PART VI

INSTABILITY MECHANISMS

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A. NON-RADIAL PULSATIONS – MAGNETIC FIELDS

NON-RADIAL OSCILLATIONS

P. LEDOUX

Institut d'Astrophysique, Université de Liège, Belgium

Abstract. The problem of the adiabatic non-radial oscillations of spherical stars is reviewed and results recalled for a variety of models. The anomalous behaviour of the eigenfunctions for highly condensed models is related to the apparent mobile singularities depending on the eigenvalues. Tables of Q-values are provided to facilitate possible applications to variable stars.

In the case of the gravity modes, the existence of multiple spectra some stable (g^+ modes) some unstable (g^- modes) if superadiabatic and subadiabatic regions alternate is discussed and the interest of further investigations underlined.

As far as vibrational stability (effects of the non adiabatic terms) is concerned, a general expression is given for the 'damping coefficient'. The attention is drawn to the possibility for g^+ modes of becoming vibrationally unstable under the effect of various factors and in various models, including the Sun where this was advocated as a possibility of relieving the neutrinos difficulty.

Finally the present status of the most obvious candidates among variable stars for non-radial oscillations, the β Canis Majoris stars and the rapid blue variables (white dwarfs) is briefly reviewed.

1. Introduction

I shall devote this introductory lecture mainly to linear non-radial oscillations of purely spherical stars i.e. devoid of rotation or magnetic fields or of tides which, in general, would imply deviations from spherical symmetry. The effects of some of these factors will be treated in other lectures or communications during this session and I shall limit myself to a few comments on some of the simplest aspects of their influence.

There has been lately quite a renewal of interest in the response of stars to non-radial perturbations aroused either by attempts at interpreting some types of variable stars like the β Canis Majoris stars and the new white dwarf variables, or by phenomena in the external layers of the Sun like the 5-min oscillation discovered by Leighton, or by the hope to add somewhat to our knowledge of convection and its penetration in nearby convectively stable zones, or by the desire to explore some new aspects of stellar stability which may be of great importance for the evolution of the star. On the other hand, one must expect that such non-radial motions should be easily excited in a variety of close double stars with eccentric orbits and it is likely that, with the extra-ordinary progress in observational techniques, these should become observable and be identified as such pretty soon. Finally there is direct evidence in novae, perhaps even in planetary nebulae, for the presence of non-radial velocity fields.

There exist general review articles centered on the stellar case (Ledoux and Walraven, 1958; Ledoux, 1969) but the problem is one of importance in many branches of geophysics as well (meteorology, oceanography and also tidal or free oscillations of the Earth) and the literature in this field is quite extensive, a few general accounts being also available (cf. Eckart, 1960; Tolstoy, 1963; Bolt and Derr, 1969). One should also

Ledoux et al. (eds.), Stellar Instability and Evolution, 135–173.

mention the books by Chandrasekhar (1961, 1969) where many related problems are treated and methods presented which are of general interest in this context.

2. General Definitions and Equations

The simplest way to introduce the indispensable definitions and notations is still to write the general linearized equations derived from the conservation of mass, momentum and energy and from Poisson's equation. If the initial non-perturbed configuration is strictly in hydrostatic equilibrium and assuming a time dependence for all perturbations of the form $e^{i\sigma t}$, these equations can be written with fairly standard notations (cf. Ledoux and Walraven, 1958)

$$\frac{\delta\varrho}{\varrho}=\frac{\varrho'}{\varrho}+\frac{\delta r}{\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}r}=-\operatorname{div}\boldsymbol{\delta}\mathbf{r} \tag{1a}$$

$$\sigma^2\boldsymbol{\delta}\mathbf{r}-\operatorname{grad}\Phi'+\frac{\varrho'}{\varrho^2}\operatorname{grad}p-\frac{1}{\varrho}\operatorname{grad}p'=-\frac{i\sigma}{\varrho}\operatorname{div}\mathbf{P}(\boldsymbol{\delta}\mathbf{r}) \tag{1b}$$

or

$$\sigma^2\boldsymbol{\delta}\mathbf{r}-\operatorname{grad}\left(\phi'+\frac{p'}{\varrho}\right)+\frac{\mathbf{r}}{r}\mathscr{A}\frac{\Gamma_1 p}{\varrho}\operatorname{div}\boldsymbol{\delta}\mathbf{r}=$$
$$=\frac{\Gamma_3-1}{i\sigma}\frac{1}{\varrho}\operatorname{grad}\varrho\left(\varepsilon-\frac{1}{\varrho}\operatorname{div}\mathbf{F}\right)'-\frac{i\sigma}{\varrho}\operatorname{div}\mathbf{P}(\boldsymbol{\delta}\mathbf{r}) \tag{1b'}$$

$$\delta p-\frac{\Gamma_1 p}{\varrho}\delta\varrho=p'+\delta r\frac{\mathrm{d}p}{\mathrm{d}r}-\frac{\Gamma_1 p}{\varrho}\left(\varrho'+\delta r\frac{\mathrm{d}\varrho}{\mathrm{d}r}\right)=\frac{\Gamma_3-1}{i\sigma}\varrho\left(\varepsilon-\frac{1}{\varrho}\operatorname{div}\mathbf{F}\right)' \tag{1c}$$

which can also be written

$$p'-\Gamma_1 p\left(\frac{\varrho'}{\varrho}+\mathscr{A}\delta\mathbf{r}\right)=\frac{\Gamma_3-1}{i\sigma}\varrho\left(\varepsilon-\frac{1}{\varrho}\operatorname{div}\mathbf{F}\right)' \tag{1c'}$$

or in terms of the variables T and p

$$\delta T-\frac{\Gamma_2-1}{\Gamma_2}\frac{T}{p}\delta p=T'+\delta r\frac{\mathrm{d}T}{\mathrm{d}r}-\frac{\Gamma_2-1}{\Gamma_2}\frac{T}{p}\left(p'+\delta r\frac{\mathrm{d}p}{\mathrm{d}r}\right)=$$
$$=\frac{1}{i\sigma C_p}\left(\varepsilon-\frac{1}{\varrho}\operatorname{div}\mathbf{F}\right)' \tag{1c''}$$

or

$$T'-T\left(\frac{\Gamma_2-1}{\Gamma_2}\frac{p'}{p}+\delta r\mathscr{S}\right)=\frac{1}{i\sigma C_p}\left(\varepsilon-\frac{1}{\varrho}\operatorname{div}\mathbf{F}\right)' \tag{1c'''}$$

$$\nabla^2\Phi'=4\pi G\varrho', \tag{1d}$$

where a prime denotes an Eulerian perturbation and δ a Lagrangian one. The right-hand members of (1b) and (1c) represent non-conservative terms related to the generation of energy ε and its flux (**F**) and to the effects of the viscous stresses $\mathbf{P}(\boldsymbol{\delta}\mathbf{r})$. In general these terms are much smaller than the others in (1b) and (1c) and neglecting them yields the *adiabatic approximation.*

The quantity $\mathscr{A}$ defined by

$$\mathscr{A}=\frac{1}{\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}r}-\frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} \tag{2}$$

is the argument of the criterion for convection which develops or not depending on whether $\mathscr{A}>0$ or $\mathscr{A}<0$. $\mathscr{A}$ is directly related to the Brunt-Väisälä frequency N by

$$N^2=-g\mathscr{A},$$

where g is the gravity. The quantity

$$\mathscr{S}=\frac{\Gamma_2-1}{\Gamma_2}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}r}-\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r} \tag{2$'$}$$

is the argument of Schwarzschild criterion which, for constant chemical composition (mean molecular weight $\bar{\mu}=$ ct.), is sufficient to determine whether convection develops ($\mathscr{S}>0$) or not ($\mathscr{S}<0$). For a mixture of perfect gas and radiation, one has

$$\mathscr{A}=\frac{1}{\bar{\mu}}\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}r}+\frac{4-3\beta}{\beta}\mathscr{S}, \tag{2$''$}$$

where β is the ratio of the gas to the total pressure. Thus, if $\bar{\mu}$ decreases upwards, convection may not appear even if $\mathscr{S}>0$.

We shall limit ourselves first to the adiabatic case (r.h.m. of (1b) and (1c) identically zero) and note that, in spherical polar coordinates (r, θ, φ), the radial component δr of the displacement as well as p', ϱ' and Φ' can be factorized in the form

$$f'(r,\theta,\varphi)=f'(r)\,P_l^m(\cos\theta)\,e^{\mathrm{i}m\varphi},\quad -l\leqslant m\leqslant l$$

where $P_l^m(\cos\theta)$ is the associated Legendre polynomial of degree l and order m so that

$$\operatorname{div}\boldsymbol{\delta}\mathbf{r}=\frac{1}{r^2}\frac{d}{\mathrm{d}r}(r^2\delta r)-\frac{l(l+1)}{\sigma^2 r^2}\left(\Phi'+\frac{p'}{\varrho}\right) \tag{3}$$

and

$$\nabla^2\Phi'=\frac{1}{r^2}\frac{d}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right)-\frac{l(l+1)}{r^2}\Phi'=4\pi G\varrho'. \tag{4}$$

System (1) where we take only the r-component of Equation (1b′), i.e. here

$$\sigma^2\delta r=\frac{d}{\mathrm{d}r}\left(\Phi'+\frac{p'}{\varrho}\right)-\mathscr{A}\,\frac{\Gamma_1 p}{\varrho}\operatorname{div}\boldsymbol{\delta}\mathbf{r} \tag{5}$$

becomes then an ordinary differential system of the fourth order in the space variable r for $\delta r, p', \varrho', \Phi'$. Once its solution is known, the two other components of the equation of motion

$$\sigma^2 r\delta\theta = \left(\Phi' + \frac{p'}{\varrho}\right)\frac{1}{r}\frac{\partial P_l^m}{\partial\theta}\, e^{\mathrm{i}m\varphi} \tag{6}$$

$$\sigma^2 r \sin\theta\, \delta\varphi = \left(\Phi' + \frac{p'}{\varrho}\right) \mathrm{i}m \frac{P_l^m}{r\sin\theta}\, e^{\mathrm{i}m\varphi} \tag{7}$$

serve to define the horizontal components of the displacements.

If we associate with the 4th order problem, the four natural boundary conditions:
δr: finite in $r=0$ (in fact $\delta r \propto r^{l-1}$ and $p', \varrho', \Phi' \propto r^l$ as $r \to 0$)
$\delta p = p' + \delta r(\mathrm{d}p/\mathrm{d}r) = 0$ in $r=R$
$(\Phi_i')_R = (\Phi_e')_R = C/R^{l+1}$ (continuity of potential across $r=R$)
$(\mathrm{d}\Phi_i'/\mathrm{d}r)_R + (l(l+1)/R)\,(\Phi_i')_R = -(4\pi G\varrho\delta r)_R$ ($=0$ if $\varrho_R=0$)
(continuity of gravitational force across R),

we are left with a self-adjoint 4th order eigenvalue problem whose solutions, orthogonal to each other

$$\int_0^R \delta\mathbf{r}_i \cdot \delta\mathbf{r}_k^* \varrho \,\mathrm{d}\mathscr{V} = 0 \qquad i \neq k \tag{8}$$

define the stationary modes of non-radial oscillations.

As can be verified immediately from (1a, c, d), (5), (3) and (4) and the boundary conditions, this eigenvalue problem does not depend explicitly of m so that, for a given l, there are $(2l+1)$ solutions (corresponding to the various possible values of m in $-l \leqslant m \leqslant l$) associated with each eigenvalue. Fields of forces devoid of spherical symmetry can remove this $(2l+1)$-fold degeneracy either totally (for instance: rotation) or partially (for instance: magnetic field, tides).

One must also be aware that $\sigma^2=0$ is a highly degenerate trivial eigenvalue of the spherical configuration corresponding to a displacement normal to gravity with all corresponding $\varrho', p', \Phi'=0$, which may give rise to significant eigenvalues (for instance, corresponding to toroidal oscillations) in presence of fields of forces of the type considered above so that the total number of distinct modes for a given l may become larger than $(2l+1)$ (cf. for instance Perdang, 1968, for a general group-theoretical discussion of the question).

Apart from this question of degeneracy, the 4th order eigenvalue problem is rather peculiar of the type which has recently drawn a certain amount of attention from mathematicians (for problems related to ours, cf. for instance: Eisenfeld, 1968a, b; Weinberger, 1968) under the name of 'non-linear' eigenvalue problems because, contrarily to the case of the classical Sturm-Liouville problem, the eigenvalue, here $\lambda=\sigma^2$, enters non-linearly in the coefficients. This translates mathematically the fact that, in general, the spectrum of eigenvalues splits into distinct spectra with quite different physical meanings.

To illustrate the situation, let us consider the approximation obtained by neglecting Φ' which for $l>0$ is usually a fairly good approximation except perhaps for the lowest modes since Φ', solution of Poisson's equation, is an integral expression over the whole star decreasing when l and the number of nodes of ϱ' along the radius increase. This has been confirmed by numerical integrations (Robe, 1968) which furthermore show that the characteristics of the spectra are not affected. In that case, we are left with a second order differential problem which can be written

$$\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}u=\left[\frac{l(l+1)}{\sigma^2}-\frac{\varrho r^2}{\Gamma_1 p}\right]y \tag{9a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}r}+y\mathscr{A}=\frac{1}{r^2}(\sigma^2+\mathscr{A}g)\,u \tag{9b}$$

with $u=r^2\,\delta r$ and $y=\dfrac{p'}{\varrho}$.

The 'non-linear' character of the eigenvalue problem is now apparent. However, as pointed out by Cowling (1941) and ignoring for the time being possible difficulties towards the extremities of the interval, this problem with the first two boundary conditions tends to a Sturm-Liouville problem for $|\sigma^2|$ either very large or very small.

