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OPERATOR QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED WITH DAVIS–CHOI–JENSEN'S INEQUALITY FOR POSITIVE MAPS

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Abstract

In this paper we establish operator quasilinearity properties of some functionals associated with Davis–Choi–Jensen's inequality for positive maps and operator convex or concave functions. Applications for the power function and the logarithm are provided.

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1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all self-adjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive-definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [15, page 18]), we say that a map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely

$$\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$$

for any $\lambda, \mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, that is, if $A \in \mathcal{B}^+(H)$, then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, that is, $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \le B \text{ implies } \Phi(A) \le \Phi(B),$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \le A \le \beta 1_H$, then $\alpha 1_K \le \Phi(A) \le \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we see that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, that is, it is also normalised.

A real-valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge)\ (1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0, 1]$ and for all self-adjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The classical Davis–Choi–Jensen's inequality which motivates this article states that if $f : I \to \mathbb{R}$ is an operator convex function on the interval I and if $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any self-adjoint operator A with spectrum contained in I,

$$f(\Phi(A)) \le \Phi(f(A)) \tag{1.1}$$

(see [1, Theorem 2.1]).

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.1),

$$f(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)) \le \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$, we get the following *Davis–Choi–Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1_H) f(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))\Psi^{1/2}(1_H) \le \Psi(f(A)). \tag{1.2}$$

We denote by $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, that is, $\Psi(1_H)$ is a positive invertible operator in K, and define the functional $\mathbf{F} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\mathbf{F}_{f,A}(\Psi) = \Psi^{1/2}(1_H) f(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))\Psi^{1/2}(1_H),$$

where $f : I \to \mathbb{R}$ is an operator convex (concave) function on the interval I and A is a self-adjoint operator whose spectrum is contained in I.

In this paper we establish operator quasilinearity properties of some functionals associated with Davis–Choi–Jensen's inequality (1.2) for positive maps and operator convex (concave) functions. We also give applications to the power function and the logarithm.

2. The main results

THEOREM 2.1. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval I and A a self-adjoint operator whose spectrum is contained in I. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, then*

$$\mathbf{F}_{f,A}(\Psi_1 + \Psi_2) \le (\ge) \mathbf{F}_{f,A}(\Psi_1) + \mathbf{F}_{f,A}(\Psi_2), \tag{2.1}$$

that is, $\mathbf{F}_{f,A}$ is operator subadditive (superadditive) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Proof. We give the proof for operator convex functions. If $\Psi_1, \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, then $\Psi_1 + \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and

$$\begin{aligned}\mathbf{F}_{f,A}(\Psi_1 + \Psi_2) = (\Psi_1 + \Psi_2)^{1/2}(1_H) &\\ &\cdot f((\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1 + \Psi_2)(A)\\ &\cdot (\Psi_1 + \Psi_2)^{-1/2}(1_H))(\Psi_1 + \Psi_2)^{1/2}(1_H),\end{aligned}$$

where by '$\cdot$' we understand the usual operator multiplication. Observe that

$$\begin{aligned}&(\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1 + \Psi_2)(A)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\quad= (\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1(A) + \Psi_2(A))(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\quad= (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_1(A)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\qquad\quad+ (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_2(A)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\quad= (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_1^{1/2}(1_H)(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H))\\ &\qquad\quad\cdot \Psi_1^{1/2}(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\qquad\quad+ (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_2^{1/2}(1_H)(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H))\\ &\qquad\quad\cdot \Psi_2^{1/2}(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H).\end{aligned}\tag{2.2}$$

If we define

$$V := \Psi_1^{1/2}(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H) \quad \text{and} \quad U := \Psi_2^{1/2}(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H),$$

then

$$V^* = (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_1^{1/2}(1_H) \quad \text{and} \quad U^* = (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_2^{1/2}(1_H).$$

Also,

$$\begin{aligned}V^*V + U^*U &= (\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1(1_H) + \Psi_2(1_H))(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &= (\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1 + \Psi_2)(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H) = 1_K,\end{aligned}$$

and (2.2) may be written as

$$\begin{aligned}&(\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1 + \Psi_2)(A)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\quad= V^*(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H))V + U^*(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H))U.\end{aligned}$$

