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QUASI MULTIPLICATION AND K-GROUPS

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Abstract

We give a negative answer to the question raised by Mart Abel about whether his proposed definition of K_0 and K_1 groups in terms of quasi multiplication is indeed equivalent to the established ones in algebraic K-theory.

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1. Introduction

In algebraic K-theory (and also in topological K-theory), both K_0 and K_1 groups of a nonunital ring (or algebra) are defined in terms of its unitization or any unital ring containing it as an ideal [W], since invertible matrices over a ring involved are needed in their construction. This additional unitization step makes the definition of K-groups seemingly somewhat unnatural, and sometimes a little inconvenient in discussion. On the other hand, in the noncommutative ring theory, for the general study of the Jacobson radical of a ring, the notion of quasi multiplication is successfully introduced and utilized to avoid the use of unitization of a ring [P], even though the invertibility of elements in a ring is intimately related to the notion of the Jacobson radical. In [A], Abel proposed a new definition of K_0 and K_1 groups, denoted as $\overline{K_0}$ and $\overline{K_1}$ groups, utilizing the notion of quasi multiplication and hence avoiding the step of unitization for the case of a nonunital ring. Abel raised the question of whether the $\overline{K_i}$ groups are equivalent to the K_i groups at ICTAA, Tartu, 2008 and also at the International Conference on Rings and Algebras in Honor of Professor P.H. Lee, Taipei, 2011.

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Abel's definition of K-groups in term of quasi multiplication is interesting, and seems to have the potential to simplify the discussion and possibly some proofs involving nonunital rings in the study of K-groups. For example, the 'Bott element', an important object in expressing algebraically the Bott periodicity of topological K-theory, is an element of the K_0 group of the *nonunital* algebra $C_0(\mathbb{R}^2)$ of $\mathbb{C}$-valued continuous functions vanishing at ∞ on $\mathbb{R}^2$. Unfortunately, the authors find that Abel's new definition is not equivalent to the established definition, and give counterexamples in this paper. Some of the results might be known to experts, but they are not widely known and not noted in the literature as far as the authors know.

2. Algebraic K_0 and K_1 groups

In this section, we recall the established notion of K_0 and K_1 groups. For any ring R, we denote by $R^+ := \{(r, z) : r \in R, z \in \mathbb{Z}\}$ with

$$(r, z)(r', z') := (rr' + zr' + z'r, zz')$$

the unitization of R, by $M_n(R)$ the space of $n \times n$ matrices with entries in R where $n \in \mathbb{N}$, and by $a \oplus b$ the $(n + m) \times (n + m)$ matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ for matrices $a \in M_n(R)$ and $b \in M_m(R)$. For a *unital* ring R, we use $\mathrm{GL}(R)$ to denote the group of invertible elements of R.

It is easy to see that the *direct limit*

$$M_\infty(R) := \varinjlim_{n\to\infty} M_n(R)$$

of the directed system

$$a \in M_n(R) \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1}(R)$$

of ring monomorphisms can be identified as the space of all infinite matrices $(a_{ij})_{1\le i,j<\infty}$ with each $a_{ij} \in R$ such that only *finitely many* a_{ij} are *nonzero*. Note that $M_\infty(R)$ carries canonically a *nonunital* ring structure inherited from the $M_n(R)$.

Let $\mathrm{Idem}(R)$ denote the set of all *idempotents* of R, that is, those elements $x \in R$ with $x^2 = x$. Then we have a directed system

$$p \in \mathrm{Idem}_n(R) := \mathrm{Idem}(M_n(R)) \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Idem}(M_{n+1}(R))$$

with the direct limit

$$\mathrm{Idem}_\infty(R) := \varinjlim_{n\to\infty} \mathrm{Idem}(M_n(R)) \subset M_\infty(R).$$

On the other hand, for a *unital* ring R, the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices form a direct system

$$u \in \mathrm{GL}_n(R) \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R)$$

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with its direct limit

$$\mathrm{GL}_\infty(R) := \varinjlim_{n\to\infty} \mathrm{GL}_n(R) \subset M_\infty(R)^+,$$

where $\left(\begin{smallmatrix} a & 0 \\ 0 & zI_\infty \end{smallmatrix}\right)$ is identified with $\left(\begin{smallmatrix} a-zI_n & 0 \\ 0 & 0_\infty \end{smallmatrix}\right) + z \in M_\infty(R)^+$ for any $a \in M_n(R)$ with $n \in \mathbb{N}$ and $z \in \mathbb{Z}$. We can view $M_\infty(R)^+$ as consisting of $\left(\begin{smallmatrix} a & 0 \\ 0 & zI_\infty \end{smallmatrix}\right)$ and see that

$$\mathrm{GL}_\infty(R) = (M_\infty(R) + 1) \cap \mathrm{GL}(M_\infty(R)^+).$$

Note that

$$\mathrm{GL}(M_\infty(R)^+) = \mathrm{GL}_\infty(R) \sqcup (-\mathrm{GL}_\infty(R)).$$