Indeed, in the first case, eliminating u from (9a and b) and neglecting terms of the order of $1/\sigma^2$, we obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}r^2}+\frac{\mathrm{d}y}{\mathrm{d}r}\left(\frac{2}{r}+\frac{1}{\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}r}\right)+y\left[\frac{\sigma^2\varrho}{\Gamma_1 p}+\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\mathscr{A})-\frac{l(l+1)}{r^2}\right]=0. \tag{10}$$

The only subsisting term containing σ^2 being $\sigma^2\varrho/\Gamma_1 p$, the solutions must correspond to acoustical or pressure (p) modes associated with waves propagating with the velocity of sound $c=(\Gamma_1 p/\varrho)^{1/2}$. It can also be shown that, for all realistic values of the physical parameters, all these p modes have discrete positive eigenvalues, $\sigma_p^2>0$, with an accumulation point at $+\infty$. Thus no instability can arise through these modes. This is very different from the radial case ($l=0$) for which a very important dynamical instability enters through the fundamental mode if $\Gamma_1<4/3$. Nevertheless, as we shall see later, there are many close analogies with the radial modes which are also acoustical modes.

If σ^2 is sufficiently small, eliminating y between (9a and b) and neglecting terms of the order of σ^2 yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2}+\frac{\mathrm{d}u}{\mathrm{d}r}\frac{1}{\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}r}+u\left[-\frac{\mathscr{A}g}{\sigma^2}\frac{l(l+1)}{r^2}-\frac{l(l+1)}{r^2}+\frac{d}{\mathrm{d}r}\left(\frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\right]=0. \tag{11}$$

This is again of the type of a Sturm-Liouville problem for which the parameter $\lambda=1/\sigma^2$

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will have an infinite discrete spectrum with a point of accumulation at infinity, i.e. σ^2 will take an infinite number of discrete values accumulating at zero.

If we assume $\mathscr{A} < 0$ everywhere (convective stability), these solutions correspond to gravity or g modes associated with waves propagating with a velocity of the order of $r\sigma^2/[l(l+1)(-\mathscr{A})g]^{1/2}$. In that case and provided $d/dr\,[(1/\Gamma_1 p)\,dp/dr]$ be negative everywhere, which is usually realized in realistic conditions, these σ^2 are all positive and the star is dynamically stable.

However, if $\mathscr{A} > 0$ everywhere, all the eigenvalues σ^2 become negative and dynamical instability sets in giving rise to convective motions inside the star. In other words, the only dynamical instability which may enter through non-radial perturbations is the type of violent internal motions known usually as convection. It can also be shown (Lebovitz, 1965, 1966) that $\mathscr{A} > 0$ in any finite interval however small implies the existence of negative σ_g^2 and convection in that region so that $\mathscr{A} < 0$ is really a necessary and sufficient condition for convective stability.

In the same paper, Cowling (1941) established also the existence of an intermediate mode which he called fundamental or f mode with normally no nodes in δr or ϱ' and with a positive eigenvalue σ_f^2 separating the σ_g^2 and σ_p^2.

All this has been confirmed and somewhat clarified on the basis of a variational principle by Chandrasekhar (1964) and Chandrasekhar and Lebovitz (1964) who have shown that the eigenvalues σ^2 of the general 4th order problem expressed in terms of integrals on the displacement $\boldsymbol{\delta r}$ are stationary when the latter tends to an eigensolution. In other words, each eigenvalue is an extremum but we don't know whether it is a maximum or a minimum. On the other hand, the discussion above suggests directly that asymptotically the σ_p^2, as eigenvalues of a Sturm-Liouville problem should be minima. As to the σ_g^2, since their inverse for the same reason as above are minima, one may expect them to be, asymptotically at least, maxima, this applying to their absolute values in the case of negative σ_g^2. One would then suspect σ_f^2 to be neither a true maximum or minimum. The inference seems to be supported by the results of Robe and Brandt (1966) who found, by applying the Rayleigh-Ritz method to the Chandrasekhar variational principle, that the exact σ_p^2 and σ_g^2 were approached respectively by decreasing and increasing values while in the case of σ_f^2, the successive approximations, at least in a fair proportion of cases, tend to the exact value both by larger and smaller values.

3. Non-Radial Adiabatic Oscillations of Various Models

3.1. The Homogeneous Incompressible Sphere (Thomson, Lord Kelvin, 1863)

In that case, for any value of $l \geqslant 2$ ($l = 1$ has to be rejected here as it would correspond to a displacement of the centre of mass), there are $(2l+1)$ solenoidal modes with one and the same frequency

$$\sigma_l^2 = \frac{4\pi G\varrho}{3}\, l\, \frac{2(l-1)}{2l+1}.$$

In the nomenclature adopted above this is a f mode which is the only possible type in this model.

3.2. The heterogeneous incompressible sphere

It is rather surprising that, although this is related to the simplest (no viscosity, no surface tension) spherical version of the Rayleigh-Taylor problem, detailed results have only been obtained recently. Perhaps the simplest case corresponds to an equilibrium configuration built of concentric layers of incompressible fluids of different densities. This has been considered recently by a student (Camps, 1973) who has accumulated quite a number of numerical results. As expected, apart from the f mode which is always present and is the continuation of the Kelvin mode of the homogeneous sphere, one finds as many g modes as there are discontinuities of densities. To each eigenvalue σ_g^2 corresponds an eigenfunction which in general reaches maximum amplitude (in absolute value) at the associated discontinuity (surface waves). The sign of σ_g^2 is the same as that of the density discontinuity $(\varrho_{\text{in}} - \varrho_{\text{ex}})$ with which it is associated, an instability, of course, developing at each interface where a heavier fluid (ϱ_{ex}) is superposed on a lighter one (ϱ_{in}). Thus, in general, when discontinuities of both signs are present, the g spectrum is split into two which we shall denote by $\sigma_{g^+}^2$ and $\sigma_{g^-}^2$ according as they correspond to stability or instability. In the case of four layers with two unstable discontinuities $(\varrho_{\text{in}} - \varrho_{\text{ex}} < 0)$ some peculiar behaviour was noted as the position and the value $(\varrho_{\text{in}} - \varrho_{\text{ex}})$ were varied at one of the unstable discontinuities. As the two e-folding times $(\propto 1/\sqrt{|\sigma_{g^-}^2|})$ become close to each other, the eigenfunction normally associated with one discontinuity acquires a secondary extremum, sometimes quite important, at the position of the other discontinuity.

Another case was worked out by Robe (1974) as a by-product of his programme for polytropes since by letting the physical Γ_1 tend to infinity, one reduces the problem to that of the non-radial oscillations of a spherical configuration composed of an incompressible fluid with a continuously varying density. As $\mathrm{d}\varrho/\mathrm{d}r$ and $\mathscr{A}$ which reduces here to $(1/\varrho)(\mathrm{d}\varrho/\mathrm{d}r)$ are always negative in this case $(n>0)$, one gets a discrete infinite g spectrum with all positive eigenvalues $\sigma_{g^+}^2$ in addition to the f mode. Another incompressible sphere with $\varrho = \varrho_c(1 + 0.1\, r^2/R^2)$ yielded a completely negative g spectrum as expected. Of course the p spectrum is suppressed (all $\sigma_p^2 \to \infty$) by the incompressibility.

3.3. The homogeneous compressible sphere

This was the first compressible case treated in full details (Pekeris, 1938). The problem splits here into two second order differential equations, one for div $\boldsymbol{\delta r}$ and Poisson's equation which can be solved when the solution of the first equation is known. The latter can be expressed in polynomial form yielding an explicit biquadratic algebraic equation for the eigenvalues for a given l. There appear thus, in addition to the f mode which is identical here to the solenoidal Kelvin mode of the incompressible sphere with the same density (Robe, 1965), two families of modes, one with positive increasing eigenvalues as the number of nodes along r increases, the p modes, and

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one with negative eigenvalues tending to zero as the number of nodes along r increases, the g modes. That the latter are unstable $(\sigma_{g^-}^2)$ is not surprising since, in this case, $\mathscr{A}$ which reduces to $-(1/\Gamma_1 p)(\mathrm{d}p/\mathrm{d}r)$ is positive everywhere. As illustrated on Figure 1, the p modes go over smoothly into the radial modes ($l=0$) while, of course, the g and f modes have no counterparts for $l=0$.

3.4. The polytropes

The study of their non-radial oscillations was the occasion on which Cowling (1941) introduced the spectral classification used above. In their case

$$\mathscr{A}=\left(\frac{n}{n+1}-\frac{1}{\Gamma_1}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}r},$$

where n is the polytropic index. If n and Γ_1 are constant throughout the star, $\mathscr{A}$ has a constant sign: positive (convective instability) if $(n+1)/n>\Gamma_1$, negative (convective stability) in the opposite case.

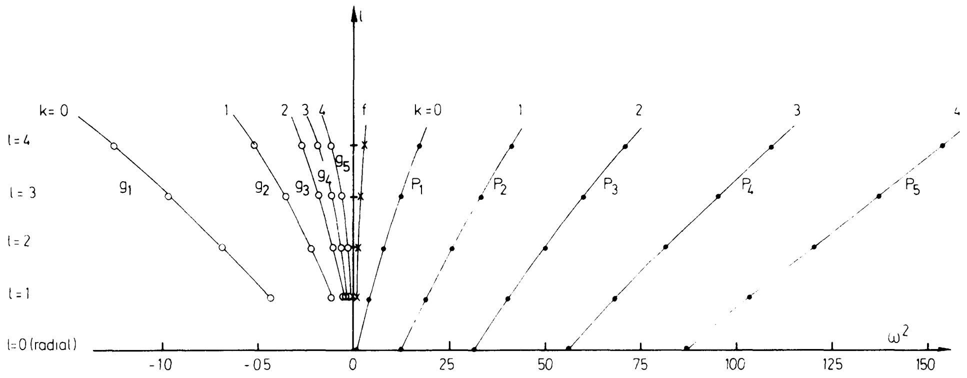

Fig. 1. Distribution of the discrete eigenvalues of the p (dots), f (crosses) and g (circles) modes for the homogeneous compressible model.

The advent of large electronic computers has made the numerical integration of even the 4th order problem a fairly simple matter especially for straight polytropes ($n=ct$) and many numerical results by Hurley *et al.* (1966) and by Robe (1968) have been added to the early results of Cowling (1941) and Kopal (1949). In particular, Robe (1968) has integrated the problem (both with and without Φ') for a whole series of polytropes ($n=0$, 1, 2, 3, 3.25, 3.5, 3.75 and 4) up to fairly high modes for various values of l and for $\Gamma_1=5/3$. Since $\Gamma_1=5/3$, only the polytropes $n=0$ and $n=1$ have $(n+1)/n>\Gamma_1$ and, as expected, they are the only one with a negative g spectrum. On the whole up to $n=3$, everything runs smoothly, the number of nodes which is always zero in δr, ϱ' and p' for the f mode goes to 1 for the same variables in the first p mode and increases regularly with the order of the mode to which it remains equal. The same is true of the g modes if they are stable. If they are unstable, the number of

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nodes in ϱ' and p' is still equal to the order of the mode but the number of nodes of δr is lower by one unit (cf. Table I).

However, some years ago, the work of Owen (1957) who had been unable to find modes without nodes for $n > 3.25$ raised the question of the existence of the f mode and of the first few p or g modes in models of high enough central condensation. However Robe's results show that all the modes continue to exist but they tend to acquire extra-nodes. For instance for $n = 3.25$, the f mode which has still zero node in δr and ϱ' acquires 2 nodes in p'. For $n = 3.5$ not only does the f mode have 2 nodes in δr and ϱ' as well as in p' but the g_1 mode has acquired three nodes in p'. The situation deteriorates further as we go to higher central condensation and, for instance, for $n = 4$, the f mode has 4 nodes in all variables and the abnormality in the number of nodes extends to the p_4 and g_4 modes. One should also note that for high enough modes (p_5 and g_5 for $n = 4$) the number of nodes becomes regular again and equal to the order of the mode (cf. Table I).

TABLE I

Eigenvalues σ^2 in units of $\pi G\bar{\varrho}$, $l = 2$, $\Gamma_1 = 5/3$

	$n=1$				$n=3$				$n=3.5$				$n=4$			
Modes		Nodes of δr	Nodes of p'	Nodes of ϱ'		Nodes of δr	Nodes of p'	Nodes of ϱ'		Nodes of δr	Nodes of p'	Nodes of ϱ'		Nodes of δr	Nodes of p'	Nodes of ϱ'
p_{10}	416	10	10	10	312	10	10	10	297	10	10	10	288	10	10	10
p_9	346	9	9	9	262	9	9	9	252	9	9	9	245	9	9	9
p_8	282.2	8	8	8	217	8	8	8	210	8	8	8	206	8	8	8
p_7	224.5	7	7	7	176.5	7	7	7	171	7	7	7	171	7	7	7
p_6	173.2	6	6	6	139.8	6	6	6	137.2	6	6	6	140.3	6	6	6
p_5	128.2	5	5	5	107.4	5	5	5	106.8	5	5	5	116.6	5	5	5
p_4	89.44	4	4	4	79.23	4	4	4	80.39	4	4	4	102.2	4	6	6
p_3	57.11	3	3	3	55.29	3	3	3	57.88	3	3	3	83.83	5	5	5
p_2	31.32	2	2	2	35.63	2	2	2	39.68	2	2	2	67.75	4	4	4
p_1	12.41	1	1	1	20.35	1	1	1	27.91	1	1	1	56.18	3	5	5
f	1.997	0	0	0	10.90	0	0	0	21.55	2	2	2	45.77	4	4	4
g_1	−0.4039	0	1	1	6.553	1	1	1	16.13	1	3	1	36.79	3	5	5
g_2	−0.1844	1	2	2	3.771	2	2	2	11.39	2	2	2	30.67	4	4	4
g_3	−0.1070	2	3	3	2.430	3	3	3	7.655	3	3	3	23.99	5	5	5
g_4	−0.07029	3	4	4	1.694	4	4	4	5.417	4	4	4	20.48	4	4	6
g_5	−0.04989	4	5	5	1.248	5	5	5	4.023	5	5	5	17.02	5	5	5
g_6	−0.03731	5	6	6	0.958	6	6	6	3.099	6	6	6	13.44	6	6	6
g_7	−0.02899	6	7	7	0.759	7	7	7	2.464	7	7	7	10.78	7	7	7
g_8	−0.02319	7	8	8	0.616	8	8	8	2.00	8	8	8	8.82	8	8	8
g_9	−0.01898	8	9	9	0.510	9	9	9	1.66	9	9	9	7.34	9	9	9
g_{10}	−0.01583	9	10	10	0.429	10	10	10	1.40	10	10	10	6.20	10	10	10

The same phenomenon has now been found in physical models sufficiently evolved to have large central condensations (Dziembowski, 1971; Scuflaire, 1973). As stressed by Dziembowski and as we shall discuss later this can have very important consequences on other aspects of the problem like the question of vibrational stability.