By applying f and using Hansen–Pedersen–Jensen's inequality for operator convex functions,

$$\begin{aligned}&f((\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1 + \Psi_2)(A)(\Psi_1 + \Psi_2)^{-1/2}(1_H))\\ &\quad\le V^* f(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H))V + U^* f(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H))U\\ &\quad= (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_1^{1/2}(1_H)f(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H))\\ &\qquad\quad\cdot \Psi_1^{1/2}(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\\ &\qquad\quad+ (\Psi_1 + \Psi_2)^{-1/2}(1_H)\Psi_2^{1/2}(1_H)f(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H))\\ &\qquad\quad\cdot \Psi_2^{1/2}(1_H)(\Psi_1 + \Psi_2)^{-1/2}(1_H).\end{aligned}\tag{2.3}$$

Finally, by multiplying both sides of (2.3) by $(\Psi_1 + \Psi_2)^{1/2}(1_H)$,

$$\begin{aligned}\mathbf{F}_{f,A}(\Psi_1 + \Psi_2) &\le \Psi_1^{1/2}(1_H) f(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H))\Psi_1^{1/2}(1_H) \\ &\quad + \Psi_2^{1/2}(1_H) f(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H))\Psi_2^{1/2}(1_H) \\ &= \mathbf{F}_{f,A}(\Psi_1) + \mathbf{F}_{f,A}(\Psi_2),\end{aligned}$$

and the proof is concluded. □

Corollary 2.2. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval I and A a self-adjoint operator whose spectrum is contained in I. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0, 1]$, then*

$$\mathbf{F}_{f,A}((1 - \lambda)\Psi_1 + \lambda\Psi_2) \le (\ge)\ (1 - \lambda)\mathbf{F}_{f,A}(\Psi_1) + \lambda\mathbf{F}_{f,A}(\Psi_2),$$

that is, $\mathbf{F}_{f,A}$ is operator convex (concave) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Proof. Suppose that $\Psi_1, \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0, 1]$. Then $(1 - \lambda)\Psi_1 + \lambda\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and, by (2.1),

$$\begin{aligned}\mathbf{F}_{f,A}((1 - \lambda)\Psi_1 + \lambda\Psi_2) &\le (\ge)\, \mathbf{F}_{f,A}((1 - \lambda)\Psi_1) + \mathbf{F}_{f,A}(\lambda\Psi_2) \\ &= (1 - \lambda)\mathbf{F}_{f,A}(\Psi_1) + \lambda\mathbf{F}_{f,A}(\Psi_2),\end{aligned}$$

since $\mathbf{F}_{f,A}$ is positive homogeneous on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, that is,

$$\mathbf{F}_{f,A}(\alpha\Psi) = \alpha\mathbf{F}_{f,A}(\Psi)$$

for any $\alpha > 0$ and $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. □

For Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, we write $\Psi_2 >_I \Psi_1$ if $\Psi_2 - \Psi_1 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. This means that $\Psi_2 - \Psi_1$ is a positive linear functional and $\Psi_2(1_H) - \Psi_1(1_H) \in \mathcal{B}^{++}(K)$. For examples of such maps, see [6].

Corollary 2.3. *Let $f : I \to [0, \infty)$ be an operator concave function on the interval I and A a self-adjoint operator whose spectrum is contained in I. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi_2 >_I \Psi_1$, then*

$$\mathbf{F}_{f,A}(\Psi_2) \ge \mathbf{F}_{f,A}(\Psi_1), \tag{2.4}$$

that is, $\mathbf{F}_{f,A}$ is operator monotonic nondecreasing in the order '$>_I$' of $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Proof. Let $\Psi_1, \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi_2 >_I \Psi_1$. By (2.1),

$$\mathbf{F}_{f,A}(\Psi_2) = \mathbf{F}_{f,A}(\Psi_1 + \Psi_2 - \Psi_1) \ge \mathbf{F}_{f,A}(\Psi_1) + \mathbf{F}_{f,A}(\Psi_2 - \Psi_1),$$

implying that

$$\mathbf{F}_{f,A}(\Psi_2) - \mathbf{F}_{f,A}(\Psi_1) \ge \mathbf{F}_{f,A}(\Psi_2 - \Psi_1).$$

Since f is positive and $\Psi_{2,1} := \Psi_2 - \Psi_1 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and since $\Psi_{2,1}(1_H) = \Psi_2(1_H) - \Psi_1(1_H) \in \mathcal{B}^{++}(K)$, it follows that

$$f(\Psi_{2,1}^{-1/2}(1_H)\Psi_{2,1}(A)\Psi_{2,1}^{-1/2}(1_H)) \ge 0$$

and, by multiplying both sides by $\Psi_{2,1}^{1/2}(1_H)$, we see that $\mathbf{F}_{f,A}(\Psi_2 - \Psi_1) \ge 0$ and the inequality (2.4) is proved. □