For a *unital* ring R, let $\approx$ be the *equivalence relation* on $\mathrm{Idem}_\infty(R)$ defined as $p \approx q$ if and only if there exists $u \in \mathrm{GL}_\infty(R)$ (or equivalently $u \in \mathrm{GL}_1(M_\infty(R)^+)$) such that $p = uqu^{-1}$, where $p, q \in \mathrm{Idem}_\infty(R)$. We denote by $[a] \in \mathrm{Idem}_\infty(R)/\approx$ the equivalence class of $a \in \mathrm{Idem}_\infty(R)$.

Note that $\oplus$ is not well defined on $\mathrm{Idem}_\infty(R)$, since $a \oplus 0_k = a$ in $\mathrm{Idem}_\infty(R)$ for any $a \in M_n(R)$ and $k \in \mathbb{N}$ where 0_k is the zero matrix in $M_k(R)$, but $a \oplus 0_k \oplus b \neq a \oplus b$ in $\mathrm{Idem}_\infty(R)$ for any nonzero $b \in M_m(R)$. However, it is easy to see that $a \oplus b \approx b \oplus a$ for any $a \in M_n(R)$ and $b \in M_m(R)$, and hence $a \oplus 0_k \oplus b \oplus 0_l \approx a \oplus b \oplus 0_k \oplus 0_l$ for any $k, l \in \mathbb{N}$. So $\oplus$ is well defined on $\mathrm{Idem}_\infty(R)/\approx$, and $(\mathrm{Idem}_\infty(R)/\approx, \oplus)$ becomes an *abelian semigroup*.

For an abelian semigroup S, we use $\mathcal{G}(S)$ to denote the *Grothendieck group* [L] of S.

In algebraic K-theory, if R is a *unital* ring, then $K_0(R)$ is defined as the Grothendieck group $\mathcal{G}(\mathrm{Idem}_\infty(R)/\approx, \oplus)$. More generally, $K_0(R)$ is defined as the *kernel* of the canonical group homomorphism $\mu_R : K_0(R^+) \to K_0(\mathbb{Z})$ induced by 'modulo R'. When R is unital, this K_0 group is isomorphic to the one that we defined first, because in this case, R^+ is isomorphic to the *ring* direct sum $R \oplus \mathbb{Z}$.

Similarly, if R is a *unital* ring, then $K_1(R)$ is defined as the abelianization

$$\mathrm{GL}_\infty(R)/[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]$$

of the group $\mathrm{GL}_\infty(R)$, where $[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]$ denotes the commutator subgroup of $\mathrm{GL}_\infty(R)$ generated by elements of the form $aba^{-1}b^{-1}$. More generally, $K_1(R)$ is defined as the *kernel* of the canonical group homomorphism $\mu_R : K_1(R^+) \to K_1(\mathbb{Z})$ induced by 'modulo R'. When R is unital, this K_1 group is isomorphic to the one that we defined first.

3. Abel's K_0 and K_1 groups

In this section, we recall the new definition of K_0 and K_1 groups proposed by Abel in terms of quasi multiplication.

First, recall that the *quasi multiplication* of a ring R is defined as $x \circ y := x + y - xy$ for $x, y \in R$, and $(R, \circ)$ becomes a *unital* semigroup with 0 as its identity element.

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For a *unital* ring R, the map $\chi \equiv \chi_R : x \in (R, \cdot) \mapsto 1 - x \in (R, \circ)$ is a *unital* semigroup *isomorphism* since $\chi(1) = 0$ and

$$\begin{aligned}\chi(x) \circ \chi(y) &= (1 - x) \circ (1 - y) = (1 - x) + (1 - y) - (1 - x)(1 - y) \\ &= 1 - xy = \chi(xy)\end{aligned}$$

for any $x, y \in R$, where $(R, \cdot)$ is R equipped with the original multiplication operation $\cdot$ as a unital semigroup. However, it should be noted that χ does not preserve the addition, but interestingly, it satisfies

$$\chi(x) + \chi(y) - \chi(z) = \chi(x + y - z).$$

In [A], observing that the condition for an element p in a ring R to be an idempotent, $p^2 = p$, is equivalent to $p \circ p = p$, which is the same as the idempotent condition for p in the semigroup $(R, \circ)$, Abel introduced the definition of a new K_0 group $\overline{K_0}(R)$ as follows.

