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A SUFFICIENT CONDITION FOR PANCYCLIC GRAPHS

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Abstract

A graph G is called an $[s, t]$-graph if any induced subgraph of G of order s has size at least t. We prove that every 2-connected $[4, 2]$-graph of order at least 7 is pancyclic. This strengthens existing results. There are 2-connected $[4, 2]$-graphs which do not satisfy the Chvátal–Erdős condition on Hamiltonicity. We also determine the triangle-free graphs among $[p + 2, p]$-graphs for a general p.

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1. Introduction

We consider finite simple graphs, and use standard terminology and notation from [3, 8]. The *order* of a graph is its number of vertices and the *size* is its number of edges. A *k-cycle* is a cycle of length k. In 1971, Bondy [1] introduced the concept of a pancyclic graph. A graph G of order n is called *pancyclic* if for every integer k with $3 \le k \le n$, G contains a k-cycle. For an account of these graphs, see [5].

DEFINITION 1.1. Let s and t be given integers. A graph G is called an *$[s, t]$-graph* if any induced subgraph of G of order s has size at least t.

Denote by $\alpha(G)$ the independence number of a graph G. We note two facts:

(1) every $[s, t]$-graph is an $[s + 1, t + 1]$-graph;
(2) $\alpha(G) \le k$ if and only if G is a $[k + 1, 1]$-graph.

Thus, the concept of an $[s, t]$-graph is an extension of the independence number. We are interested in two results about $[4, 2]$ graphs.

THEOREM 1.2 (Liu and Wang, [6]). *Every 2-connected $[4, 2]$-graph of order at least 6 is Hamiltonian.*

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THEOREM 1.3 (Liu, Wang and Gao, [7]). *Let G be a 2-connected $[4, 2]$-graph of order n with $n \geq 7$. If G contains a k-cycle with $k < n$, then G contains a $(k + 1)$-cycle.*

We strengthen Theorem 1.3 by proving that every 2-connected $[4, 2]$-graph of order at least 7 is pancyclic (Theorem 2.5). To do so, we will determine the triangle-free graphs among $[p + 2, p]$-graphs. This preliminary result (Lemma 2.4) is of independent interest.

2. Main results

We denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph G, respectively, and denote by $|G|$ and $e(G)$ the order and size of G, respectively. Thus, $|G| = |V(G)|$ and $e(G) = |E(G)|$. For a vertex subset $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S. The neighbourhood of a vertex x is denoted by $N(x)$ and the closed neighbourhood of x is $N[x] \triangleq N(x) \cup \{x\}$. The degree of x is denoted by $\deg(x)$. For $S \subseteq V(G), N_S(x) \triangleq N(x) \cap S$ and the degree of x in S is $\deg_S(x) \triangleq |N_S(x)|$. Given two vertex subsets S and T of G, we denote by $[S, T]$ the set of edges having one endpoint in S and the other in T. The degree of S is $\deg(S) \triangleq |[S, \overline{S}]|$, where $\overline{S} = V(G) \setminus S$. We denote by C_n and K_n the cycle of order n and the complete graph of order n, respectively. Finally, $\overline{G}$ denotes the complement of a graph G.

We will need the following two lemmas on integral quadratic forms.

LEMMA 2.1. *Given positive integers $n \geq k \geq 2$, let $x_1, x_2, \ldots, x_k$ be positive integers such that $\sum_{i=1}^{k} x_i = n$. Then,*

$$n - 1 \leq \sum_{i=1}^{k-1} x_i x_{i+1} \leq \begin{cases} \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil & \text{if } k = 2, 3, \\ ab + k - 5 & \text{if } k \geq 4, \end{cases} \tag{2.1}$$

where $a = \lfloor (n - k + 4)/2 \rfloor$ and $b = \lceil (n - k + 4)/2 \rceil$. For any n and k, the lower and upper bounds in (2.1) can be attained.

PROOF. Define a quadratic polynomial $f(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k-1} x_i x_{i+1}$. We first prove the left-hand inequality in (2.1). Let $x_j = \min\{x_i \mid 1 \leq i \leq k\}$. We have

$$\begin{aligned} f(x_1, x_2, \ldots, x_k) &\geq x_1 x_j + \cdots + x_{j-1} x_j + x_j x_{j+1} + x_j x_{j+2} + \cdots + x_j x_k \\ &= x_j (x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k) \\ &= x_j (n - x_j) \\ &\geq n - 1. \end{aligned}$$

This proves the first inequality in (2.1). The lower bound $n - 1$ is attained for $x_1 = n - k + 1, x_2 = \cdots = x_k = 1$.