The origin of this curious behaviour is however the same in physical models and polytropes for which it has been discussed by Robe (1968). This can be understood most easily on the basis of the approximate ($\Phi' = 0$) second order system (9a–b) which, in the new variables

$$v = u p^{1/\Gamma_1} \qquad w = y \varrho p^{-1/\Gamma_1}$$

becomes

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \left[\frac{l(l+1)}{\sigma^2} - \frac{\varrho r^2}{\Gamma_1 p}\right] \frac{p^{2/\Gamma_1}}{\varrho} w \tag{12a}$$

$$\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{1}{r^2} (\sigma^2 + \mathscr{A} g) \frac{\varrho}{p^{2/\Gamma_1}} v \tag{12b}$$

Let us first note that in general whatever the value of σ^2, both brackets in these equations will vanish at some point in the interval $0 < r < R$ since $\varrho r^2/\Gamma_1 p$ and $\mathscr{A} g$ vary from 0 at the centre to respectively $+\infty$ and $+$ or $-\infty$ (depending on the sign of $\mathscr{A}$) at the surface. This introduces some extra-singularities in the second order problem which tend to the surface or the centre as σ^2 becomes very large or very small. These singularities are however regular in Fuchs sense and, furthermore, they are apparent in the sense that all independent solutions and their derivatives remain finite and continuous across them. In particular, as the detailed discussion of the asymptotic behaviour of the eigensolutions shows (Vandakurow, 1967; Tassoul, 1967; Tassoul and Tassoul, 1968; Smeyers, 1968), the presence of these singularities does not invalidate our simplified arguments following Equations (9a–b) resting in particular on the neglect of terms in $1/\sigma^2$ (Equation (10)) or in σ^2 (Equation (11)) used to establish the characteristics of the p and g spectra. However, they may affect the behaviour of the solutions: whenever one of the involved bracket vanishes, the derivative of v (or w) vanishes too, just as it vanishes at the nodes of w (or v) and this can play havoc with the number and the distribution of zeros of consecutive modes at least for the first few modes in very condensed configurations. Indeed as the central condensation increases, $\mathscr{A} g$ develops a deeper and deeper minimum (cf. Figures 2 and 3) fairly close to the centre which implies that a greater and greater number of the low modes, starting with the f modes, have σ^2 values which make the factor $(\sigma^2 + \mathscr{A} g)$ vanish at three points in the star instead of one for low central condensation models. According to (12b), this forces w (or p') to change the sense of its variation and will, in favourable circumstances, bring about an extra zero. But this in turn, affects v (or δr) in a similar way and again an extra node may appear as discussed in detail for the polytrope $n = 4$ in Robe (1968). However for high enough p modes (σ^2

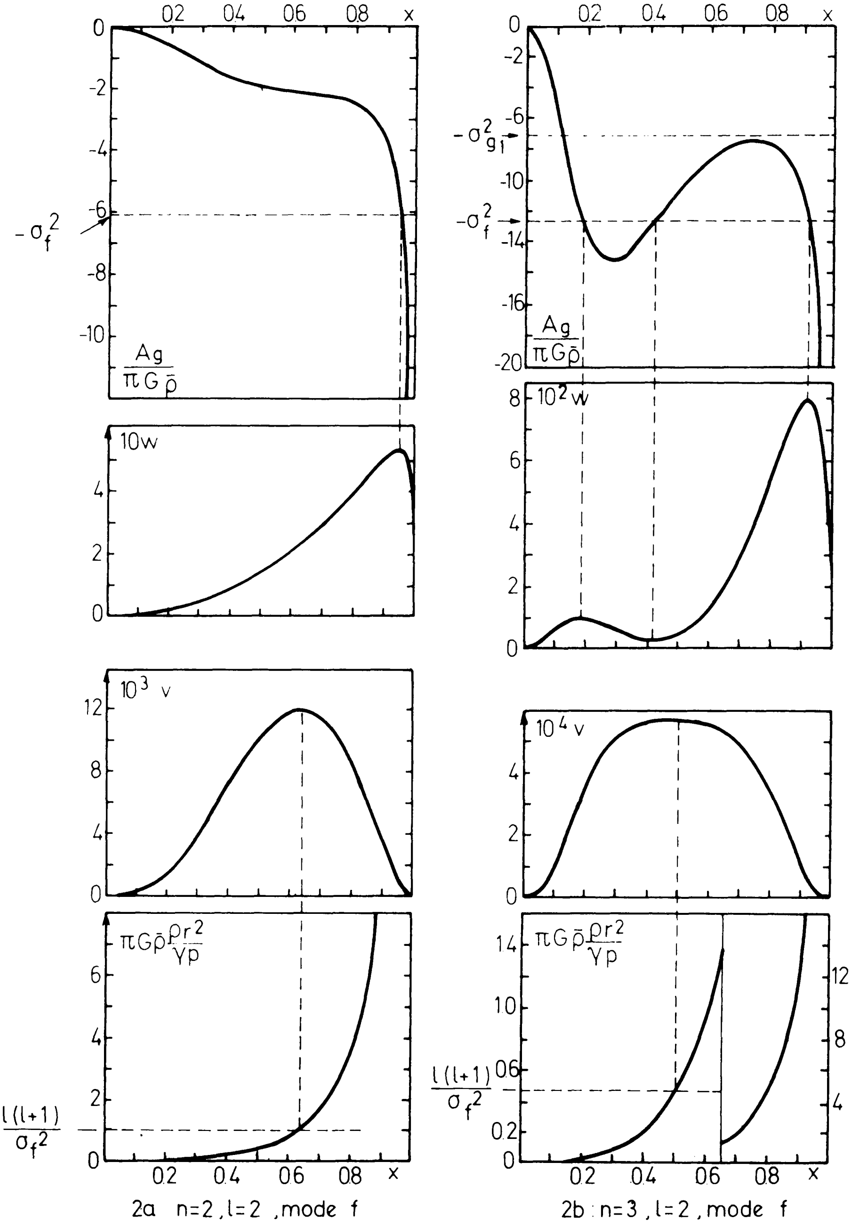

Fig. 2. Illustration of the behaviour of the apparent singularities, in various polytropes of increasing central condensation ($n = 2$ and 3) and of the apparition of extra-nodes.

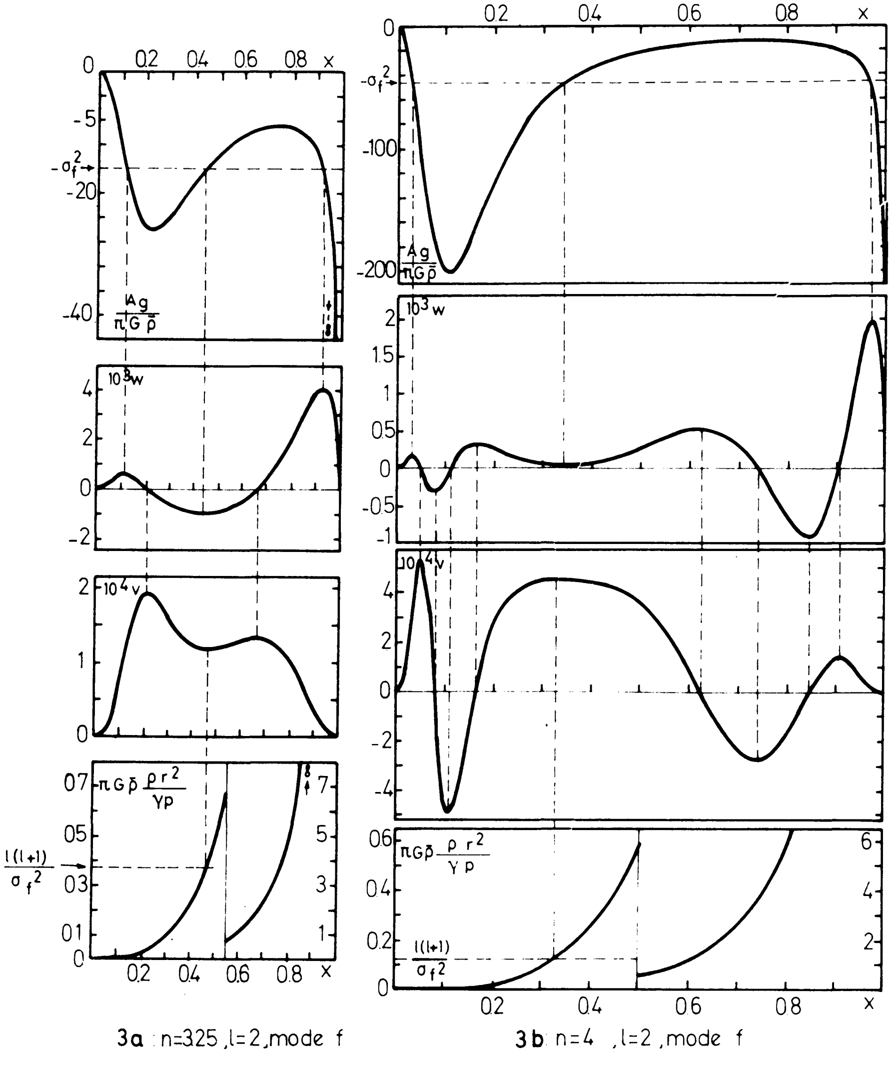

Fig. 3. Same as Figure 2 for polytropes $n=3$ and 4.

sufficiently large) or g modes (σ^2 sufficiently small) the situation always becomes normal again because, as in models with small central condensation, the factor $(\sigma^2 + \mathscr{A}g)$ finally vanishes again at one point only.

The minimum of $\mathscr{A}g$ tends to become deeper and to approach the centre as the central condensation increases as illustrated on Figure 4 where the behaviour of $\varrho r^2/\Gamma_1 p$ is also represented as well as the positions of a few apparent singularities for the polytrope $n=4.2$ and a fairly highly evolved model.

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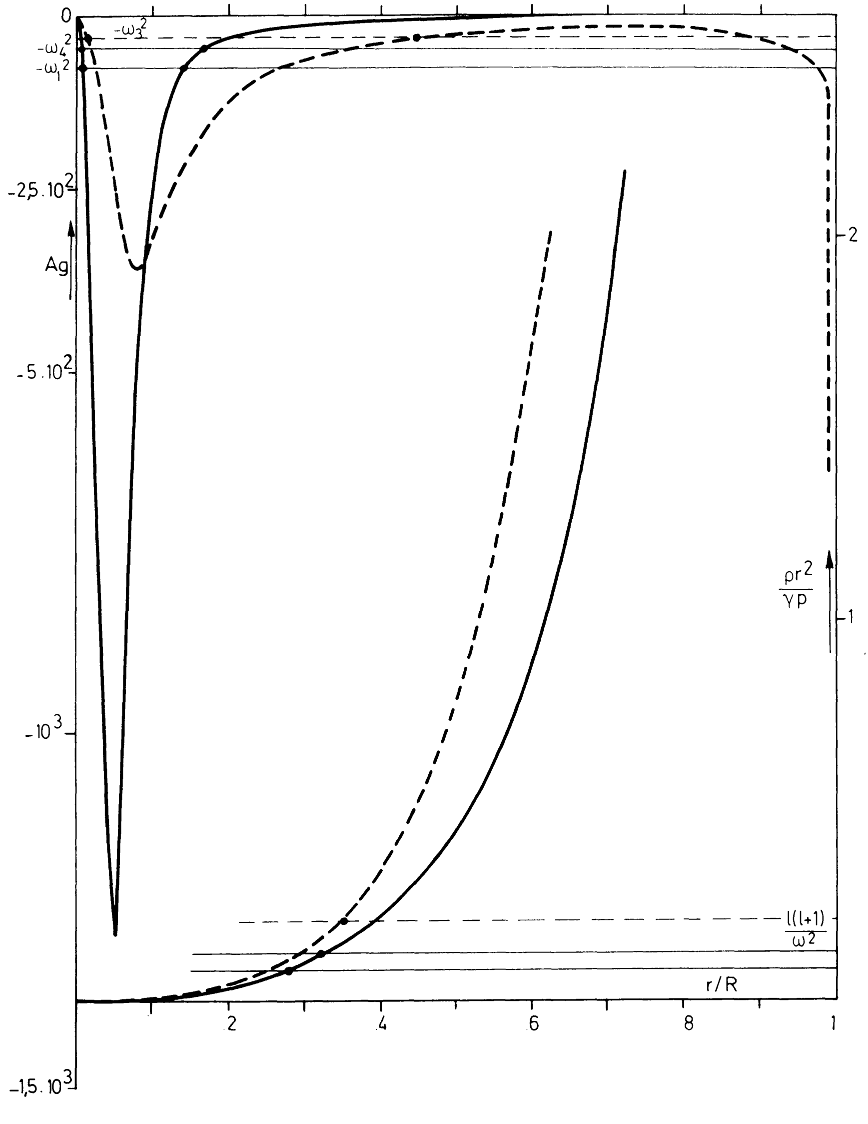

Fig. 4. Illustration of the behaviour of $\mathscr{A}g$ and $\varrho r^2/\Gamma_1 p$ in highly condensed configurations with the positions of a few apparent singularities. Polytrope $n=4.2$: broken lines; evolved model: full lines. In the latter $\mathscr{A}g$ tends to remain close to zero up to very near the surface because of the extensive external convection zone (after Scuflaire, unpublished).