First, note that the quasi multiplication $\circ$ on matrix rings $M_n(R)$ and $M_\infty(R)$ is compatible with the canonical inclusions between them. We denote the semigroups $(M_n(R), \circ)$ and $(M_\infty(R), \circ)$ as $M_n^\circ(R)$ and $M_\infty^\circ(R)$, respectively, and denote the set of $\circ$-idempotents (that is, idempotents with respect to the operation $\circ$) by $\mathrm{Idem}_n^\circ(R) \subset M_n^\circ(R)$ and $\mathrm{Idem}_\infty^\circ(R) \subset M_\infty^\circ(R)$. So $\mathrm{Idem}_\infty^\circ(R) = \mathrm{Idem}_\infty(R)$.

Note that in a *unital* ring R, an element p is an idempotent, $p^2 = p$ or equivalently $p \circ p = p$, if and only if $1 - p$ is an idempotent,, $(1 - p)^2 = 1 - p$ or equivalently $(1 - p) \circ (1 - p) = 1 - p$. Furthermore, $p \approx q$ for two elements p, q, that is, $upu^{-1} = q$ for some invertible element $u \in R$, if and only if $1 - p \approx 1 - q$, because $u(1 - p)u^{-1} = 1 - upu^{-1}$. On the other hand, since $\chi : x \in (R, \cdot) \mapsto 1 - x \in (R, \circ)$ is a semigroup isomorphism, $1 - p \approx 1 - q$ for two elements $p, q \in R$ if and only if $p \approx_\circ q$, that is, there is a *quasi invertible* element $v \in (R, \circ)$, that is, $v \circ \hat{v} = 0 = \hat{v} \circ v$ for some $\hat{v} \in R$, such that $v \circ p \circ \hat{v} = q$, which is equivalent to $1 - \hat{v} = (1 - v)^{-1}$ and $(1 - v)(1 - p)(1 - \hat{v}) = 1 - q$. We define

$$\mathrm{GL}^\circ(R) \equiv \mathrm{GL}(R, \circ)$$

to be the set of all *quasi invertible* elements of R, and note that

$$\mathrm{GL}^\circ(R) = 1 - \mathrm{GL}(R) \equiv \{1 - u : u \in \mathrm{GL}(R)\}.$$

Applying the above discussion, we get, for $p, q \in \mathrm{Idem}_\infty(R) = \mathrm{Idem}_\infty^\circ(R)$ over a *unital* ring R, that $p \approx q$ or equivalently $I_\infty - p \approx I_\infty - q$, if and only if $p \approx_\circ q$, that is, there exists by definition

$$v \in \mathrm{GL}_\infty^\circ(R) := \mathrm{GL}^\circ(M_\infty(R)) \equiv \mathrm{GL}(M_\infty(R), \circ) = 1 - \mathrm{GL}_\infty(R)$$

such that $v \circ p \circ \hat{v} = q$ where $v \circ \hat{v} = 0$. So

$$[p] \in \mathrm{Idem}_\infty(R)/\approx \;\mapsto\; [p] \in \mathrm{Idem}_\infty^\circ(R)/\approx_\circ$$

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is a bijection that preserves the $\oplus$ operation, and hence $K_0(R) \cong \mathcal{G}(\mathrm{Idem}^\circ_\infty(R)/\approx_\circ, \oplus)$ for a *unital* ring R.

However, the above definitions of $\mathrm{GL}^\circ_\infty(R) := \mathrm{GL}^\circ(M_\infty(R)) \equiv \mathrm{GL}(M_\infty(R), \circ)$, $\mathrm{Idem}^\circ_\infty(R)$, and the equivalence relation $\approx_\circ$ are still *valid* for any *nonunital* ring R. So Abel [A] introduced the new K_0 group defined as

$$\overline{K_0}(R) := \mathcal{G}(\mathrm{Idem}^\circ_\infty(R)/\approx_\circ, \oplus)$$

and raised the question whether $K_0(R) \cong \overline{K_0}(R)$ for all rings R.

Similarly, for a *unital* ring R, since

$$\chi : u \in (\mathrm{GL}_\infty(R), \cdot) \mapsto v := 1 - u \in (\mathrm{GL}^\circ_\infty(R), \circ)$$

is a group isomorphism,

$$K_1(R) = \mathrm{GL}_\infty(R)/[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)] \cong \mathrm{GL}^\circ_\infty(R)/[\mathrm{GL}^\circ_\infty(R), \mathrm{GL}^\circ_\infty(R)]_\circ$$

where the subscript in $[\cdot, \cdot]_\circ$ reminds us that the group operation is $\circ$ and the identity element is 0.