Corollary 2.4. *Let $f : I \to [0,\infty)$ be an operator concave function on the interval I and A a self-adjoint operator whose spectrum is contained in I. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$, then*

$$T\mathbf{F}_{f,A}(\Upsilon) \geq \mathbf{F}_{f,A}(\Psi) \geq t\mathbf{F}_{f,A}(\Upsilon).$$

Proof. The result follows from (2.4) on taking first $\Psi_2 = T\Upsilon$, $\Psi_1 = \Psi$ and then $\Psi_2 = \Psi$, $\Psi_1 = t\Upsilon$ and by the positive homogeneity of $\mathbf{F}_{f,A}$. □

We consider now the functional $\mathbf{J}_{f,A} : \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)] \to \mathcal{B}(K)$ defined by

$$\begin{aligned}\mathbf{J}_{f,A}(\Psi) &:= \Psi(f(A)) - \mathbf{F}_{f,A}(\Psi)\\ &= \Psi(f(A)) - \Psi^{1/2}(1_H)f(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))\Psi^{1/2}(1_H).\end{aligned}$$

Theorem 2.5. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval I and A a self-adjoint operator whose spectrum is contained in I. Then the functional $\mathbf{J}_{f,A}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, positive homogeneous and concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ and superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.*

Proof. We consider only the operator convex case. The positivity of $\mathbf{J}_{f,A}$ on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ is equivalent to Davis–Choi–Jensen's inequality for general positive linear maps (1.2). The positive homogeneity follows by the same property of $\mathbf{F}_{f,A}$ and the definition of $\mathbf{J}_{f,A}$.

If $\Psi_1, \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ and $\lambda \in [0,1]$, then, by Corollary 2.2,

$$\begin{aligned}&\mathbf{J}_{f,A}((1-\lambda)\Psi_1 + \lambda\Psi_2)\\ &\quad= ((1-\lambda)\Psi_1 + \lambda\Psi_2)(f(A)) - \mathbf{F}_{f,A}((1-\lambda)\Psi_1 + \lambda\Psi_2)\\ &\quad\geq (1-\lambda)\Psi_1(f(A)) + \lambda\Psi_2(f(A)) - (1-\lambda)\mathbf{F}_{f,A}(\Psi_1) - \lambda\mathbf{F}_{f,A}(\Psi_2)\\ &\quad= (1-\lambda)[\Psi_1(f(A)) - \mathbf{F}_{f,A}(\Psi_1)] + \lambda[\Psi_2(f(A)) - \mathbf{F}_{f,A}(\Psi_2)]\\ &\quad= (1-\lambda)\mathbf{J}_{f,A}(\Psi_1) + \lambda\mathbf{J}_{f,A}(\Psi_2),\end{aligned}$$

which proves the operator concavity of $\mathbf{J}_{f,A}$. The operator superadditivity follows in a similar way and we omit the details. □

Corollary 2.6. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and A a self-adjoint operator whose spectrum is contained in I. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, $t, T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$, then*

$$T\mathbf{J}_{f,A}(\Upsilon) \geq \mathbf{J}_{f,A}(\Psi) \geq t\mathbf{J}_{f,A}(\Upsilon)$$

or, equivalently,

$$T(\Upsilon(f(A)) - \mathbf{F}_{f,A}(\Upsilon)) \geq \Psi(f(A)) - \mathbf{F}_{f,A}(\Psi) \geq t(\Upsilon(f(A)) - \mathbf{F}_{f,A}(\Upsilon)) \geq 0. \quad (2.5)$$

The inequality (2.5) has been obtained in [6] in an equivalent form for an operator concave function f and normalised functionals Ψ and Υ.