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The behaviour of p and g modes, taken from Robe's study, is illustrated on Figure 5 for various polytropes and $l=2$. The p modes behave very much like the radial modes with the ratio of the amplitudes at the surface to that near the centre increasing very much as the central condensation increases. On the other hand, as l increases, the

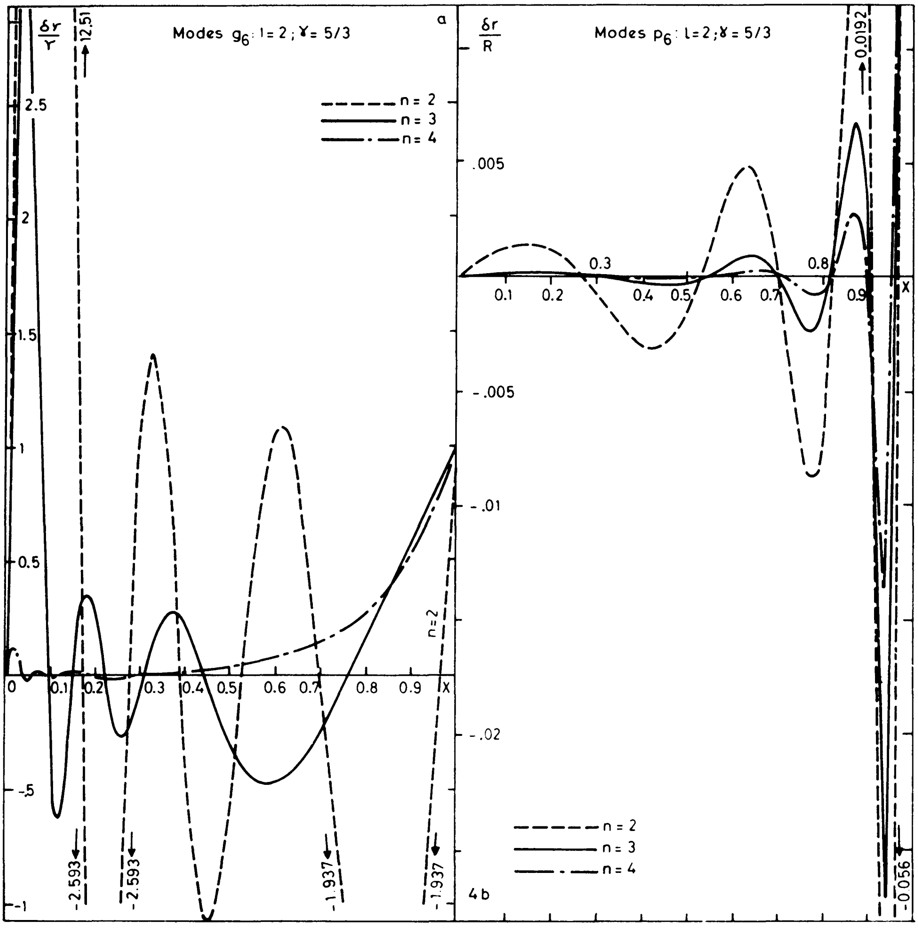

Fig. 5a' b. Behaviour of the radial displacement for the g_6 and p_6 modes for $l=2$ and various polytropes.

nodes of the p modes tend to concentrate in the external layers as already noted by Smeyers (1967). This is illustrated on Figure 6 for the p_3 mode of the polytrope $n=3$ according to Robe (1973b). The g modes offer more peculiar characteristics, the amplitude being susceptible to reach appreciably higher values in the interior than at the surface at least as long as the central condensation is not too large. On the other hand, in the case of very high central condensation, the occurrence of extra-nodes

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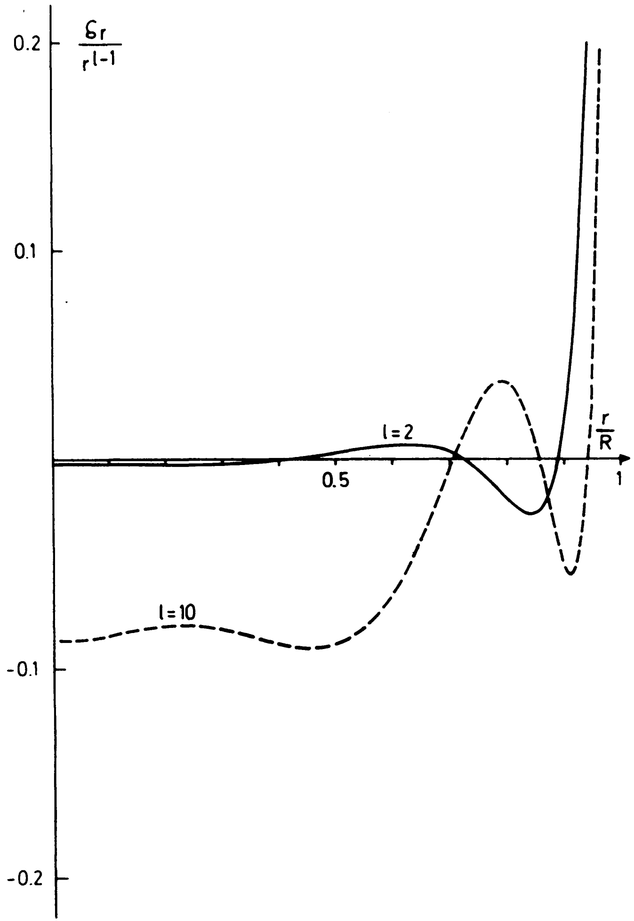

Fig. 6. Behaviour of p modes for increasing values of the degree l of the spherical harmonic.

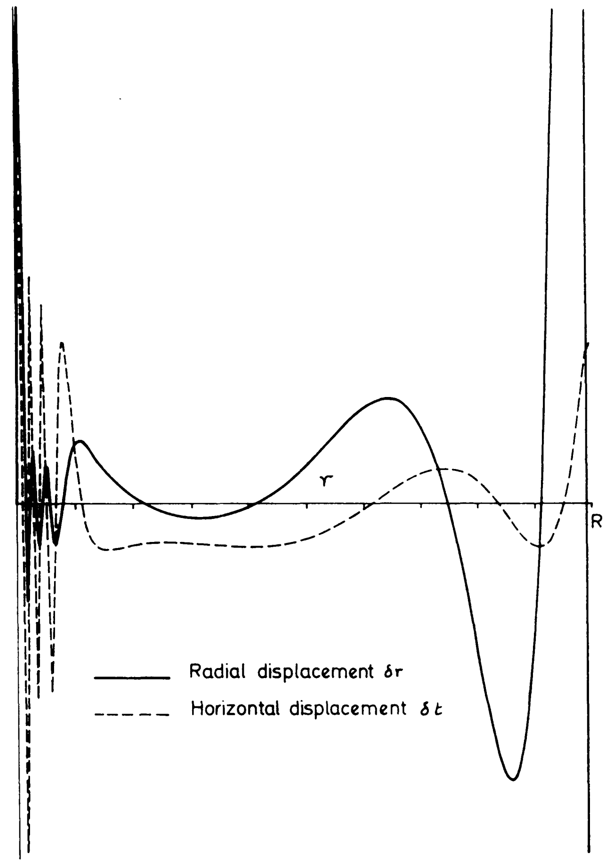

Fig. 7. Run of the amplitude for the radial and horizontal displacements in the case of the g_4 mode of a highly condensed physical model ($\varrho_c/\bar{\varrho} \simeq 3 \times 10^3$) (after Scuflaire, unpublished).

close to the centre seems to favour also a pick-up of the amplitude in these regions (cf. Figure 7 from Scuflaire's work).

3.5. Physical Models

The problem has been integrated numerically for main-sequence models of fairly high mass by Van der Borght and Wan Fook Sun (1965) and by Smeyers (1967). Since these models of fairly low central condensation possess very extensive convective cores where $\mathscr{A}$ is essentially equal to zero, it is natural to find, apart from the f and

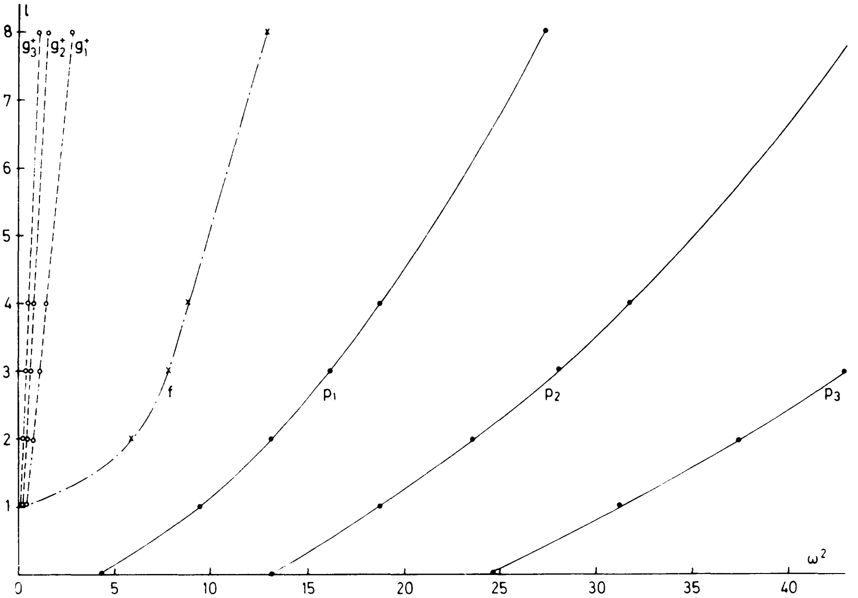

Fig. 8. Distribution of the eigenvalues of the p (dots), f (crosses) and g (circles) modes for a main sequence model of fairly high mass.

p modes, which behave very much as in polytropes of index $n = 3$, families of g^+ modes with eigenvalues concentrated in a very small interval around zero (cf. Figure 8) since if $\mathscr{A}$ was zero everywhere all the σ_g^2 would vanish. The behaviour of these g modes is also peculiar, the amplitude oscillating only in the radiative envelope ($\mathscr{A} > 0$) while it decreases more or less exponentially in the convective core (cf. Figure 9).

As already mentioned, the non-radial oscillations of quite a variety of stellar models at different stages of evolution have been studied in recent years (Dziembowski, 1971; Robe *et al.*, 1972; Dziembowski and Sienkiewicz, 1973; Scuflaire, 1973; Osaki, 1973) mostly in connection with various aspects of vibrational stability towards such perturbations but examples of various adiabatic modes will be found there.

Table II summarizes in general terms the various types of possible spectra for the various models considered here, illustrating their origin and their lineage.

For those interested in the possible applications of non-radial oscillations to the interpretation of periodic variability in certain stars, Tables III and IV present re-

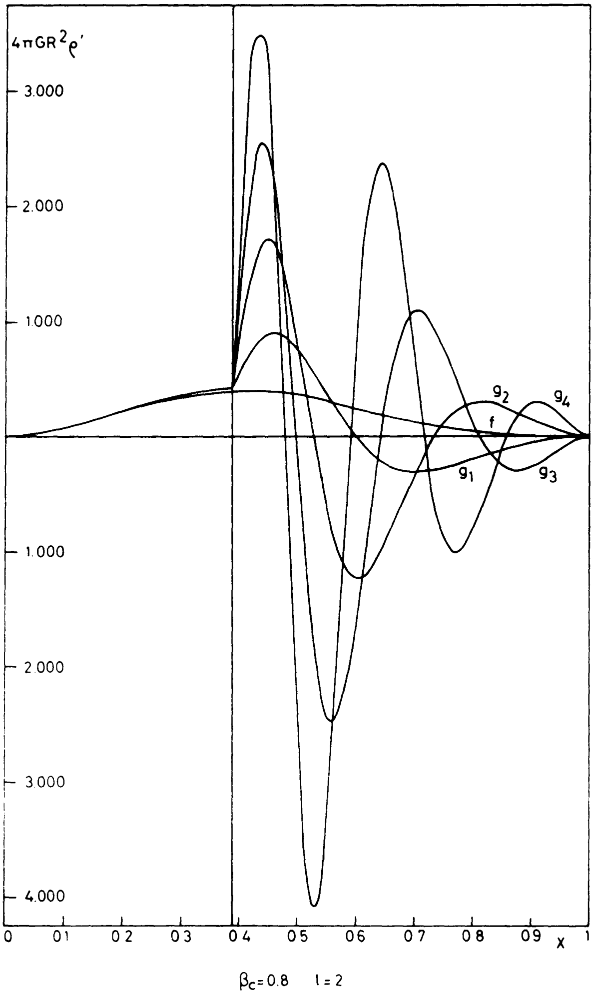

Fig. 9. Run of ϱ' for various g modes in a main sequence model of fairly high mass with a large convective core.

spectively values of $Q = P_{\text{days}}(\bar{\varrho}/\bar{\varrho}_{\odot})^{1/2}$ for massive stars (Smeyers, 1967, cf. also Figure 8) for different values of l and for various polytropes (Robe, 1968) for $l=2$. In some cases, as we shall see later, the jump in period from the f and p modes to the g modes may be much larger (cf. Table VI).