Since $\mathrm{GL}^\circ_\infty(R) := \mathrm{GL}(M_\infty(R), \circ)$ is a well-defined group for any *nonunital* ring R, Abel [A] introduced the new K_1 group defined as

$$\overline{K_1}(R) := \mathrm{GL}^\circ_\infty(R)/[\mathrm{GL}^\circ_\infty(R), \mathrm{GL}^\circ_\infty(R)]_\circ$$

and raised the question whether $K_1(R) \cong \overline{K_1}(R)$ for all rings R.

4. Counterexamples

In this section, we give a negative answer to both of Abel's questions by showing an example of a concrete nonunital ring R with $K_0(R) \not\cong \overline{K_0}(R)$ and another example with the canonical natural homomorphism $\overline{K_1}(R) \to K_1(R)$ induced by the isomorphisms $v \in \mathrm{GL}^\circ_n(R^+) \mapsto 1 - v \in \mathrm{GL}_n(R^+)$ not being an isomorphism.

In the following discussion, for a $\mathbb{C}$-algebra $\mathcal{A}$, we denote by $\mathcal{A}^\sim = \mathcal{A} + \mathbb{C}$ the $\mathbb{C}$-algebra unitization of $\mathcal{A}$, while $\mathcal{A}^+ = \mathcal{A} + \mathbb{Z}$ still denotes the ring unitization of $\mathcal{A}$.

4.1. Counterexample for K_0 group. Let $R := C_0(\mathbb{R}^2)$ be the algebra of all $\mathbb{C}$-valued continuous functions f on $\mathbb{R}^2$ that vanish at infinity, that is, $\lim_{\|(x,y)\|\to\infty} f(x, y) = 0$.

We note that $M_n(C_0(\mathbb{R}^2))$ consists of $n \times n$ matrices with entries in $C_0(\mathbb{R}^2)$, and hence can be identified with the algebra $C_0(\mathbb{R}^2, M_n(\mathbb{C}))$ of $M_n(\mathbb{C})$-valued continuous functions f on $\mathbb{R}^2$ with $\lim_{\|(x,y)\|\to\infty} \|f(x, y)\| = 0$ where for any matrix $A \in M_n(\mathbb{C})$, $\|A\|$ denotes the operator norm

$$\|A\| := \sup\{\|AX\| : X \in \mathbb{C}^n \text{ with } \|X\| = 1\}.$$

Note that if $A \in M_n(\mathbb{C})$ is a nonzero idempotent, that is, $A^2 = A \neq 0$, then $\|A\| \geq 1$ since for any unit vector X in the nonzero range of A, we have $AX = X$ and hence $\|A\| \geq \|AX\| = \|X\| = 1$.

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We claim that $\mathrm{Idem}_\infty^\circ(C_0(\mathbb{R}^2)) = \{0\}$, and hence

$$\overline{K_0}(R) = \mathcal{G}(\mathrm{Idem}_\infty^\circ(R)/\approx_\circ, \oplus) = 0.$$

Indeed, $\mathrm{Idem}_\infty^\circ(C_0(\mathbb{R}^2)) = \mathrm{Idem}_\infty(C_0(\mathbb{R}^2))$. If

$$p \in \mathrm{Idem}_n(C_0(\mathbb{R}^2)) = \mathrm{Idem}(M_n(C_0(\mathbb{R}^2))) = \mathrm{Idem}(C_0(\mathbb{R}^2, M_n(\mathbb{C}))),$$

then $p^2 = p$ as $M_n(\mathbb{C})$-valued functions on $\mathbb{R}^2$ imply that $p(x, y)^2 = p(x, y)$ for all $(x, y) \in \mathbb{R}^2$, and hence $p(x, y)$ is an idempotent in $M_n(\mathbb{C})$, which then implies that either $\|p(x, y)\| = 0$, that is, $p(x, y) = 0$, or $\|p(x, y)\| \geq 1$. By the continuity of $(x, y) \mapsto p(x, y)$ and hence of $(x, y) \mapsto \|p(x, y)\|$, the condition $\lim_{\|(x,y)\|\to\infty} \|p(x, y)\| = 0$ implies that $p(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$, that is, $p = 0$.

On the other hand, it is known that $K_0^{\mathrm{top}}(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$ where K_0^{top} denotes the topological K_0 group for Banach algebras [T]. For completeness, we sketch the proof of this fact.

First we note that $C_0(\mathbb{R}^2)^\sim \cong C(\mathbb{S}^2)$ since $\mathbb{S}^2$ is the one-point compactification of $\mathbb{R}^2$.

It is well known that for a *unital* ring R, the maps

$$p \in \mathrm{Idem}_n(R) \mapsto p(R^n) \in \mathcal{P}(R)$$

induce a canonical semigroup isomorphism $(\mathrm{Idem}_\infty(R)/\approx, \oplus) \to (\mathcal{P}(R)/\cong, \oplus)$, where $\mathcal{P}(R)$ is the collection of all finitely generated projective modules over R [W]. So $K_0(R) \cong \mathcal{G}(\mathcal{P}(R)/\cong, \oplus)$.