Now we prove the second inequality in (2.1). The case $k = 2$ is an elementary fact: $f(x_1, x_2) = x_1 x_2 \leq \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$, where equality holds when $x_1 = \lfloor n/2 \rfloor$ and $x_2 = \lceil n/2 \rceil$. The case $k = 3$ reduces to the case $k = 2$ as follows:

$$f(x_1, x_2, x_3) = x_2 (x_1 + x_3) \leq \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil,$$

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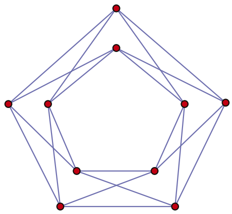

FIGURE 1. The 2-blow-up of C_5.

where equality holds when $x_2 = \lfloor n/2 \rfloor$ and $x_1 + x_3 = \lceil n/2 \rceil$. Next, suppose $k \geq 4$. Denote by $f_{\max}$ the maximum value of f. If $x_1 > 1$, with $x_1' = 1, x_2' = x_2, x_3' = x_3 + x_1 - 1$ and $x_i' = x_i$ for $i \geq 4$, then

$$f(x_1', x_2', \dots, x_k') - f(x_1, x_2, \dots, x_k) = (x_1 - 1)x_4 > 0.$$

Similarly analysing the variable x_k, we deduce that $f_{\max}$ can only be attained at some $x_1, \dots, x_k$ when $x_1 = x_k = 1$, which we now assume. With $x_2' = 1, x_3' = x_3$, $x_4' = x_4 + x_2 - 1$ and $x_i' = x_i$ for $i = 5, \dots, k-1$,

$$f(1, 1, x_3', x_4', \dots, x_{k-1}', 1) - f(1, x_2, x_3, \dots, x_{k-1}, 1) = (x_2 - 1)(x_5 - 1) \geq 0.$$

Hence, $f_{\max}$ can be attained at a certain $(1, 1, x_3, \dots, x_{k-1}, 1)$. Successively applying this argument, we deduce that $f_{\max}$ can be attained at $(1, 1, \dots, 1, x_{k-2}, x_{k-1}, 1)$. Now $(x_{k-2} + 1) + (x_{k-1} + 1) = n - k + 4$ and so

$$\begin{aligned} f(1, 1, \dots, 1, x_{k-2}, x_{k-1}, 1) &= (x_{k-2} + 1)(x_{k-1} + 1) + k - 5 \\ &\leq \lfloor (n - k + 4)/2 \rfloor \cdot \lceil (n - k + 4)/2 \rceil + k - 5. \end{aligned}$$

This proves the second inequality in (2.1). The upper bound is attained at $x_1 = x_2 = \cdots = x_{k-3} = x_k = 1, x_{k-2} = \lfloor (n - k + 2)/2 \rfloor$ and $x_{k-1} = \lceil (n - k + 2)/2 \rceil$. □

LEMMA 2.2 [9, Theorem 1]. *Given positive integers $n \geq k \geq 2$, let $x_1, x_2, \dots, x_k$ be positive integers such that $\sum_{i=1}^{k} x_i = n$. Then,*

$$2n - k \leq \sum_{i=1}^{k} x_i x_{i+1}, \tag{2.2}$$

where $x_{k+1} \triangleq x_1$. For any n and k, the lower bound in (2.2) can be attained.

A sharp upper bound on the quadratic form in (2.2) is also determined in [9], but we do not need it here.

DEFINITION 2.3. Given a graph H and a positive integer k, the *k-blow-up of H*, denoted by $H^{(k)}$, is the graph obtained by replacing every vertex of H with k different vertices where a copy of u is adjacent to a copy of v in the blow-up graph if and only if u is adjacent to v in H.

For example, $C_5^{(2)}$ is depicted in Figure 1.

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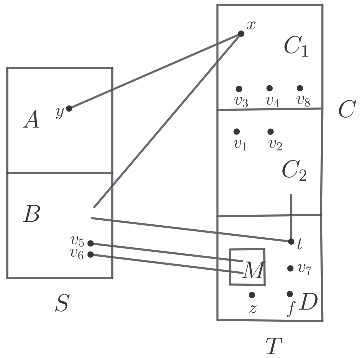

FIGURE 2. The structure of G.