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TABLE II

Types of spectra for various models

(X: exists; (X): exists but has to be rejected; O: does not exist)

Models	Incompressible				Compressible			
	Homogeneous		Heterogeneous		Homogeneous		Non-homogeneous	
	$l=1$	$l>1$	$l=1$	$l>1$	$l=1$	$l>1$	$l=1$	$l>1$
p modes non-solenoidal infinite discrete for *l* fixed	O	O	O	O	X	X	X	X
f mode solenoidal	(X)	X	(X)	X	(X)	X	(X)	O
non-solenoidal, one mode only for *l* and *m* fixed	O	O	O	O	O	O	O	X
g modes solenoidal	O	O	(X)	X	O	O	O	O
non-solenoidal	O	O	O	O	X	X	X	X
			discrete finite *or* infinite			infinite	discrete	

TABLE III

Q values. $(M/M_\odot)\,\bar{\mu}^2=9.85,\ \beta_c=0.8,\ \varrho_c/\bar{\varrho}=20.07$

Degree sph. harm. / Modes	$l=0$ (radial)	$l=1$	$l=2$	$l=3$	$l=4$	$l=8$
g_4		0.6630	0.3882	0.2794	0.2208	0.1288
g_3	–	0.5244	0.3093	0.2245	0.1791	0.1088
g_2	–	0.3829	0.2292	0.1693	0.1376	0.0898
g_1	–	0.2378	0.1477	0.1137	0.0963	0.0718
f	–		0.0514	0.0427	0.0392	0.0323
p_1	0.0566 (fund.)	0.0377	0.0322	0.0289	0.0268	0.0223
p_2	0.0318 (1st mode)	0.0268	0.0238	0.0219	0.0206	0.0174
p_3	0.0233 (2nd mode)	0.0208	0.0190	0.0177	0.0168	0.0145
p_4		0.0170	0.0158	0.0149	0.0142	0.0124

TABLE IV

Q values for polytropes ($\Gamma_1 = 5/3$, $l = 0$: radial and $l = 2$)

$n=2$, $\varrho_c/\bar{\varrho}=11.40$		$n=3$, $\varrho_c/\bar{\varrho}=54.18$		$n=3.5$, $\varrho_c/\bar{\varrho}=152.9$		$n=4$, $\varrho_c/\bar{\varrho}=622.4$	
Radial	Non-radial	Radial	Non-radial	Radial	Non-radial	Radial	Non-radial
	g_{10}: 0.664		g_{10}: 0.204		g_{10}: 0.113		g_{10}: 0.0536
	g_5 : 0.384		g_5 : 0.120		g_5 : 0.0668		g_5 : 0.0325
	g_4 : 0.327		g_4 : 0.103		g_4 : 0.0575		g_4 : 0.0296
	g_3 : 0.270		g_3 : 0.0851		g_3 : 0.0484		g_3 : 0.0273
	g_2 : 0.213		g_2 : 0.0681		g_2 : 0.0397		g_2 : 0.0242
	g_1 : 0.154		g_1 : 0.0523		g_1 : 0.0333		g_1 : 0.0221
	f : 0.06572		f : 0.04056		f : 0.02884		f : 0.01979
fund: 0.0563	p_1 : 0.0341	fund: 0.0381	p_1 : 0.0297	fund: 0.0326	p_1 : 0.0253	fund: 0.0298	p_1 : 0.0179
(1): 0.0317	p_2 : 0.0236	(1): 0.0281	p_2 : 0.0224	(1): 0.0252	p_2 : 0.0213	(1): 0.0232	p_2 : 0.0163
(2): 0.0226	p_3 : 0.0182	(2): 0.0217	p_3 : 0.0180	(2): 0.0205	p_3 : 0.0178	(2): 0.0190	p_3 : 0.0146
(3): 0.0175	p_4 : 0.0148	(3): 0.0176	p_4 : 0.0150	(3): 0.0171	p_4 : 0.0149	(3): 0.0162	p_4 : 0.0132
(4): 0.0143	p_5 : 0.0125	(4): 0.0148	p_5 : 0.0129	(4): 0.0147	p_5 : 0.0130	(4): 0.0140	p_5 : 0.0124

3.6. Models with two or more regions of opposite signs in $\mathscr{A}$

We have already recalled Lebovitz's result (1965, 1966) according to which the presence in a star of any finite region, however small, with $\mathscr{A}>0$ implies the existence of negative g eigenvalues ($\sigma^2_{g^-}$). But it would be surprising that, in such cases, positive $\sigma^2_{g^+}$ would not also subsist. Indeed it is easy to verify (Ledoux and Smeyers, 1966) that Equation (11) can be written in self-adjoint form as

$$\frac{d}{dr}\left(\varrho\frac{du}{dr}\right)+u(\lambda s-t)=0 \tag{13}$$

with

$$\lambda=-\frac{1}{\sigma^2},\quad s=\frac{l(l+1)\,\mathscr{A}g\varrho}{r^2},\quad t=\frac{l(l+1)\,\varrho}{r^2}-\varrho\frac{d}{dr}\left(\frac{1}{\Gamma_1 p}\frac{dp}{dr}\right)$$

the solution u in which we are interested and its derivative having to be regular in the complete interval $0\leqslant r\leqslant R$. But it is well known (cf. Ince, 1964) that, if in this equation s changes sign in the interval $0<r<R$, there are indeed two spectra. If t is everywhere positive, which is generally true in stellar models, then one spectrum is entirely positive while the other is entirely negative and in both cases $|\lambda|$ tends to infinity by discrete values. Converting to σ^2 $(=-1/\lambda)$, we keep of course two spectra, entirely negative or positive, but converging by discrete values to zero.

As far as the eigensolutions are concerned, Equation (13) shows that the g^+ modes (stable, $\lambda<0$, $\sigma^2>0$) do not oscillate in the unstable region ($\mathscr{A}>0$, $s>0$) where the amplitude tends to decrease exponentially while all the nodes tend to accumulate in the stable region ($\mathscr{A}<0$, $s<0$). The reverse is true of the g^- modes (unstable, $\lambda>0$, $\sigma^2<0$) which do not oscillate in the stable region ($\mathscr{A}<0$, $s<0$) (cf. Figure 10).

This behaviour was checked numerically by Smeyers (1966) for the complete 4th order problem on artificial models, for instance with a strongly convectively unstable core ($\mathscr{A}>0$) and a strongly convectively stable envelope ($\mathscr{A}<0$).

The analysis of asymptotic non-radial modes by Tassoul and Tassoul (1968) which is certainly adequate in the case of only two regions of opposite signs for $\mathscr{A}$ (only one turning point at the junction) gives another very clear illustration of the existence and the behaviour of these two g spectra. Let us recall also that even the first g^+ modes in the large mass stars of Smeyers with massive cores in convective equilibrium ($\mathscr{A}\simeq 0$) had also a similar behaviour. In this case, using the small but positive values of $\mathscr{A}$ in the core, necessary to insure the heat transfer, it was also possible to isolate the negative g spectrum corresponding to unstable g^- modes oscillating only in the core and decaying monotonically and rapidly in the external radiative envelope. Such g^- modes have also been studied by Saslaw and Schwarzschild (1965) (cf. also Fowley, 1972) to discuss the penetration of convection from the core into the surrounding stable envelope. Even if this penetration is very small, there are still interesting questions related to the possible excitation by these unstable g^- modes of other modes (g^+ and p modes) and the possible transport of heat by these modes in the

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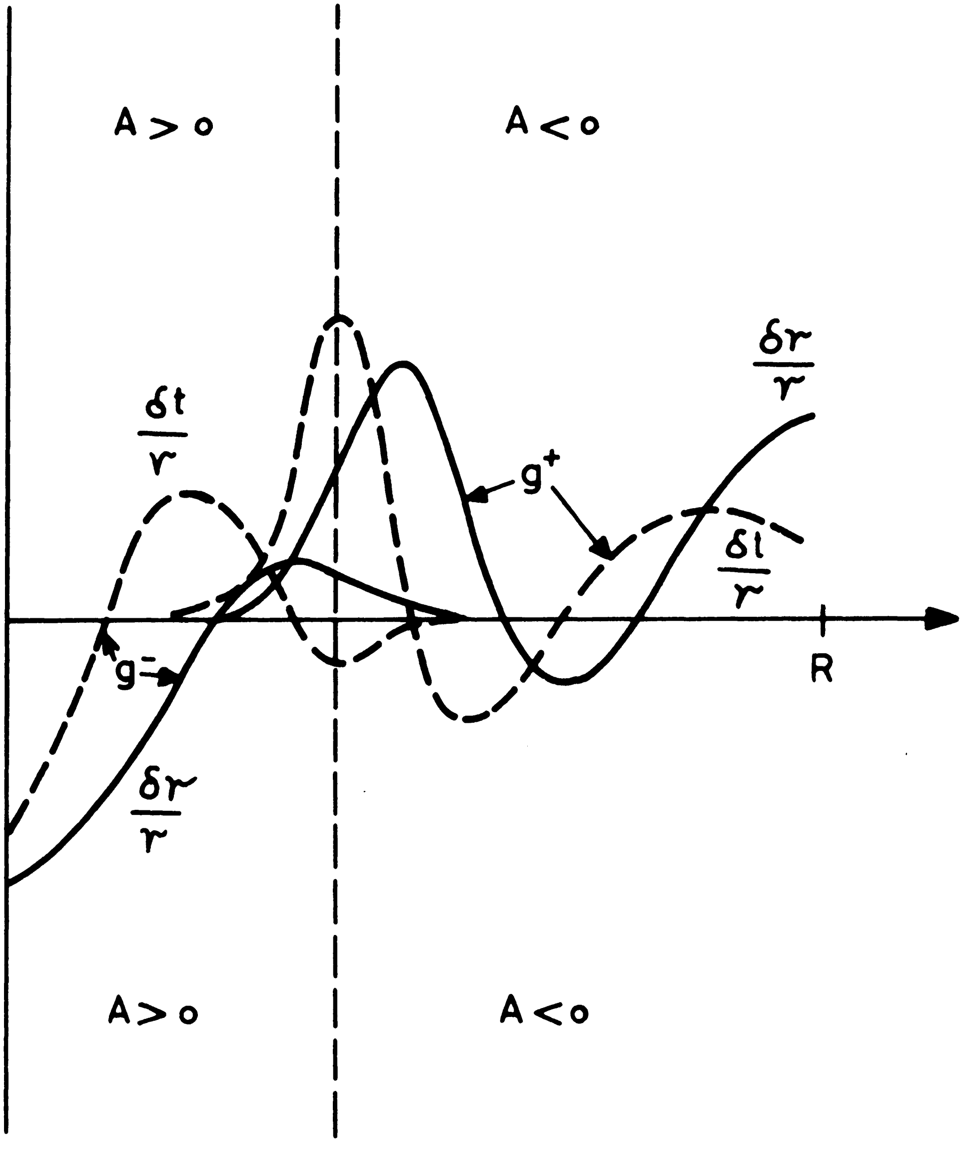

Fig. 10. Stable (g^+) and unstable (g^-) modes in a star with a super adiabatic core and a radiative envelope.

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transition zone in the stable region across which they get damped. A particularly interesting problem would arise if some of these other modes turned out to be vibrationally unstable locally (cf. for instance Spiegel (1964), for such a vibrational instability of p modes in a superadiabatic region, also Moore and Spiegel (1966); for g^+ modes, Souffrin and Spiegel (1967)) so that they could pick up extra energy in that region enabling them to move further out before being damped, achieving a more efficient heat transfer. A similar idea has been discussed by Ulrich (1970) in another context to explain the heating of the solar chromosphere and the corona by dissipation, in these external layers, of the 5-min oscillation considered as a standing acoustic wave trapped at subphotospheric levels.

On the other hand, it is well known (Cowling and Newing, 1949; Ledoux, 1949, 1951) that in presence of a slow rotation $\mathbf{\Omega}$, the $(2l+1)$ degeneracy of the eigenvalues is lifted for all the modes so that the eigenvalues in a rotating frame become

$$\sigma_{l,m} = \sigma_l + mC_l\mathbf{\Omega} \quad -l \leqslant m \leqslant l,$$

where σ_l is the square root of the eigenvalue (σ_l^2) of the non-rotating star for the same value of the degree l of the spherical harmonic and C_l, a constant which can be evaluated from the corresponding eigensolution. If the σ_l considered is that of a g^- mode $(\sigma_l^2 < 0)$ it is pure imaginary and the resulting motion tends to be oscillatory with a period proportional to that of rotation and an increasing amplitude fed by the available buoyancy energy. If the convection zone in which these g^- modes are localized is in the external layers, Ledoux (1967) has suggested that an oscillatory theory of magnetic variable stars could be developed on this idea.

Of course, resonance between one of these overstable convective mode and another dynamically stable g^+ mode for instance could help stabilize both the period and the motion. This last possibility has been suggested recently in a very interesting paper by Osaki (1974) in an effort to find an explanation to the origin of the variability of the β-Cephei stars for which up to now all other sources of pulsational instability for radial as well as non-radial modes have failed. However, in this case, the unstable g^- modes considered by Osaki are localized in the important and fairly rapidly spinning convective core of models in the late stages of the main sequence which is not unreasonable for β-Cephei stars.

He shows that a resonance between the overstable convective mode with a period determined by that of the rotation of the core and the non-radial f mode for $l=2$, $m=2$ is possible. On the other hand, he finds that, due to the already fairly high central condensation of the model, this f mode has one node somewhat outside the convective core and fairly large amplitudes in the core. Of course, this is favourable since it is the coupling, which occurs linearly in presence of rotation, with the g^- mode localized in the core which is supposed to drive the f mode. This would lead to a preferentially excited wave traveling around the equator in the same direction as the rotation as suggested initially by Ledoux (1951), a hypothesis still found to be the most plausible in a previous paper of Osaki (1971) as far as the beat phenomenon and the characteristic variations in line width of these stars are concerned. The Q value, 0.033, given by Osaki for the f mode seems reasonable for a somewhat condensed model (cf. Table IV, n_{eff} between $n=3$ and $n=4$) but it may still be somewhat high since direct evaluations from observations seem to lean towards smaller values (0.027 in Ledoux and Walraven, 1958; 0.021 in Hitotuyanagi and Takeuti, 1964). Nevertheless, it does not seem excluded that the same mechanism could work with one of the first p mode which have lower Q values.

If the convective zone is in the very external layers of small density and small heat capacity, the subsisting superadiabatic gradient, once turbulent convection is established at a dominant relatively small wavelength, may be much larger than in the central core. In such cases, one might expect erratic excitation of g^- modes of long wavelengths giving rise to transient phenomena observable at the surface. As above, in presence of rotation or magnetic field providing a restoring force, this may also give rise to transient or semi-permanent oscillatory motions with periods determined by the rotation or the magnetic field.