On the other hand, for compact Hausdorff spaces X, by Swan's theorem [S], the map $E \mapsto \Gamma(E)$ induces a canonical semigroup isomorphism $(\mathcal{VB}(X)/\cong, \oplus) \to (\mathcal{P}(C(X))/\cong, \oplus)$ where $\mathcal{VB}(X)$ is the collection of all complex vector bundles E over X and $\Gamma(X)$ is the $C(X)$-module of all continuous cross sections of the vector bundle E.

So

$$K_0(C(\mathbb{S}^2)) \cong \mathcal{G}(\mathcal{VB}(\mathbb{S}^2)/\cong, \oplus) \cong \mathbb{Z} \oplus \mathbb{Z}$$

identifying each isomorphism class $[E]$ of a vector bundle E over $\mathbb{S}^2$ with a corresponding $(n, r) \in \mathbb{Z} \oplus \mathbb{Z}$, where $r \geq 0$ is the complex dimension of (each fiber of) E, and $n \in \mathbb{Z}$ records the winding or twisting number along the equator of $\mathbb{S}^2$ when constructing E by gluing together two trivial vector bundles over the upper and lower hemispheres along the equator [H]. For example, the tangent bundle $T\mathbb{S}^2$ of $\mathbb{S}^2$ is identified with $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}$, and is closely related to the *Bott periodicity* in topological K-theory [T].

By definition, $K_0^{\mathrm{top}}(\mathcal{A})$ of a Banach algebra $\mathcal{A}$ is the kernel of the canonical homomorphism

$$K_0(\mathcal{A}^\sim) \mapsto K_0(\mathbb{C}) \cong \mathbb{Z}$$

induced by 'modulo $\mathcal{A}$'. In the case of $\mathcal{A} = C_0(\mathbb{R}^2)$,

$$(n, r) \in \mathbb{Z} \oplus \mathbb{Z} \cong K_0(C(\mathbb{S}^2)) \cong K_0(C_0(\mathbb{R}^2)^\sim) \mapsto r \in K_0(\mathbb{C}) \cong \mathbb{Z}$$

and hence the kernel $K_0^{\mathrm{top}}(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$.

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We claim that there is a surjective homomorphism from $K_0(C_0(\mathbb{R}^2))$ to $K_0^{\mathrm{top}}(C_0(\mathbb{R}^2))$ and hence $K_0(C_0(\mathbb{R}^2)) \not\cong 0$. Indeed, we have the following general result.

Proposition 4.1. *Let $R^\sim = R + \mathbb{F}$ and $R^+ = R + \mathbb{Z}$ be respectively the algebra unitization and the ring unitization of a nonunital algebra R over a field $\mathbb{F}$ of characteristic 0. In the following commuting diagram, all vertical arrows are surjective and the rows are exact:*

$$\begin{array}{ccccccccccc} 0 & \to & K_0(R) \equiv \ker(\mu_*^+) & \to & K_0(R^+) & \xrightarrow{\mu_*^+} & K_0(\mathbb{Z}) & = & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow_{\iota_*} & & \downarrow_{\kappa_*} & & \| & & \\ 0 & \to & \ker(\mu_*^\sim) & \to & K_0(R^\sim) & \xrightarrow{\mu_*^\sim} & K_0(\mathbb{F}) & = & \mathbb{Z} & \to & 0 \end{array}$$

where the homomorphisms $\mu_*^\sim : K_0(R^\sim) \to K_0(\mathbb{F})$ and $\mu_*^+ : K_0(R^+) \to K_0(\mathbb{Z})$ are canonically induced by the 'modulo R' maps $\mu^\sim : R^\sim \to \mathbb{F}$ and $\mu^+ : R^+ \to \mathbb{Z}$, and ι_* and κ_* are induced by the inclusion maps $\iota : R^+ \to R^\sim$ and $\kappa : \mathbb{Z} \to \mathbb{F}$ respectively.

Proof. The diagram is clearly commuting and the exactness of rows is easy to see.

We claim that the inclusion $\iota : \mathrm{Idem}_\infty(R^+) \subset \mathrm{Idem}_\infty(R^\sim)$ induces a semigroup isomorphism

$$\phi : (\mathrm{Idem}_\infty(R^+)/\approx, \oplus) \to (\mathrm{Idem}_\infty(R^\sim)/\approx, \oplus).$$

First, ϕ is clearly a well-defined homomorphism. Furthermore, finitely generated projective modules over either $\mathbb{F}$ or $\mathbb{Z}$ are well known to be classified as finitely generated free modules, or equivalently, any $n \times n$ matrix idempotent over $\mathbb{F}$ or $\mathbb{Z}$ can be conjugated to $I_k \oplus 0_{n-k}$ for some $k \le n$ by an invertible matrix over $\mathbb{F}$ or $\mathbb{Z}$. So $K_0(\mathbb{Z}) \cong \mathbb{Z} \cong K_0(\mathbb{F})$ with $\kappa_* = \mathrm{id}_{\mathbb{Z}}$.