Now we are ready to determine the triangle-free graphs among $[p+2, p]$-graphs. For a graph G, we denote by $\delta(G)$ and $\Delta(G)$ its minimum and maximum degrees, respectively. We regard isomorphic graphs as the same graph. Thus, for two graphs G and H, the notation $G = H$ means that G and H are isomorphic.

LEMMA 2.4. *Let G be a $[p+2, p]$-graph of order n with $\delta(G) \geq p \geq 2$ and $n \geq 2p+3$. Then, G is triangle-free if and only if p is even, $p \geq 6$ and $G = C_5^{(p/2)}$.*

PROOF. We will repeatedly use the condition that G is a $[p+2, p]$-graph without necessarily mentioning it. Denote $\Delta = \Delta(G)$ and choose a vertex $x \in V(G)$ such that $\deg(x) = \Delta$. Let $S = N(x)$ and $T = V(G) \setminus S$. Then, $|S| = \Delta$.

Suppose that G is triangle-free. Then, S is an independent set. Since G is a $[p+2, p]$-graph, $\Delta \leq p+1$. We assert that $\Delta = p$ and hence G is p-regular, since $\delta(G) \geq p$ by the assumption. Otherwise, $\Delta = p+1$. Since $n \geq 2p+3$, $|T| \geq p+2$. Thus, $G[T]$ contains an edge uv and $|\{u\} \cup S| = p+2$ implies that $\deg_S(u) \geq p$. Similarly, $\deg_S(v) \geq p$. Since $p + p = 2p > p+1 = |S|$, we have $N_S(u) \cap N_S(v) \neq \emptyset$. Let $w \in N_S(u) \cap N_S(v)$. Then, $wuvw$ is a triangle, which is a contradiction. This shows that G is p-regular.

Let $y \in S$ and denote $C = N(y)$. Then, C is an independent set and $|C| = p$. Denote $D = T \setminus C$. The structure of G is illustrated in Figure 2.

Since $n \geq 2p+3$, we have $|D| = n - 2p \geq 3$. Thus, D is not a clique, since G is triangle-free. Let z and f be any two distinct nonadjacent vertices in D. Since $|\{z, f\} \cup S| = p+2$, $|\{z, f\} \cup C| = p+2$, and S and C are independent sets,

$$\deg_S(z) + \deg_S(f) = \|[\{z, f\}, S]\| \geq p \quad \text{and} \quad \deg_C(z) + \deg_C(f) = \|[\{z, f\}, C]\| \geq p.$$

Note that $S \cap C = \emptyset$ and $\deg(z) = \deg(f) = p$. We must have

$$\|[\{z, f\}, S]\| = p \quad \text{and} \quad \|[\{z, f\}, C]\| = p. \tag{2.3}$$

We assert that D is an independent set. Otherwise, D contains two adjacent vertices u_1 and u_2. Let $u_3 \in D \setminus \{u_1, u_2\}$. Since G is triangle-free, u_3 is nonadjacent to at

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least one vertex in $\{u_1, u_2\}$, say, u_1. Setting $z = u_1$ and $f = u_3$ in (2.3), we deduce that $|[\{u_1, u_3\}, S \cup C]| = 2p$. However, since both u_1 and u_3 have degree p, and u_1 already has a neighbour $u_2 \notin S \cup C$, we have $|[\{u_1, u_3\}, S \cup C]| \leq 2p - 1$, which is a contradiction.

Observe that now (2.3) holds for any two distinct vertices z and f in D. Equation (2.3) has the equivalent form

$$\deg_S(z) + \deg_S(f) = p \quad \text{and} \quad \deg_C(z) + \deg_C(f) = p. \tag{2.4}$$

Then, (2.4) and $|D| \geq 3$ imply that for any vertex $z \in D$,

$$\deg_S(z) = \deg_C(z) = p/2. \tag{2.5}$$

To see this, in contrast, first suppose $\deg_S(z) > p/2$. Then, by the first equality in (2.4), for any two other vertices $f, r \in D$, we have $\deg_S(f) < p/2$ and $\deg_S(r) < p/2$, yielding $\deg_S(f) + \deg_S(r) < p$, which contradicts (2.4). If $\deg_S(z) < p/2$, the same argument gives a contradiction. A similar analysis with C in place of S shows $\deg_C(z) = p/2$. Thus, we have proved (2.5). In particular, $q \triangleq p/2$ is a positive integer, that is, p is even. Now choose an arbitrary but fixed vertex $t \in D$ and denote $B = N_S(t)$, $C_2 = N_C(t)$, $A = S \setminus B$ and $C_1 = C \setminus C_2$ (see the illustration in Figure 2). We have

$$|A| = |B| = |C_1| = |C_2| = q.$$

Since G is p-regular of order $n \geq 2p + 3$, it is impossible that $p = 2$. Otherwise, G would be a 2-regular graph of order ≥ 7, which is not a $[4, 2]$-graph. Thus, $p \geq 4$ and $q \geq 2$.