In the sun, excitation, by these g^- modes in the hydrogen convection zone, of acoustic p waves which get dissipated as shock-waves in the tenuous layers at higher altitude is often advocated as the heating mechanism of the chromosphere and corona but we cannot go here into the many and complex aspects of these theories. Another interesting problem for the Sun atmosphere has been raised by the discovery by Leighton (1961) of a 5-min periodicity of the gas velocity in the photosphere and the chromosphere. It seems, however, that the complexity of the external layers of the Sun has allowed many different suggestions, all with a certain degree of plausibility and some with special advantages (cf. the review by Schatzman and Souffrin, 1967). The last four known to the present author ranges from trapped acoustic waves well below the photosphere (Ulrich, 1970) or even straight standing p modes (Wolff, 1972) to trapped gravity waves (g^+ modes) between the top of the hydrogen convection zone and the hydrogen ionization zone in the chromosphere (Thomas *et al.*, 1971) through trapped acoustic waves in the chromosphere (McKenzie, 1971) all four recovering at least the 5-min periodicity. A careful numerical analysis of all the possible modes in as good a model as possible of a fairly extensive outer envelope of the Sun perhaps including a good part or the whole convection zone of hydrogen and taking into account the variation of the mean molecular weight $\bar{\mu}$ as well as the new ionization of hydrogen in the upper chromosphere would be very welcome.

Cases of more than two regions of alternating signs in $\mathscr{A}$ are also of interest as such situations arise in many models in the course of stellar evolution. The heterogeneous incompressible model composed of superposed layers of different densities discussed previously offers the simplest example and we already noted there that, in the case of two unstable discontinuities $(\varrho_{\text{in}}-\varrho_{\text{ex}})<0$ separated by a stable one, the behaviour of the eigenfunctions associated with the two corresponding negative eigenvalues $\sigma_{g^-}^2$ can drastically depend on the closeness of these eigenvalues. While, in general, i.e. as long as they are not very close, each of the eigenfunctions has a single maximum at the unstable interface with which it is associated, one of them may acquire a secondary important maximum at the other discontinuity, the minimum amplitude between the two remaining appreciable, when its eigenvalue becomes very close to that associated with the other discontinuity.

Goosens and Smeyers (1974), on the other hand, have studied a compressible composite polytropic model consisting of two convectively stable zones separated by a convectively unstable one. Apart from the unstable g spectrum ($\sigma_{g^-}^2$) associated with the intermediate zone, they also find two stable g spectra, one essentially associated with the core, the other with the envelope. Again, in general, a mode associated with one given region has an appreciable amplitude in that region only and stable modes do not oscillate in the intermediate unstable region while unstable modes do not oscillate in the surrounding stable regions. However, for the stable modes, here again they find more or less accidental 'resonances' giving rise to stable modes with an appreciable amplitude in the two stable regions which contribute nearly equally to the eigenvalues of these modes.

This last phenomenon does not appear in Tassoul and Tassoul (1968) asymptotic discussion which associates with each region of a given sign for $\mathscr{A}$ a spectrum of g modes (stable or unstable depending on the sign of $\mathscr{A}$) with amplitude large in that region only and decreasing exponentially outside. But if we look back at Equation (13) we see that the points where $\mathscr{A}$ vanishes are turning points of the equation in Langer's terminology, i.e. points where the coefficient (s) of the eigenvalue vanishes. Now the method used by Tassoul and Tassoul which is valid when there is only one turning point (2 regions of opposite signs for $\mathscr{A}$) may lead to partly erroneous or incomplete results in the case of two turning points (three regions of alternating signs for $\mathscr{A}$) according to Langer (1959) who, in this case, develops the solution in terms of Weber functions rather than Bessel functions as in the case of one turning point. It may be that this is the reason why the more or less resonant interaction between two zones does not appear in the discussion of Tassoul and Tassoul. Indeed, a preliminary investigation by Iweins and Ledoux (1971) although not using Weber functions but treating the various junctions more completely than in Tassoul and Tassoul, already suggested something of the sort when the eigenvalues of the two regions were commensurable.

I believe that this is an important question which should be pursued because of its possible applications to the case encountered in stellar evolution when two convective zones approach each other sometimes very closely since the penetration through the narrow intermediate stable zone may be sufficiently enhanced to lead really to a complete remixing of the whole layer which could have important effects on the evolution.

4. Vibrational Stability Towards Non-Radial Modes

As we mentioned at the beginning, it is only relatively recently that active interest in this question arose. As is well known, the problem here is to evaluate the effects of the non-conservative terms in Equations (1b) and (1c) which have been neglected up to now. They are usually very small compared to the other terms in the equations except perhaps in a fairly narrow external region whose heat capacity is often sufficiently small to render its influence negligible. In a first order theory, they introduce a damping factor in the time dependence which becomes $e^{i\sigma_a t}\cdot e^{-\sigma' t}$, the perturbation being vibrationally stable or unstable depending on whether σ' is positive or negative.

As long as the σ_a^2 of the adiabatic mode considered does not become too small (which may happen for the highest g modes), one may, as in the case of radial modes, use a perturbation method to evaluate σ'. As shown by Simon (1957) when the unperturbed model is in hydrostatic and thermal equilibrium, this yields

$$\sigma' = -\frac{1}{2\sigma_a^2}\frac{\iiint (\Gamma_3 - 1)\left(\frac{\delta\varrho}{\varrho}\right)_a \left(\varepsilon_N - \varepsilon_\nu - \frac{1}{\varrho}\operatorname{div}\mathbf{F}\right)'_a \varrho r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi}{\iiint (\boldsymbol{\delta}\mathbf{r}\cdot\boldsymbol{\delta}\mathbf{r}^*)_a \, \varrho r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi}, \tag{14}$$

where the suffix 'a' denotes solutions of the adiabatic problem or terms to be evaluated by means of these solutions.

In this expression, we have written explicitly $\varepsilon = \varepsilon_N - \varepsilon_\nu$ where ε_N is the rate at which nuclear energy is transformed into heat and ε_ν represents the net rate of neutrinos emission at the expense of the thermal energy. However, we have still neglected the effects of viscosity which is certainly justified as far as molecular and radiative viscosity are concerned but may not be the case at all if turbulent viscosity comes into play (Counson *et al.*, 1956). On the other hand, in the latter case, the transfer of momentum would also probably be sufficiently important to modify appreciably the distribution of the amplitudes in the star and this leads to a much more difficult problem.

Let us assume that **F** reduces to the radiative flux

$$\mathbf{F}_R = -\frac{c}{\kappa\varrho}\operatorname{grad}\left(\tfrac{1}{3}aT^4\right)$$

and let us denote the sensitivity of ε_N and κ to ϱ and T by the coefficients

$$(\varepsilon_N)_\varrho \equiv \varepsilon_\varrho = \left(\frac{\partial \log \varepsilon_N}{\partial \log \varrho}\right)_T, \quad \varepsilon_T = \left(\frac{\partial \log \varepsilon_N}{\partial \log T}\right)_\varrho, \quad \kappa_\varrho = \left(\frac{\partial \log \kappa}{\partial \log \varrho}\right)_T, \quad \kappa_T = \left(\frac{\partial \log \kappa}{\partial \log T}\right)_\varrho$$

the neutrino losses being neglected in the following although they can easily be reintroduced by analogy with ε_N.

As all perturbations in (14) depend on θ and φ through known functions $P_l^m(\cos\theta)\times \times e^{\mathrm{i}m\varphi}$ and their derivatives, the integration on these variables may be carried out yielding, for a mode corresponding to a spherical harmonic of degree l

$$\sigma_l'(2\sigma_{l,a}^2 I) = -\int_0^R \left[\left(\frac{\delta T}{T}\right)_{l,a}^2 \left(\frac{\varepsilon_\varrho}{\Gamma_3 - 1} + \varepsilon_T\right) + \left(\frac{\delta T}{T}\right)_{l,a} \frac{l(l+1)}{\sigma_{l,a}^2 r^2}\chi_{l,a}\right] 4\pi\varrho\varepsilon r^2 \, \mathrm{d}r +$$

$$+ \int_0^R \left(\frac{\delta T}{T}\right)_{l,a} \frac{d}{\mathrm{d}r}\left\{L(r)\left[4\left(\frac{\delta r}{r}\right)_{l,a} + (4 - \kappa_T)\left(\frac{\delta T}{T}\right)_{l,a} - \kappa_\varrho\left(\frac{\delta\varrho}{\varrho}\right)_{l,a} + \right.\right.$$

$$+\frac{(\mathrm{d}/\mathrm{d}r)\,(\delta T/T)_{l,a}}{(1/T)\,(\mathrm{d}T/\mathrm{d}r)}-\frac{l(l+1)\,\chi_{l,a}}{\sigma^2 r^2}\Bigg]\Bigg\}\,\mathrm{d}r+$$

$$+\int_0^R\left(\frac{\delta T}{T}\right)_{l,a}(\delta r)_{l,a}\,\frac{l(l+1)\,16\pi acT^4}{3\kappa\varrho}\left(\frac{\Gamma_2-1}{\Gamma_2}-\frac{d\log T}{d\log p}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}r}\,\mathrm{d}r+$$

$$+\int_0^R\left(\frac{\delta T}{T}\right)_{l,a}\left(\frac{p'}{p}\right)_{l,a}\frac{\Gamma_2-1}{\Gamma_2}\,l(l+1)\,\frac{16\pi acT^4}{3\kappa\varrho}\,\mathrm{d}r, \tag{15}$$

where

$$I=\int_0^R\left(\delta r_{l,a}^2+\frac{l(l+1)}{\sigma_a^4 r^2}\,\chi_{l,a}^2\right)4\pi\varrho r^2\,\mathrm{d}r$$

and

$$\chi=\left(\frac{p'}{\varrho}+\Phi'\right),\quad L(r)=4\pi r^2 F_r(r),$$

where Φ' is often negligible.

This expression recalls the familiar form of the damping coefficient for radial pulsations, the essential differences coming from the terms in $l(l+1)$ which correspond to the effects of the inequalities of ϱ' and T' on a level surface introduced by the factor $P_l^m(\cos\theta)\,e^{\mathrm{i}m\varphi}$ and the resulting horizontal gradients.

The first term corresponds to the effects of the energy generation and, if it is concentrated around the centre as is often the case, its importance, as already pointed out by Simon (1957), is less than in the radial case as all non-radial amplitudes δT, $\delta\varrho$, δp tend to zero at the centre while they remain finite in radial oscillations. One may then expect that the vibrational stability of ordinary stellar models and especially of main sequence models towards non-radial perturbations will be easily secured by the generally stabilizing influence of the radiative conductivity represented by the second term, as confirmed by Wan's discussion (1966).

The third term is really characteristic of non-radial perturbations. In particular, it is proportional to the argument $\mathscr{S}$ (cf. 2′) of Schwarzschild's criterion according to which, in case of uniform chemical composition, radiative equilibrium is stable if $\mathscr{S}<0$. In this case, this term favours vibrational stability since $(\delta T/T)_a$ and $(\delta r)_a$ being generally of opposite signs while $\mathrm{d}p/\mathrm{d}r$ is negative, its contribution to σ' is positive. However, in a superadiabatic region the sign of $\mathscr{S}$ is inversed and this term may contribute negatively to σ' favouring vibrational instability. This must correspond to some of the effects discussed, as already mentioned, by Spiegel (1964) and by Souffrin and Spiegel (1967).

On the other hand, in presence of a gradient of chemical composition, the mean molecular weight $\bar{\mu}$ decreasing towards the exterior, stable radiative equilibrium subsists as long as $\mathscr{A}<0$ even if $\mathscr{S}>0$ (cf. 2″). Thus such a zone where

$$\frac{\Gamma_2-1}{\Gamma_2}<\frac{\mathrm{d}\log T}{\mathrm{d}\log p}<\frac{\Gamma_2-1}{\Gamma_2}+\frac{\beta}{4-3\beta}\frac{\mathrm{d}\log\bar{\mu}}{\mathrm{d}\log p} \tag{16}$$

should contribute to vibrational instability as shown by Kato (1966) using a local discussion.

As far as the last term is concerned, it is negligible for high g modes (p', small) and it remains probably always rather small, contributing mainly to vibrational stability, since more often than not p' and δT are of the same sign.

When radial modes are unstable as in Cepheids and RR Lyrae stars due to the special behaviour of κ and Γ in the ionization zones of H and He in the very external layers, one might expect (cf. Zahn, 1968) that non-radial p modes should also be vibrationally unstable since the run of the amplitude is generally similar for the two types of modes. However, as pointed out by Dziembowski (1971) the central condensation of the appropriate models is already very high and causes, as we have recalled previously, the apparition in the p eigenfunctions of extra-nodes in the central region with an increase of the amplitude which thus varies very rapidly, enhancing quite strongly the conductive dissipation in that region. According to Dziembowski the last effect is strong enough to damp the distabilizing influence of the external layers. However this may not be true of all p-modes especially those of large l as discussed above (cf. Figure 6).

On the other hand, as we have stressed before, g^+ modes can have fairly large amplitudes in the interior favouring for instance the distabilizing effects of nuclear reactions especially if the latter are not concentrated at the centre. In particular, according to their behaviour in models comprising two regions of opposite signs in $\mathscr{A}$, g^+ modes should have large amplitudes in the small radiative core of a small mass star decreasing rapidly in its convective envelope so that the effect of the core where the energy generation takes place should be dominant. The vibrational stability of such a model for $M=0.5\ M_\odot$ in its early main sequence phase was studied by Robe *et al.* (1972). Although, as expected, the vibrational stability of the g^+ modes, especially g_1^+ for $l=2$, decreased with the radius of the core, it still subsisted for the smallest core radius reached ($r/R=0.354$) although the positive contribution to σ' of term (2) in (15) exceeded only very slightly the absolute value of the negative contribution of term (1). In fact, a somewhat higher sensitivity of ε to T or ϱ or a lower sensitivity of κ could easily have brought about vibrational instability. In fact Noels *et al.* (1974) find that the effective ε_T is appreciably larger than the one used by Robe *et al.* and sufficient to lead to instability. One should note however that as the horizontal component of the g^+ modes considered is also important, the corresponding unstable motion could at most lead, when sufficiently amplified, to some kind of forced turbulent semi-convection. In the case of an inhomogeneous core, the corresponding mixing could be significant because of the induced turbulence and might have important evolutionary consequences.