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Now assume that A is a self-adjoint operator whose spectrum is contained in $[m, M]$ for some real constants $M > m$. If f is convex, then for any $t \in [m, M]$,

$$f(t) \leq \frac{(M-t)f(m) + (t-m)f(M)}{M-m}. \tag{2.6}$$

If A is a self-adjoint operator whose spectrum is contained in $[m, M]$, then $m1_H \leq A \leq M1_H$ and by applying the map Ψ we get $m\Psi(1_H) \leq \Psi(A) \leq M\Psi(1_H)$ for $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. This is equivalent to

$$m1_K \leq \Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H) \leq M1_K.$$

From (2.6) and the continuous functional calculus,

$$\begin{aligned} f(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)) &\leq \frac{1}{M-m}[f(m)(M1_K - \Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))] \\ &\quad + \frac{1}{M-m}[f(M)(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H) - m1_K)]. \end{aligned}$$

Multiplying this inequality on both sides by $\Psi^{1/2}(1_H)$ yields

$$\mathbf{F}_{f,A}(\Psi) \leq \mathbf{T}_{f,A}(\Psi),$$

where

$$\mathbf{T}_{f,A}(\Psi) := \frac{f(m)(M\Psi(1_H) - \Psi(A)) + f(M)(\Psi(A) - m\Psi(1_H))}{M-m}$$

is a trapezoidal-type functional. We observe that $\mathbf{T}_{f,A}$ is *additive* and *positive homogeneous* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

We define the functional $\mathbf{D}_{f,A} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\begin{aligned} \mathbf{D}_{f,A}(\Psi) &:= \mathbf{T}_{f,A}(\Psi) - \mathbf{F}_{f,A}(\Psi) \\ &= \frac{f(m)(M\Psi(1_H) - \Psi(A)) + f(M)(\Psi(A) - m\Psi(1_H))}{M-m} \\ &\quad - \Psi^{1/2}(1_H)f(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))\Psi^{1/2}(1_H). \end{aligned}$$

We observe that if f is convex (concave) on $[m, M]$ and $m1_H \leq A \leq M1_H$, then

$$\mathbf{D}_{f,A}(\Psi) \geq (\leq)\, 0 \quad \text{for any } \Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)].$$

THEOREM 2.7. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval $[m, M]$ and A a self-adjoint operator whose spectrum is contained in $[m, M]$. Then the functional $\mathbf{D}_{f,A}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, positive homogeneous and operator concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and operator superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.*

The proof is similar to the proof of Theorem 2.5 and we omit the details.

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COROLLARY 2.8. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and A a self-adjoint operator whose spectrum is contained in I. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, $t, T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$, then*

$$T\mathbf{D}_{f,A}(\Upsilon) \geq \mathbf{D}_{f,A}(\Psi) \geq t\mathbf{D}_{f,A}(\Upsilon)$$

or, equivalently,

$$\begin{aligned} T&\bigg[\frac{f(m)(M\Upsilon(1_H) - \Upsilon(A)) + f(M)(\Upsilon(A) - m\Upsilon(1_H))}{M - m} - \mathbf{F}_{f,A}(\Upsilon)\bigg] \\ &\geq \frac{f(m)(M\Psi(1_H) - \Psi(A)) + f(M)(\Psi(A) - m\Psi(1_H))}{M - m} - \mathbf{F}_{f,A}(\Psi) \\ &\geq t\bigg[\frac{f(m)(M\Upsilon(1_H) - \Upsilon(A)) + f(M)(\Upsilon(A) - m\Upsilon(1_H))}{M - m} - \mathbf{F}_{f,A}(\Upsilon)\bigg] \geq 0. \end{aligned}$$

3. Examples for the power function and the logarithm

It is well known that the function $f_\nu : [0, \infty) \to [0, \infty)$ given by $f_\nu(x) = x^\nu$ for $\nu \in (0, 1)$ is operator concave and positive on $[0, \infty)$. We consider the functional on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ defined by

$$\mathbf{F}_{\nu,A}(\Psi) = \Psi^{1/2}(1_H)(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))^\nu \Psi^{1/2}(1_H),$$

where A is a positive operator on H.

Assume that C, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notation for operators as in [11]:

$$C\nabla_\nu B := (1 - \nu)C + \nu B,$$

the *weighted operator arithmetic mean*, and

$$C\sharp_\nu B := C^{1/2}(C^{-1/2}BC^{-1/2})^\nu C^{1/2},$$

the *weighted operator geometric mean,* where $\nu \in [0, 1]$. When $\nu = \frac{1}{2}$, we write $C\nabla B$ and $C\sharp B$ for brevity, respectively. The definition of $C\sharp_\nu B$ can be extended to any real number ν.