Choose any two distinct vertices $v_1, v_2 \in C_2$. Then, $|\{v_1, v_2\} \cup S| = p + 2$ implies that $|[\{v_1, v_2\}, S]| \geq p$. Since G is triangle-free, $N(v_i) \cap B = \emptyset$ for $i = 1, 2$. Hence, $N_S(v_i) = N_A(v_i)$ for $i = 1, 2$. However, $|A| = q$. We have $N_S(v_i) = A$ and $\deg_A(v_i) = q$ for $i = 1, 2$, implying that every vertex in C_2 is adjacent to every vertex in A.

Choose any two distinct vertices $v_3, v_4 \in C_1$. Then, $|\{v_3, v_4\} \cup C_2 \cup B| = p + 2$. Since $\{v_3, v_4\} \cup C_2$ is an independent set and $[C_2, B] = \emptyset$, we have $|[\{v_3, v_4\}, B]| \geq p$. However, $|B| = q$. Hence, $N_B(v_j) = B$ for $j = 3, 4$. This shows that every vertex in C_1 is adjacent to every vertex in B. Consequently, every vertex in B has exactly q neighbours in D.

Choose any vertex $v_5 \in B$. Denote $M = N_D(v_5)$. We have $|M| = q$. Since G is triangle-free and every vertex in B is adjacent to every vertex in C_1, the neighbourhood of any vertex in M is disjoint from C_1. Thus, the q neighbours of any vertex of M in C are exactly the vertices of C_2, implying that every vertex in M is adjacent to every vertex in C_2. The neighbourhood of any vertex in C_2 is $A \cup M$. For the same reason, for any vertex $v_6 \in B$ with $v_6 \neq v_5$, we must have $N_D(v_6) = M$. Hence, the neighbourhood of any vertex in M is $B \cup C_2$.

We assert that $M = D$. Otherwise, let $v_7 \in D \setminus M$. Take a vertex $v_8 \in C_1$. Note that $B \cup C_2$ is an independent set of cardinality p and $[v_7, B \cup C_2] = \emptyset$. Denote

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$R = \{v_7, v_8\} \cup B \cup C_2$. Then, $|R| = p + 2$ and hence $G[R]$ has size at least p. However, the size of $G[R]$ is at most $|[v_8, B]| + 1 = q + 1 < p$, which is a contradiction. Finally, since G is p-regular, every vertex in A must be adjacent to every vertex in C_1. Denote $V_1 = A, V_2 = C_1, V_3 = B, V_4 = D, V_5 = C_2$ and set $V_6 = V_1$. Then, each V_i is an independent set of cardinality $q = p/2$ and every vertex in V_i is adjacent to every vertex in V_{i+1} for $i = 1, 2, \ldots, 5$. This proves that $G = C_5^{(q)}$. Note that we have shown above that $q = |D| \geq 3$, implying that $p = 2q \geq 6$.

Conversely, let $H = C_5^{(q)}$, where $q = p/2$ and $p \geq 6$ is even. We will prove that H is a triangle-free $[p + 2, p]$-graph. Write $H = H_1 \vee H_2 \vee H_3 \vee H_4 \vee H_5 \vee H_1$, where each $H_i = \overline{K_q}$ and $\vee$ is the join operation on two vertex-disjoint graphs. If H contains a triangle, it must lie in $H[V(H_i) \cup V(H_{i+1})]$ for some i ($H_6 \triangleq H_1$). However, this is a bipartite graph, containing no triangle.

Let $U \subseteq V(H)$ with $|U| = p + 2$. We need to show $e(H[U]) \geq p$. Denote $I = \{i \mid U \cap V(H_i) \neq \emptyset, 1 \leq i \leq 5\}$. Since $|H_i| = q$ for $1 \leq i \leq 5$ and $|U| = p + 2$, we have $|I| \geq 3$. Denote $x_i = |U \cap V(H_i)|$ for $1 \leq i \leq 5$. Then, $0 \leq x_i \leq q$. We distinguish three cases.