One might think also that late evolutionary phases, when the models acquire large convective envelopes again and develop shell-source of energy at some distance from the centre, would offer even better conditions for vibrational instability of these g^+ modes. An investigation of this problem has been started in Liège, but it ran into the already discussed difficulty of the appearance of a great many extra-nodes in the central region due to the very high central condensation which, as pointed out by Dziembowski (1971) will increase considerably the dissipation there. Nevertheless, until computations are more advanced it is impossible to come to a definite conclusion in this case.

5. The Solar Neutrino Problem and Local Analysis

The question of mixing as a result of the vibrational instability of some g^+ modes has been raised by Dilke and Gough (1972) in an attempt to find a solution to the solar neutrino problem. Following a suggestion by Fowler (1972), different authors (Rood, 1972; Ezer and Cameron, 1972) have shown that if, after the Sun has evolved sufficiently to possess an appreciable positive gradient of H and especially ^{3}He, the central half or so of it is suddenly mixed, the following readjustments lead finally to an expansion of the central part and a lowering of the neutrino flux to something like the present observed upper limit, a situation which should not last very much longer than a few million years while the luminosity is around its minimum. While the cause of the mixing was left undetermined in these two investigations, Dilke and Gough on the contrary attempted to relate it to the vibrational instability of a g^+ mode.

Of course, this instability must originate here in the gradient of chemical composition set up by the evolution and more particularly in the gradients of H and especially ^{3}He. This would have the great advantage that, the critical values of these gradients necessary for initiating mixing being fixed by the vibrational instability criterion, it would become possible to estimate the time elapsing between successive mixings, i.e. the time necessary to build these critical gradients, and perhaps compare this to the time separating major glaciations since the sun luminosity would also drop to a minimum after each mixing.

The qualitative argument of Dilke and Gough in favour of this g^+ mode vibrational instability was based on a local discussion. This type of approach has been used a few times for different purposes in recent years and since it may at least point to interesting factors affecting the stability, we may as well summarize the results. In such approaches, the general equations are usually particularized to a narrow layer of negligible curvature ($r \to \infty$), $l(l+1)/r^2 \to k_{\mathrm{H}}^2$, where k_{H} is the horizontal wave-number in the plane (x, y). One may then to a very good approximation neglect Φ' and, for fairly high g modes, p' as well except in the equation of motion. Usually the fluid is treated as incompressible

$$\operatorname{div} \boldsymbol{\delta}\mathbf{r} = 0 \tag{17}$$

except for the effects of thermal dilatation (Boussinesq approximation). Some sim-

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plifying assumption is often made also concerning the conductivity, but here we shall let K in the heat equation

$$\mathbf{F} = -K \operatorname{grad} T$$

vary quite generally with ϱ and T.

So as to cover the case treated by Dilke and Gough (pp-chain), we shall write

$$\varepsilon' = \varepsilon\left(\nu\frac{T'}{T} + \frac{\varrho'}{\varrho}\right) + \mu_1\frac{X_1'}{X_1} + \mu_3\frac{X_3'}{X_3},$$

where ν is the usual effective exponent characterizing the sensitivity of the nuclear reactions to changes in T on the time-scale of the perturbation (our previous ε_T; for the pp-chain $\varepsilon_\varrho = 1$); X_1 and X_3 denote the abundances in mass of H and ^{3}He entering the reactions $p(p, \beta^+\nu)\,D(p,\gamma)\,{}^3\mathrm{He}$ and ${}^3\mathrm{He}({}^3\mathrm{He}, 2p)\,{}^4\mathrm{He}$ retained as the most significant by these authors and releasing energy at the rate ε_{11} and ε_{33} respectively. We then have

$$\nu = (\nu_{11}\varepsilon_{11} + \nu_{33}\varepsilon_{33})/\varepsilon, \quad \mu_1 = 2\varepsilon_{11}/\varepsilon, \quad \mu_3 = 2\varepsilon_{33}/\varepsilon$$

and the energy Equation (1c″) may be written

$$T' - T\,\delta r\,\mathscr{S} = \frac{1}{sC_p}\left(\varepsilon + \frac{1}{\varrho}\operatorname{div}(K \operatorname{grad} T)\right) \tag{18}$$

where $i\sigma$ has been replaced by s since we shall adopt here a time dependence of the form e^{st}.

As the variation of the chemical composition from point to point is an important factor in the problem, we must add explicitly the following three equations

$$\bar{\mu}' + \delta z\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}z} = 0, \quad X_1' + \delta z\frac{\mathrm{d}X_1}{\mathrm{d}z} = 0, \quad X_3' + \delta z\frac{\mathrm{d}X_3}{\mathrm{d}z} = 0 \tag{19}$$

which express the fact that the chemical composition does not vary following the motion which is too fast for transmutations to be significant. Finally, the equation of state (gas + radiation) taking into account the variations of chemical composition yields:

$$\frac{\beta\varrho'}{\varrho} - \beta\frac{\bar{\mu}'}{\bar{\mu}} + (4 - 3\beta)\frac{T'}{T} = 0, \tag{20}$$

where $\beta = p_G/p$ and μ has its usual definition

$$\bar{\mu} = \frac{2}{1 + 3X_1 + 0.5X_4}$$

from which $\bar{\mu}'$ may be computed in terms of X_1'. Together, Equations (17) to (20) with the three components of the equation of motion (1b) (with $i\sigma = s$ and in which we take the viscosity into account with the kinematic coefficient of viscosity η constant) constitute a system of 9 independent equations in 9 variables

$$(\varrho', p', T', \bar{\mu}', X_1', X_3', \delta x, \delta y, \delta z).$$

If the space dependence is of the form

$$f' = f e^{i(k_x x + k_y y + k_z z)}$$

introducing a new wave number k_z along the vertical axis z opposed to g, the compatibility condition, assuming all coefficients constant, yields the dispersion relation

$$\begin{aligned} & s^4 + s^3(2\eta k^2 + \delta) + s^2\left(\eta^2 k^4 - \frac{k_H^2}{k^2} g\mathscr{A} + 2\eta k^2 \delta\right) + \\ & s\left[\frac{k_H^2}{k^2} g\delta_1\left(\frac{\mu_1}{X_1}\frac{dX_1}{dz} + \frac{\mu_3}{X_3}\frac{dX_3}{dz}\right) + \frac{k_H^2}{k^2} g\frac{1}{\bar{\mu}}\frac{d\bar{\mu}}{dz}\delta + \right. \\ & \left. + \eta^2 k^4 \delta - \eta k_H^2 g\mathscr{A}\right] + \\ & + \left[\eta k_H^2 g\delta_1\left(\frac{\mu_1}{X_1}\frac{dX_1}{dz} + \frac{\mu_3}{X_3}\frac{dX_3}{dz}\right) + \eta k_H^2 g\frac{1}{\bar{\mu}}\frac{d\bar{\mu}}{dz}\delta_2\right] = 0, \end{aligned} \tag{21}$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2, \quad k_H^2 = k_x^2 + k_y^2$$

$$\delta = \frac{4-3\beta}{\beta}\left(\frac{\varepsilon}{C_p T} - m\right) - \left(\frac{\nu\varepsilon}{C_p T} - \frac{K k^2}{C_p \varrho} + n\right) = \delta_1 - m\frac{4-3\beta}{\beta} - \delta_2$$

with

$$m = \frac{1}{C_p T}\left(\frac{K}{\varrho} - K_\varrho\right)\frac{d^2 T}{dz^2} + \frac{1}{C_p T}\frac{dT}{dz}\left(\frac{1}{\varrho}\frac{dK}{dz} - \frac{dK_\varrho}{dz} - ik_z K_\varrho\right)$$

$$n = \frac{1}{C_p T} K_T \frac{d^2 T}{dz^2} + \frac{1}{C_p \varrho}\frac{dT}{dz}\left[\frac{dK_T}{dz} + ik\, K_T\right] + \frac{ik_z}{C_p \varrho}\frac{dK}{dz}$$

$$K_\varrho = \left(\frac{\partial K}{\partial \varrho}\right)_T \qquad K_T = \left(\frac{\partial K}{\partial T}\right)_\varrho$$

If the chemical composition is constant and all non-conservative terms negligible, (21) reduces to

$$s^2 = \frac{k_H^2}{k^2} g\mathscr{A}$$

which shows that dynamical instability leading to proper convection appears only if $\mathscr{A} > 0$ and that, in this simple case, the latter is the stronger the higher the ratio of the vertical to the horizontal wavelengths.

If we neglect changes in chemical composition, we are left with

$$s^3 + s^2(2\eta k^2 + \delta) + s\left(\eta^2 k^4 - \frac{k_H^2}{k^2} g\mathscr{A} + 2\eta k^2 \delta\right) + (\eta^2 k^4 \delta - \eta k_H^2 g\mathscr{A}) = 0$$

which to the first order in the small quantities δ, ηk^2 reduces, apart from a trivial

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secular root, to the problem studied by Defouw (1970) with his $\mathscr{L}_T = -\nu\varepsilon/T$, $\mathscr{L}_\varrho = -\varepsilon/\varrho$ and $m=n=0$, $\beta=1$; of course, without rotation or magnetic field.

If, on the other hand, we neglect the energy generation ($\varepsilon=0$) and viscosity ($\eta=0$) and treat $\mathrm{d}T/\mathrm{d}z$ and K/ϱ as constant in perturbing $(1/\varrho\ \mathrm{div}\,\mathbf{F})$ (i.e. $m=0$, $n=0$), we obtain

$$s^3 + s^2 \frac{Kk^2}{C_p\varrho} - s\frac{k_{\mathrm{H}}^2}{k^2} g\mathscr{A} - k_{\mathrm{H}}^2 g \frac{K}{C_p\varrho}\frac{1}{\bar{\mu}}\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}z} = 0$$

which is Kato's equation (1966). Ignoring again the secular root and keeping only first order terms, this reduces to:

$$s^2 + \frac{Kk^2}{C_p\varrho}\frac{4-3\beta}{\beta}\frac{\mathscr{S}}{\mathscr{A}} s - \frac{k_{\mathrm{H}}^2}{k^2} g\mathscr{A} = 0$$

with roots

$$s_{1,2} = -\frac{Kk^2}{2C_p\varrho}\frac{4-3\beta}{\beta}\frac{\mathscr{S}}{\mathscr{A}} \pm \sqrt{\frac{k_{\mathrm{H}}^2}{k^2} g\mathscr{A}}.$$

It can be seen immediately that if $\mathscr{A}<0$ but $\mathscr{S}>0$, i.e. if condition (16) is satisfied, there are oscillating solutions with increasing amplitudes. It might be interesting to generalize the discussion to include the effects of viscosity and let K depend on ϱ and T but the corresponding dispersion equation which is of the 4th degree is harder to handle.

Of course, in this context, a local instability is only an indication and in any definite situation one should have to check whether it can actually give rise to a global instability for the star or whether stabilizing factors in other regions can overcome the local instability (cf. for instance Gabriel, 1969).

Finally, in the same conditions as above, but if we keep the terms in ε we get Dilke and Gough's (1972) equation

$$s^3 + s^2\left(\frac{\varepsilon}{C_pT} - \frac{\nu\varepsilon}{C_pT} + \frac{Kk^2}{C_p\varrho}\right) - s\frac{k_{\mathrm{H}}^2}{k^2} g\mathscr{A} + \\ + \frac{k_{\mathrm{H}}^2}{k^2} g\frac{\varepsilon}{C_pT}\left(\frac{\mu_1}{X_1}\frac{\mathrm{d}X_1}{\mathrm{d}z} + \frac{\mu_3}{X_3}\frac{\mathrm{d}X_3}{\mathrm{d}z}\right) + \frac{k_{\mathrm{H}}^2}{k^2} g\frac{1}{\bar{\mu}}\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}z}\left(\frac{\nu\varepsilon}{C_pT} - \frac{Kk^2}{C_p\varrho}\right) = 0$$

which, again discarding the secular root, has a solution which can be written in a first approximation

$$s_{1,2} = \pm\sqrt{\frac{k_{\mathrm{H}}^2}{k^2} g\mathscr{A}} - \\ -\frac{1}{2}\left[\frac{\varepsilon}{C_pT}\left(1 + \frac{1}{\mathscr{A}}\left(\frac{\mu_1}{X_1}\frac{\mathrm{d}X_1}{\mathrm{d}z} + \frac{\mu_3}{X_3}\frac{\mathrm{d}X_3}{\mathrm{d}z}\right) - \frac{\mathscr{S}}{\mathscr{A}}\left(\frac{\nu\varepsilon}{C_pT} - \frac{Kk^2}{C_p\varrho}\right)\right)\right].$$

If $\mathscr{A}<0$, we can indeed have amplified oscillating motions provided the square bracket is negative and abundances X_1 and X_3 increasing with z tend certainly to make this easier. If also $\mathscr{S}<0$ which is probably the case in the Sun, the term in $\nu\varepsilon$ is also destabilizing while the effect of Kato's term is now in the opposite direction. Dilke and Gough estimated that, at least locally, modes could be unstable on this basis. Of course, this does not yet, of itself, prove that this would lead to an efficient mixing (cf. Ulrich and Rood, 1973).

