduate Texts in Mathematics, 247 (Springer, New York, 2008).

[LP93] D. D. Long and M. Paton, 'The Burau representation is not faithful for $n \geq 6$', *Topology* **32**(2) (1993), 439–447.

[MK99] K. Murasugi and B. I. Kurpita, *A Study of Braids*, Mathematics and Its Applications, 484 (Kluwer Academic Publishers, Dordrecht, 1999).

[Moo91] J. A. Moody, 'The Burau representation of the braid group B_n is unfaithful for large n', *Bull. Amer. Math. Soc. (N.S.)* **25**(2) (1991), 379–384.

[MP69] W. Magnus and A. Peluso, 'On a theorem of V. I. Arnol'd', *Comm. Pure Appl. Math.* **22** (1969), 683–692.

[Nak21] T. Nakamura 'The cohomology class of the mod 4 braid group', *Int. J. Algebra Comput.* **32** (2021), 1125–1159.

[PV96] B. Perron and J. P. Vannier, 'Groupe de monodromie géométrique des singularités simples', *Math. Ann.* **306**(1) (1996), 231–245.

[Rag04] M. S. Raghunathan, 'The congruence subgroup problem', *Proc. Math. Sci.* **114** (2004), 299–308.

[Sty18] C. Stylianakis, 'Congruence subgroups of braid groups', *Internat. J. Algebra Comput.* **28**(2) (2018), 345–364.

[Tit66] J. Tits, 'Normalisateurs de tores i: Groupes de coxeter étendus', *J. Algebra* **4** (1966), 96–116.

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