Furthermore, classes in either $\mathrm{Idem}_\infty(R^+)/\approx$ or $\mathrm{Idem}_\infty(R^\sim)/\approx$ are represented by elements p in $\mathrm{Idem}_n(R^+)$ or $\mathrm{Idem}_n(R^\sim)$ such that $\mu^+(p) = I_k \oplus 0_{n-k}$ or $\mu^\sim(p) = I_k \oplus 0_{n-k}$ for some k, either of which implies that $p \in \mathrm{Idem}_\infty(R^+)$ and hence ϕ is surjective. Indeed, for any $q \in \mathrm{Idem}_n(R^\sim)$, since $\mu^\sim(q) \in \mathrm{Idem}_n(\mathbb{F})$, there is $u \in \mathrm{GL}_n(\mathbb{F}) \subset \mathrm{GL}_n(R^\sim)$ such that $u\mu^\sim(q)u^{-1} = I_k \oplus 0_{n-k}$ for some k, and hence we get $[q] = [p]$ for $p := uqu^{-1} \in \mathrm{Idem}_n(R^\sim)$ with $\mu^\sim(p) = u\mu^\sim(q)u^{-1} = I_k \oplus 0_{n-k}$. A similar argument can be applied to any $q \in \mathrm{Idem}_n(R^+)$.

Thus

$$\iota_* : K_0(R^+) = \mathcal{G}(\mathrm{Idem}_\infty(R^+)/\approx, \oplus) \to K_0(R^\sim) = \mathcal{G}(\mathrm{Idem}_\infty(R^\sim)/\approx, \oplus)$$

induced by ϕ is surjective, which then induces a surjective homomorphism from $K_0(R) \equiv \ker(\mu_*^+)$ to $\ker(\mu_*^\sim)$ because κ_* is an isomorphism in the commuting diagram. □

Thus we get that

$$\overline{K_0}(C_0(\mathbb{R}^2)) = 0 \not\cong K_0(C_0(\mathbb{R}^2))$$

which shows that even for commutative C*-algebras $\mathcal{A}$, the K_0-groups $\overline{K_0}(\mathcal{A})$ and $K_0(\mathcal{A})$ are in general not isomorphic.

Before moving on to the case of K_1-groups, we would like to mention the following counterexample for the case of K_0-groups that the referee kindly pointed out to us. This example gives a negative answer to Abel's question for general, or even commutative, rings (but not for the more specialized class of C*-algebras).

For an integer $n \geq 2$, there are clearly no nontrivial idempotents in $\mathrm{Idem}_\infty(n\mathbb{Z})$ for the nonunital ring $n\mathbb{Z}$ and hence $\overline{K_0}(n\mathbb{Z}) = 0$. On the other hand, $K_0(n\mathbb{Z})$ cannot always be 0 (for example, $n = 5$), by the well-known exact sequence [W]

$$\mathrm{GL}_\infty(\mathbb{Z}) \to \mathrm{GL}_\infty(\mathbb{Z}/(n\mathbb{Z})) \to K_0(n\mathbb{Z}) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Z}/(n\mathbb{Z}))$$

in algebraic K-theory associated with the short exact sequence

$$0 \to n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(n\mathbb{Z}) \to 0.$$

4.2. Counterexample for K_1 group. Let $R \subset C_0(\mathbb{R}) \subset C_0(\mathbb{R})^\sim \cong C(\mathbb{S}^1)$ be the algebra of all *smooth* $\mathbb{C}$-valued functions f on the unit circle $\mathbb{S}^1$ such that $f(1) = 0$, where $1 \in \mathbb{S}^1 \subset \mathbb{C}$, the multiplicative unit of $\mathbb{C}$, is viewed as the point ∞ when $\mathbb{S}^1$ is viewed as the one-point compactification of $\mathbb{R}$, and the isomorphism $C_0(\mathbb{R})^\sim \cong C(\mathbb{S}^1)$ identifies each $f \in C(\mathbb{S}^1)$ with the element

$$(f - f(1)) + f(1) \in C_0(\mathbb{R}) + \mathbb{C} \equiv C_0(\mathbb{R})^\sim.$$

Note that we have $R^+ \subset C_0(\mathbb{R})^\sim \cong C(\mathbb{S}^1)$ identified with the subring of $C(\mathbb{S}^1)$ consisting of all *smooth* functions $f \in C^\infty(\mathbb{S}^1)$ with $f(1) \in \mathbb{Z}$.