Using this notation, we observe that

$$\mathbf{F}_{\nu,A}(\Psi) = \Psi(1_H)\sharp_\nu\Psi(A).$$

In particular, for $\nu = \frac{1}{2}$,

$$\mathbf{F}_{1/2,A}(\Psi) = \Psi(1_H)\sharp\Psi(A).$$

Using the results from the previous section for the operator concave function f_ν, we see that $\mathbf{F}_{\nu,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order '$>_I$' and we have the inequality

$$T\Upsilon(1_H)\sharp_\nu\Upsilon(A) \geq \Psi(1_H)\sharp_\nu\Psi(A) \geq t\Upsilon(1_H)\sharp_\nu\Upsilon(A),$$

where $\Psi,\ \Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, $t,\ T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$. The operator concavity, superadditivity and monotonicity may also be derived from the corresponding properties of the weighted operator geometric mean (see [15, page 146]).

If we consider the functional

$$\mathbf{J}_{\nu,A}(\Psi) := \Psi(A^{\nu}) - \Psi(1_H)\sharp_{\nu}\Psi(A),$$

then $\mathbf{J}_{\nu,A}$ is *negative, operator convex and operator subadditive* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. Also, if $0 < m1_H \le A \le M1_H$, then we can consider the functional

$$\mathbf{D}_{\nu,A}(\Psi) := \frac{m^{\nu}(M\Psi(1_H) - \Psi(A)) + M^{\nu}(\Psi(A) - m\Psi(1_H))}{M - m} - \Psi(1_H)\sharp_{\nu}\Psi(A)$$

and from the above section we conclude that $\mathbf{D}_{\nu,A}$ is *negative, operator convex and operator subadditive* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Now consider the function $\Phi_p(t) = t^p$ which is operator convex on $(0, \infty)$ if either $1 \le p \le 2$ or $-1 \le p \le 0$. We consider the functional on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ defined by

$$\begin{aligned}\mathbf{F}_{p,A}(\Psi) &= \Psi^{1/2}(1_H)(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))^p\Psi^{1/2}(1_H)\\ &= \Psi(1_H)\sharp_p\Psi(A),\end{aligned}$$

where A is a positive-definite operator on H. In particular,

$$\begin{aligned}\mathbf{F}_{2,A}(\Psi) &= \Psi^{1/2}(1_H)(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))^2\Psi^{1/2}(1_H)\\ &= \Psi(A)\Psi^{-1}(1_H)\Psi(A)\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_{-1,A}(\Psi) &= \Psi^{1/2}(1_H)(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))^{-1}\Psi^{1/2}(1_H)\\ &= \Psi(1_H)\Psi^{-1}(A)\Psi(1_H).\end{aligned}$$

From the previous section, we can infer that $\mathbf{F}_{p,A}$ is *positive, operator convex and subadditive* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

For $\Psi_1,\ \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, we have by the properties of $\mathbf{F}_{p,A}$ that the scalar-valued function

$$\rho_{p,A}(\Psi) := \|\mathbf{F}_{p,A}(\Psi)\| = \|\Psi(1_H)\sharp_p\Psi(A)\|$$

is subadditive and positive homogeneous on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Consider the functional

$$\mathbf{J}_{p,A}(\Psi) := \Psi(A^p) - \Psi(1_H)\sharp_p\Psi(A).$$

If A is positive definite and either $1 \le p \le 2$ or $-1 \le p \le 0$, then the functional $\mathbf{J}_{p,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order '$>_I$' and we have the inequality

$$\begin{aligned}T[\Upsilon(A^p) - \Upsilon(1_H)\sharp_p\Upsilon(A)] &\ge \Psi(A^p) - \Psi(1_H)\sharp_p\Psi(A)\\ &\ge t[\Upsilon(A^p) - \Upsilon(1_H)\sharp_p\Upsilon(A)] \ge 0,\end{aligned}$$

where $\Psi,\ \Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, $t,\ T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$. Also, if $0 < m1_H \le A \le M1_H$, then the functional

$$\mathbf{D}_{p,A}(\Psi) := \frac{m^p(M\Psi(1_H) - \Psi(A)) + M^p(\Psi(A) - m\Psi(1_H))}{M - m} - \Psi(1_H)\sharp_p\Psi(A)$$

is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order '$>_I$' and we have the inequality

$$\begin{aligned} &T\Big[\frac{m^p(M\Upsilon(1_H) - \Upsilon(A)) + M^p(\Upsilon(A) - m\Upsilon(1_H))}{M - m} - \Upsilon(1_H)\sharp_p\Upsilon(A)\Big] \\ &\quad \ge \frac{m^p(M\Psi(1_H) - \Psi(A)) + M^p(\Psi(A) - m\Psi(1_H))}{M - m} - \Psi(1_H)\sharp_p\Psi(A) \\ &\quad \ge t\Big[\frac{m^p(M\Upsilon(1_H) - \Upsilon(A)) + M^p(\Upsilon(A) - m\Upsilon(1_H))}{M - m} - \Upsilon(1_H)\sharp_p\Upsilon(A)\Big] \ge 0, \end{aligned}$$

where $\Psi, \Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, $t, T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$.