Case 1: $|I| = 3$. There are at least two consecutive integers in I (1 and 5 are regarded as consecutive here). Without loss of generality, suppose $1, 2 \in I$. Then, $1 \leq x_1, x_2 \leq q$ and $x_1 + x_2 \geq p + 2 - q = q + 2$. Hence, $e(H[U]) \geq x_1 x_2 \geq 2q = p$.

Case 2: $|I| = 4$. Without loss of generality, suppose $I = \{1, 2, 3, 4\}$. Then, $e(H[U]) = x_1x_2 + x_2x_3 + x_3x_4$, where each x_i is a positive integer and $x_1 + x_2 + x_3 + x_4 = p + 2$. Applying Lemma 2.1, we have $e(H[U]) \geq (p + 2) - 1 = p + 1 > p$.

Case 3: $|I| = 5$. Now, $e(H[U]) = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$, where each x_i is a positive integer and $x_1 + x_2 + x_3 + x_4 + x_5 = p + 2$. Applying Lemma 2.2, we have $e(H[U]) \geq 2(p + 2) - 5 = 2p - 1 > p$.

In every case, H is a $[p + 2, p]$-graph. This completes the proof. □

THEOREM 2.5. *Every* 2*-connected* $[4, 2]$*-graph of order at least* 7 *is pancyclic.*

PROOF. Let G be a 2-connected $[4, 2]$-graph of order at least 7. Since G is 2-connected, $\delta(G) \geq 2$. By the case $p = 2$ of Lemma 2.4, G contains a triangle C_3. Then, successively applying Theorem 1.3, we deduce that G is pancyclic. □

REMARK 2.6. The Chvátal–Erdős theorem on Hamiltonian graphs [1, 4, 8] states that for a graph G, if $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian, where κ and α denote the connectivity and independence number, respectively. Bondy [2] proved that if a graph satisfies Ore's condition, then it satisfies the Chvátal–Erdős condition. A computer search for graphs of lower orders shows that there are many graphs which satisfy the condition in Theorem 2.5, but do not satisfy the Chvátal–Erdős condition. There are exactly 398 such graphs of order 9. For every integer $n \geq 7$, we give an example. Let $G_1 = K_{n-3}^-$ be the graph obtained from K_{n-3} by deleting one edge xy and let $G_2 = uvw$

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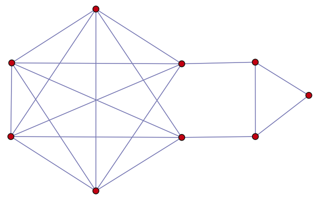

FIGURE 3. The graph Z_9.

be a triangle that is vertex-disjoint from G_1. Construct a graph Z_n from G_1 and G_2 by adding two edges xu and yv. The graph Z_9 is depicted in Figure 3.

Clearly, Z_n is a 2-connected $[4, 2]$-graph of order n, but $2 = \kappa(Z_n) < \alpha(Z_n) = 3$.

References

[1] J. A. Bondy, 'Pancyclic graphs. I', *J. Combin. Theory Ser. B* **11** (1971), 80–84.

[2] J. A. Bondy, 'A remark on two sufficient conditions for Hamilton cycles', *Discrete Math.* **22**(2) (1978), 191–193.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, 244 (Springer, London, 2008).

[4] V. Chvátal and P. Erdős, 'A note on Hamiltonian circuits', *Discrete Math.* **2** (1972), 111–113.

[5] J. C. George, A. Khodkar and W. D. Wallis, *Pancyclic and Bipancyclic Graphs* (Springer, Cham, 2016).

[6] C. Liu and J. Wang, '$[s, t]$-graphs and their hamiltonicity', *J. Shandong Normal Univ. Nat. Sci.* **20**(1) (2005), 6–7 (in Chinese).

[7] X. Liu, J. Wang and G. Gao, 'Cycles in 2-connected $[4, 2]$-graphs', *J. Shandong Univ. Nat. Sci.* **42**(4) (2007), 32–35 (in Chinese).

[8] D. B. West, *Introduction to Graph Theory* (Prentice Hall, Upper Saddle River, NJ, 1996).

[9] X. Zhan, 'Extremal numbers of positive entries of imprimitive nonnegative matrices', *Linear Algebra Appl.* **424**(1) (2007), 132–138.

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