By definition, $K_1(R)$ is the *kernel* of the canonical group homomorphism $\mu_R : K_1(R^+) \to K_1(\mathbb{Z})$ induced by the 'modulo R' map $\mu^+ : f \in R^+ \mapsto f(1) \in \mathbb{Z}$ which is extended canonically to a ring homomorphism $M_n(R^+) \to M_n(\mathbb{Z})$, still denoted as μ^+. More explicitly, since $[\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z})] = E_n(\mathbb{Z})$ [W],

$$\begin{aligned} K_1(R) &= \varinjlim_{n\to\infty} \frac{\{u \in \mathrm{GL}_n(R^+) : \mu^+(u) \equiv u(1) \in [\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z})]\}}{[\mathrm{GL}_n(R^+), \mathrm{GL}_n(R^+)]} \\ &= \varinjlim_{n\to\infty} \frac{(\mu^+)^{-1}(E_n(\mathbb{Z}))}{[\mathrm{GL}_n(R^+), \mathrm{GL}_n(R^+)]} \end{aligned}$$

where $E_n(\mathcal{R})$ for any unital ring $\mathcal{R}$ denotes the subgroup of $\mathrm{GL}_n(\mathcal{R})$ generated by the *elementary matrices* $I_n + re_{ij}$ with $i \neq j$ and $r \in \mathcal{R}$, and

$$(\mu^+)^{-1}(E_n(\mathbb{Z})) \equiv \{u \in \mathrm{GL}_n(R^+) : \mu^+(u) \equiv u(1) \in E_n(\mathbb{Z})\}.$$

On the other hand, under the group isomorphism

$$u \in \mathrm{GL}_n(R^+) \mapsto I_n - u \in \mathrm{GL}_n^\circ(R^+),$$

we have

$$\mathrm{GL}_n^\circ(R) = \{v \in \mathrm{GL}_n^\circ(R^+) : \mu^+(v) \equiv v(1) = 0_n\} \subset \mathrm{GL}_n^\circ(R^+)$$

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identified with

$$(\mu^+)^{-1}(I_n) \equiv \{u \in \mathrm{GL}_n(R^+) : \mu^+(u) \equiv u(1) = I_n\} \subset \mathrm{GL}_n(R^+)$$

and hence

$$\begin{aligned}\overline{K_1}(R) = \frac{\mathrm{GL}^\circ_\infty(R)}{[\mathrm{GL}^\circ_\infty(R), \mathrm{GL}^\circ_\infty(R)]_\circ} &= \varinjlim_{n\to\infty} \frac{\mathrm{GL}^\circ_n(R)}{[\mathrm{GL}^\circ_n(R), \mathrm{GL}^\circ_n(R)]_\circ} \\ &\cong \varinjlim_{n\to\infty} \frac{(\mu^+)^{-1}(I_n)}{[(\mu^+)^{-1}(I_n), (\mu^+)^{-1}(I_n)]}.\end{aligned}$$

Note that obviously, $(\mu^+)^{-1}(I_n) \subset (\mu^+)^{-1}(E_n(\mathbb{Z}))$. On the other hand, since

$$E_n(\mathbb{Z}) = [\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z})] \subset [\mathrm{GL}_n(R^+), \mathrm{GL}_n(R^+)],$$

we have, for any $u \in (\mu^+)^{-1}(E_n(\mathbb{Z}))$, the well-defined $(\mu^+(u))^{-1}u \in (\mu^+)^{-1}(I_n)$ such that

$$[(\mu^+(u))^{-1}u] = [u] \quad \text{in } \frac{(\mu^+)^{-1}(E_n(\mathbb{Z}))}{[\mathrm{GL}_n(R^+), \mathrm{GL}_n(R^+)]}.$$

Thus the canonical homomorphism

$$\frac{(\mu^+)^{-1}(I_n)}{[(\mu^+)^{-1}(I_n), (\mu^+)^{-1}(I_n)]} \to \frac{(\mu^+)^{-1}(E_n(\mathbb{Z}))}{[\mathrm{GL}_n(R^+), \mathrm{GL}_n(R^+)]}$$

and, hence, the induced canonical homomorphism

$$\overline{K_1}(R) \to K_1(R)$$

are *surjective*. (This result is valid for any ring R since the above argument only utilizes the 'modulo R' map μ^+ and not the special property that $\mu^+(u) = u(1)$. Abel indicated in his talk that he also had reached this conclusion.)

Next we show that the canonical natural homomorphism $\overline{K_1}(R) \to K_1(R)$ is *not injective*. In the following, we *fix a local coordinate* system on a neighborhood of 1 in $\mathbb{S}^1$ so that the notion of derivative $f'(1)$ is well-defined without ambiguity, for any smooth function f on $\mathbb{S}^1$.