It is well known that the function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$ is operator concave on $(0, \infty)$. We consider the functional on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ defined by

$$\mathbf{F}_{\ln,A}(\Psi) = \Psi^{1/2}(1_H)\ln(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H))\Psi^{1/2}(1_H),$$

where A is a positive-definite operator on H.

Fujii and Kamei [7, 8] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{1/2}(\ln A^{-1/2}BA^{-1/2})A^{1/2},$$

a relative version of the operator entropy considered by Nakamura and Umegaki [13].

If $B \ge A$ and A is positive and invertible, then $A^{-1/2}BA^{-1/2} \ge I$ and, by the continuous functional calculus, $\ln(A^{-1/2}BA^{-1/2}) \ge 0$, which implies by multiplying both sides with $A^{1/2}$ that $S(A|B) \ge 0$. For some recent results on relative operator entropy, see [2–6, 9, 10, 12, 14].

Using the relative operator entropy notation,

$$\mathbf{F}_{\ln,A}(\Psi) = S(\Psi(1_H)|\Psi(A)),$$

where A is a positive-definite operator on H and $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. From the properties established in the previous section applied to the operator concave function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, we see that $\mathbf{F}_{\ln,A}$ is *operator concave and operator superadditive* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. These properties may also be derived from the corresponding properties of the relative operator entropy (see [15, page 153]).

Moreover, if $\Psi >_I \Upsilon$, then

$$\mathbf{F}_{\ln,A}(\Psi) - \mathbf{F}_{\ln,A}(\Upsilon) \ge \mathbf{F}_{\ln,A}(\Psi - \Upsilon)$$

and, in addition, if $\Psi(A) + \Upsilon(1_H) \ge \Upsilon(A) + \Psi(1_H)$, then

$$\mathbf{F}_{\ln,A}(\Psi) \ge \mathbf{F}_{\ln,A}(\Upsilon).$$

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The function $f(t) = -\ln t$, $t > 0$, is operator convex. If we consider now the functional

$$\mathbf{J}_{-\ln,A}(\Psi) := S(\Psi(1_H)|\Psi(A)) - \Psi(\ln(A)),$$

then from the previous section we can infer that $\mathbf{J}_{-\ln,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order '$>_I$' and we have the inequality

$$\begin{aligned} T(S(\Upsilon(1_H)|\Upsilon(A)) - \Upsilon(\ln(A))) &\geq S(\Psi(1_H)|\Psi(A)) - \Psi(\ln(A)) \\ &\geq t(S(\Upsilon(1_H)|\Upsilon(A)) - \Upsilon(\ln(A))) \geq 0 \end{aligned}$$

provided that Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$. Consider also the functional

$$\mathbf{D}_{-\ln,A}(\Psi) := S(\Psi(1_H)|\Psi(A)) - \frac{\ln m(M\Psi(1_H) - \Psi(A)) + \ln M(\Psi(A) - m\Psi(1_H))}{M - m}$$

for $\Psi \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. Then $\mathbf{D}_{-\ln,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order '$>_I$' and we have the inequality

$$\begin{aligned} &T\Big[S(\Upsilon(1_H)|\Upsilon(A)) - \frac{\ln m(M\Upsilon(1_H) - \Upsilon(A)) + \ln M(\Upsilon(A) - m\Upsilon(1_H))}{M - m}\Big] \\ &\quad\geq S(\Psi(1_H)|\Psi(A)) - \frac{\ln m(M\Psi(1_H) - \Psi(A)) + \ln M(\Psi(A) - m\Psi(1_H))}{M - m} \\ &\quad\geq t\Big[S(\Upsilon(1_H)|\Upsilon(A)) - \frac{\ln m(M\Upsilon(1_H) - \Upsilon(A)) + \ln M(\Upsilon(A) - m\Upsilon(1_H))}{M - m}\Big] \geq 0 \end{aligned}$$

provided that Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon >_I \Psi >_I t\Upsilon$.

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