On the other hand, as already pointed out above, only a global analysis can confirm whether or not the local tendency to instability is sufficiently strong to overcome the stabilizing influence of the surrounding layers which implies solving the problem of the vibrational stability of g^+ mode in a realistic solar model. Very recently Dziembowski and Sienkievicz (1973) have evaluated the total σ' as defined by (15) for models in the vicinity of the Sun. Neglecting the external convection zone, they find that all the g^+ modes should be stable. In particular, the g_1^+ mode which is the less stable with a period of the order of one hour has a damping time of the order of a few million years. A similar investigation has been made in Liège (Boury *et al.*, 1974) and the results confirm those of Dziembowski and Sienkievicz yielding, if one stops the integral for σ' at the bottom of the convective envelope, an even somewhat shorter damping time. However if this integral, taking still into account only the radiative flux, is carried on close to the surface through the convection zone, the results is reversed and σ' becomes negative corresponding to an *e*-folding time of the order of 10^6 yr. Of course this is probably not significant because the convective flux and its variations including the horizontal components should be taken into account which is not too easy and a better approach should be used above the level where the quasi-adiabatic approximation breaks down. This work is now in progress in Liège. In any case, if this instability would persist it would be only slightly related to the Dilke and Gough mechanism and would be operative continuously and not only at definite intervals depending on the building up of gradients of X_1 and X_3.*

6. Variable Stars

As we have already recalled in Section (3.6), the β Canis Majoris stars still remain among the best candidates for an explanation in terms of non-radial pulsations and we have summarized there the new interesting suggestion of Osaki (1974) as to a possible origin of the latter. If, according to this idea, they result from a coupling through rotation with some overstable g^- modes it is likely that there would be appreciable energy from the buoyancy to drive them on and we would not have to depend on vibrational instability to excite them. Nevertheless, it would be very interesting also to compute the appropriate σ' to see how much of the original available energy would be dissipated and whether a balance at a reasonable amplitude can be reached. A somewhat similar idea (Ledoux, 1967) but applied this time to a narrow external convective zone could still be useful in the discussion of magnetic variables.

The characteristics of a recently discovered new type of variables, the so-called

* **Note added in proof.** Since then, early evolutionary solar models (2.4×10^8 to 3×10^9 yr) have been found unstable (Boury *et al.*, 1974; Christensen-Dalsgaard and Gough, 1974), but the 'present' Sun, with a better treatment of convection, is stable.

white-dwarf variables, also strongly suggest non-radial pulsations. The first one DQ Her, was discovered by Walker (1956) and with a period of 71 seconds it remained the shortest period variable known for quite a long time. Its recent study by Warner *et al.* (1972) has brought direct support to the non-radial oscillation hypothesis by showing that the observed phase variations of the pulsation during eclipse are difficult to explain except in terms of a grazing eclipse of a white dwarf oscillating in a non-radial quadrupole ($l=2$) mode. Since then, Warner finds that agreement is even improved if the oscillation is not axisymetric but corresponds to the mode $l=2$, $m=2$ and another star UX Ma has been shown to have similar characteristics.

In the last few years, thanks mainly to the work of Warner, many stars have been added, some even with shorter periods (cf. Table V). For instance Warner and Robinson (1972) reported oscillations with periods near 17, 24 and 31 s respectively during outbursts of the dwarf novae Z Cam, CN Ori and AH Her and one with a period of 29 s in UX Ma. More recently Warner and Harwood (1973) added another of these fast pulsators to the list as they found brightness variations of considerable amplitude with a period of 28.15 s in another dwarf nova VW Hyi while it was declining from its recent outburst.

The list was also extended towards longer periods, first with T Cor Bor with a period of 98.2 s (Lawrence *et al.*, 1967) and then a whole group cited here by order of increasing periods: G 61–29: 105 s (Richer *et al.*, 1973), HZ 29 (Am C Vn): 115 s (Warner and Robinson, 1972), R 548: 213, 273 s (Lasker and Hesser, 1971b), G 44–32: 600 s (Lasker and Hesser, 1971a), HL Tau: 746 s (Landolt, 1968; Warner and Robinson, 1972). But this last star was recognized as multi-periodic and Fitch (1973) has given a very nice analysis which isolates another period of 494 s coupled to the first and both modulated by a long period of 3.24 h. Still other candidates have been presented like BD 14341 with a period of 840 s (Williams, 1966) and SS Cyg, 960 s (Zuckerman, 1961) where the pulsation, however, has a definitely transient character although this may be true of a large fraction of these objects.

On the one hand, such short periods can only arise in small dense stars having either reached the white-dwarf stage or well on the way to it. But, on the other hand, the work on radial pulsations of such stars (cf. for instance Vila, 1970; Van Horn *et al.*, 1972) shows that the periods in this case are much shorter than the observed ones except perhaps the very shortest ones. The same is true of the f and p modes of non-radial oscillations but the g modes offer interesting possibilities, as far as longer periods are concerned especially in white dwarfs. In fact the structure of a classical cold white dwarf completely degenerate is barotropic and $\mathscr{A}\to 0$ so that the periods of all g modes tend to infinity.

In realistic white dwarfs models $\mathscr{A}$ will no longer be zero. While still fairly small in the degenerate interior, it will become negative and appreciable in an external radiative envelope until in the very thin outer convective layers (Böhm, 1968; Van Horn, 1970), it tends towards zero again or to a very small positive value. The region with $\mathscr{A}$ negative and relatively large in absolute value will be the more extensive and the g periods the shorter the hotter the white dwarfs. Thus one may expect the periods of g modes to

TABLE V

Some short-period variables

Star	Nature	Period	Remarks
Z Cam	dw. nova	17 s	
CN Ori	dw. nova	24 s	
VW Hyi		28.1 s	
UX U Ma	DA	29 s	Eclipsing binary 4^h43^m
AH Her	dw. nova	31 s	
DQ Her	nova	71 s	Eclipsing binary 4^h39^m
T Cor Bor		98.2 s	
G 61-29	DB	105 s	Eclipsing binary 6^h16^m
HZ-29 (AM CVn)	DB_p	115 s 1015 s (orbital?)	
R 548	DA	213 s 273 s	
G 44-32	DC	600 s 822 s 1638 s (orbital?)	
HL Taur	DA	746 s 494 s Long mod. (3.24 h)	

increase as the star cools. All this was already apparent from the discussion by Baglin and Schatzman (1969) and from the computations of Harper and Rose (1970) or from general approximate expressions of the $\sigma^2_{g^+}$ as given in Ledoux and Walraven (1958, Section 79). This was essentially the basis for the proposed interpretation by Chanmugam (1972) and by Warner and Robinson (1972) in terms of g^+ modes. Multiple periodicities and the direct evidence of DH Her and UX Ma during eclipses added new arguments.

The theoretical expectations have now been confirmed in details by Osaki and Hansen (1973) who have studied the non-radial oscillations of various cooling white dwarfs models for two masses: 0.398 and 1 $M_\odot$. They find, for instance, that the period $P_{g_1^+}$ of the g_1^+ mode varies, in the first case, from 49.02 s to 209.8 s as R varies from 4.14×10^{-2} to 1.39×10^{-2} $R_\odot$ while, for the higher mass, $P_{g_1^+}$ varies from 8.15 to 111.9 s as R varies from 1.12×10^{-2} to 6×10^{-3} $R_\odot$. The lower range of observed periods is thus reasonably covered. However for the longer periods observed, one would either have to go to rather unrealistic high g^+ modes or find some factor in the structure of the star which could reduce $\mathscr{A}$ again. Osaki and Hansen suggest that the external convection zone ($\mathscr{A} \simeq 0$) which they have neglected could play this rôle but it would have to extend fairly deep in order to bring agreement with the longest observed periods. In fact, in collaboration with Böhm *et al.* (1973) we have also computed the periods of various non-radial modes for a 0.6 $M_\odot$ white dwarf model with a fitted outer convection zone and $T_{\text{eff}} = 12000$ K. The periods found are reproduced in Table VI as well as the Q values. The jump in period and Q value in going from the p and f modes to the g modes is particularly well marked in this model which is somewhat similar in this respect to the coolest models of Osaki and Hansen. However here the external convection zone is so thin that, although it reduces the increase of the amplitude of the g_1 mode somewhat towards the surface, it has a relatively small influence on its period.

TABLE VI

Periods (in sec) of various modes of non-radial oscillations for a white dwarf model of $0.6 M_\odot$, $T_{\text{eff}} = 12000$ K and a narrow outer convection zone

Modes	p_4	p_3	p_2	p_1	f	g_1	g_2	g_3	g_4	g_5	g_6
Period	2.38	2.93	3.85	5.73	11.9	200	250	347	423	528	646
Q	0.0483	0.0594	0.0761	0.1162	0.2414	4.057	5.071	7.039	8.581	10.711	13.105

As Osaki and Hansen point out, since most of these variables are novae or dwarf novae or members of close binaries, the excitation of the non-radial modes need not necessarily correspond to an intrinsic vibrational instability since they could be excited by their outbursts or by tidal action. Nevertheless, they computed the damping coefficient σ' and since they assumed no nuclear energy generation they found only positive σ' corresponding to damping times due to radiative transfer and neutrino emission of the order of 10^{11} yr for the f mode of the smaller mass (0.4 $M_\odot$) to values as small as 10^5 and 3000 yr for the g_1 mode of respectively the 0.4 $M_\odot$ and 1.0 $M_\odot$ star. While the longest damping time is of the same order as for radial modes (Van Horn *et al.*, 1972), the damping times of the g modes are much shorter. This can be qualitatively understood since these g modes have appreciable amplitudes only in the external part of the white dwarf (cf. Figure 11) which contains relatively little mass, so that the total energy of the pulsations K is relatively small. On the other hand, this is the main region of conductive dissipation so that $(\Delta K)_P$ per period is relatively large and $\sigma' = -(1/2)(\Delta K)_P/K$ is large.

Another factor which is not a priori negligible for these dense stars is the emission of gravitational waves studied first by Thorne and Campolattaro (1967), Thorne (1969), Ipser and Thorne (1973). Osaki and Hansen (1973) show that indeed this can reduce drastically the damping time of the p modes which, for the 1 $M_{\odot}$ star, can then become as small as 10 to 100 yr but hardly affects the g modes.

On the other hand, as stressed by Warner (1972), it is likely that outbursts in all the cataclysmic variables originate in the white dwarf components in the form of non-radial modes corresponding to spherical harmonics of various degrees. Is this connected as suggested by Rose (1968), Rose and Smith (1972) to vibrational instability following a thermal runaway at the surface of the degenerate core of an accreting

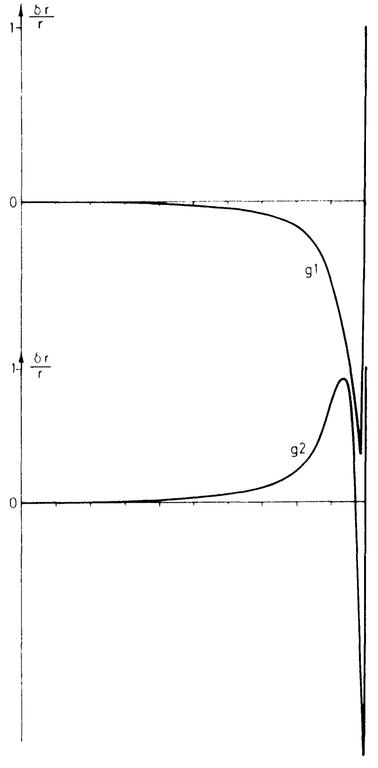

Fig. 11. Run of $\delta r/r$ for g_1 and g_2 modes ($l=2$) in a white dwarf of $0.6M_{\odot}$ with a narrow outer convection zone.

white dwarf? In fact Rose found vibrational instability for radial modes and his paper with Harper (1970) was in part aimed at discussing the secondary excitation of g modes by coupling. But with the distribution of the amplitudes of g modes in white dwarfs as described above, it is very likely that these modes could themselves become easily and strongly vibrationally unstable in the case considered by Rose. In our work with Böhm and Robe, we want to check this and preliminary calculations show that indeed very little hydrogen in the external layers generating only a small fraction of the energy radiated would be sufficient to lead to vibrational instability for the *g* modes.

It is clear however that these rapid pulsators with often many close periodicities and their rapidly changing periods offer many other interesting and puzzling problems.

We cannot here, even summarily, go into the problem of pulsars and variable X-ray sources but non-radial pulsations may still be relevant and, in any case, they may play an important rôle in the final dense stages of evolution by the emission of gravitational radiation, especially through *p* modes (cf. Hartle *et al.*, 1972).

In ending, let us draw the attention again to the general possibility of variability in binaries. In a recent paper, Herbst (1973) has reviewed this problem in binaries with strong Ca II emission and has added at least two new cases. The light period does not always agree exactly with the orbital period but is often of the same order and they certainly set a nice new problem to the theorists.

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DISCUSSION

Warner: Have you computed a C_L for a white dwarf yet?

Ledoux: I have not done it myself, no.

Warner: Do you have any feeling for what it could be? It is 0.15 for your calculation for a main sequence star.

Ledoux: I have no idea. I wouldn't like to give any number.

Tayler: It is a different problem asking whether instability will occur and discussing what happens once it is developed. As to the penetration question, do you think that with a linear treatment you get really useful information on what will happen once the convection is developed from this or just the very crude first idea?

Ledoux: I think you get only a crude first idea. Of course, what you do is to evaluate how much the gradient has to be superadiabatic in the turbulent picture, to transfer the heat. Then on that basis you compute g modes for that case and you find out how far they can penetrate the stable region. It certainly is not going to give you very precise results but at least some indications, probably as good as by any other approach.

Maeder: What typical characteristic of the light curve might result from the presence of some non-radial pulsation in the star?

Ledoux: In general non-radial pulsation would contribute very little to the light curves because you have parts of the star which are brighter, parts which are darker, and so on. So it is not a very efficient way of producing light variation, but there will be some. It will depend on the detail, which mode is excited and so on. If you go to high modes, of course, you would not see anything because you would have spots on the star, light and dark spots. But for $l=2$ for example, you would have a region which is bright and a region which is dark. This will combine with rotation too and it may be quite complicated to compute the light curve.

Buscombe: I think we do not know nearly enough about the rotational speed of stars in which there are other instabilities. I would make a plea to observers to work on this a bit because I feel that this is a parameter which is turning out to be very significant about stars. For example, in the MK classification, without detracting the great success that it has had, I think that this is completely suppressed and that we need this additional information. Part of this is confused with the macro turbulence. I think with such devices as the Fabry-Perot interferometer, there are possibilities.