Taking any $f \in R^+$ with $f(1) = 1$ and $g \in R$ with $g(1) = 0$ but $g'(1) \neq 0$, that is, *smooth* functions f, g on $\mathbb{S}^1$ with $f(1) = 1$ and g having a *simple zero* at 1, we note that

$$U := \begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}^{-1} \in [\mathrm{GL}_2(R^+), \mathrm{GL}_2(R^+)]$$

and also

$$U \equiv \begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}^{-1} \in (\mu^+)^{-1}(I_2)$$

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that is, $I_2 - U \in M_2(R)$, since

$$\begin{aligned}\mu^+\left(\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}^{-1}\right)\\ = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

However, we claim that

$$U \equiv \begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 - gf & gfg\\ -fgf & 1 + fg + fgfg\end{pmatrix}$$

cannot be in $[(\mu^+)^{-1}(I_n), (\mu^+)^{-1}(I_n)]$ for any n, and hence

$$[U] \neq 0 \quad \text{in } \overline{K_1}(R) \cong \varinjlim_{n\to\infty} \frac{(\mu^+)^{-1}(I_n)}{[(\mu^+)^{-1}(I_n), (\mu^+)^{-1}(I_n)]}$$

while

$$[U] = 0 \quad \text{in } K_1(R) \cong \varinjlim_{n\to\infty} \frac{(\mu^+)^{-1}(E_n(\mathbb{Z}))}{[\mathrm{GL}_n(R^+), \mathrm{GL}_n(R^+)]}$$

which shows that the canonical homomorphism $\overline{K_1}(R) \to K_1(R)$ is *not injective*.

Our claim is proved by the observation that the $(2, 1)$th entry fgf of $I_2 - U \in M_2(R)$ has a *simple zero* at 1 since $f(1) = 1$ while g has a simple zero at 1, and by the following lemma.

Lemma 4.2. *All entries of $I_n - V \in M_n(R)$ have zeros of order at least 2 for any $V \in [(\mu^+)^{-1}(I_n), (\mu^+)^{-1}(I_n)]$ and any $n \in \mathbb{N}$.*

Proof. Any element of $(\mu^+)^{-1}(I_n)$ is of the form $I_n + A$ with $A \in M_n(R)$, and its inverse $(I_n + A)^{-1}$ also clearly belongs to $(\mu^+)^{-1}(I_n)$. Hence $(I_n + A)^{-1} = I_n - \tilde{A}$ for some $\tilde{A} \in M_n(R)$. Note that all entries of $A - \tilde{A}$ have a zero at 1 of order at least 2, because

$$I_n = (I_n + A)(I_n - \tilde{A}) = I_n + (A - \tilde{A}) - A\tilde{A}$$

that is, $A - \tilde{A} = A\tilde{A}$, where all entries of $A\tilde{A}$ are sums of products of smooth functions having a zero at 1 and hence have a zero of order at least 2 at 1.

So for any $I_n + A, I_n + B \in (\mu^+)^{-1}(I_n)$,

$$\begin{aligned}&(I_n + A)(I_n + B)(I_n + A)^{-1}(I_n + B)^{-1}\\ &\quad = (I_n + A)(I_n + B)(I_n - \tilde{A})(I_n - \tilde{B})\\ &\quad = I_n + A + B - \tilde{A} - \tilde{B} + AB - A\tilde{A} - A\tilde{B} - B\tilde{A} - \cdots\\ &\quad = I_n + (A - \tilde{A}) + (B - \tilde{B}) + AB - A\tilde{A} - A\tilde{B} - B\tilde{A} - \cdots\end{aligned}$$

where all summands except I_n are matrices with all entries having a zero of order at least 2 at 1, that is,

$$(I_n + A)(I_n + B)(I_n + A)^{-1}(I_n + B)^{-1} = I_n + C$$

for some $C \in M_n(R)$ with all entries having a zero of order at least 2 at 1.

Now any element V of the commutator subgroup $[(\mu^+)^{-1}(I_n), (\mu^+)^{-1}(I_n)]$ is of the form

$$V = \prod_{i=1}^{m} (I_n + A_i)(I_n + B_i)(I_n + A_i)^{-1}(I_n + B_i)^{-1} = \prod_{i=1}^{m} (I_n + C_i)$$
$$= I_n + \sum_{i=1}^{m} C_i + \sum_{1 \le i < j \le m} C_i C_j + \cdots$$

for some $I_n + A_i, I_n + B_i \in (\mu^+)^{-1}(I_n)$, $1 \le i \le m$, where all summands except I_n are matrices with all entries having a zero of order at least 2 at 1. Thus $I_n - V$ is as described in the statement. □

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