13
The non-relativistic limit

In some branches of physics, such as condensed matter and quantum optics, one deals exclusively with non-relativistic models. However, there are occasionally advantages to using a relativistic formulation in quantum theory; by embedding a theory in a larger framework, one often obtains new insights. It is therefore useful to be able to take the non-relativistic limit of generally covariant theories, both as an indication of how large or small relativistic effects are and as a cultural bridge between covariant physics and non-relativistic quantum theory.

13.1 Particles and anti-particles

There is no unified theory of particles and anti-particles in the non-relativistic field theory. Formally there are two separate theories. When we take the non-relativistic limit of a relativistic theory, it splits into two disjoint theories: one for particles, with only positive definite energies, and one for anti-particles, with only negative definite energies. Thus, a non-relativistic theory cannot describe the interaction between matter and anti-matter.

The Green functions and fields reflect this feature. The positive frequency Wightman function goes into the positive energy particle theory, while the negative frequency Wightman function goes into the negative energy anti-particle theory. The objects which one then refers to as the Wightman functions of the non-relativistic field theory are asymmetrical. In normal Schrödinger field theory for matter, one says that the zero temperature negative frequency Wightman function is zero.[1]

[1] At finite temperature it must have a contribution from the heat bath for consistency.

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13.2 Klein–Gordon field

13.2.1 The free scalar field

We begin by considering the Klein–Gordon action for a real scalar field, since this is the simplest of the cases and can be treated at the level of the action. It also reveals several subtleties in the way quantities are defined and the names various quantities go by. In particular, we must recall that relativistic theories have an indefinite metric, while non-relativistic theories can be thought of as having a Euclidean, definite metric. Since one is often interested in the non-relativistic limit in connection with atomic systems, we illustrate the emergence of atomic levels by taking a two-component scalar field, in which the components have different potential energy in the centre of mass frame of the field. This is incorporated by adopting an effective mass $m_A = m + E_A/c^2$.

Consider the action:

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{2}\hbar^2 c^2 (\partial^\mu \phi_A)(\partial_\mu \phi_A) + \frac{1}{2} m_A^2 c^4 \phi_A \phi_A \right\}. \tag{13.1}$$

The variation of our action, with respect to the atomic variables, leads to

$$\delta S = \int (\mathrm{d}x) \delta\phi_A (-\hbar^2 c^2 \Box + m_A^2 c^4)\phi_A + \hbar^2 c \int \mathrm{d}\sigma_x^\mu (\phi_A \partial_\mu \phi_A). \tag{13.2}$$

The vanishing of the first term leads to the field equation

$$\hbar^2 c^2 \left(-\Box + \frac{m_A^2 c^2}{\hbar^2} \right) \phi_A(x) = 0. \tag{13.3}$$

The second (surface) term in this expression shows that any conserved probability must transform like an object of the form $\phi_A \partial_\mu \phi_A$. In fact, the real scalar field has no conserved current from which to derive a notion of locally conserved probability, but we may note the following. Any *complex* scalar field φ has a conserved current, which allows one to define the inner product

$$(\varphi_A, \varphi_B) = \mathrm{i}\hbar c \int \mathrm{d}\sigma_x^\mu (\varphi_A^* \partial_\mu \varphi_B - (\partial_\mu \varphi_A^*)\varphi_B), \tag{13.4}$$

where $\mathrm{d}\sigma_x^\mu$ is the volume element on a spacelike hyper-surface through space-time. This result is central even to the real scalar field, since a real scalar field does have a well defined probability density in the non-relativistic limit. To see this, we observe that the real scalar field $\phi(x)$ may be decomposed into positive and negative frequency parts:

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \tag{13.5}$$

where $\phi^{(+)}(x)$ is the positive frequency part of the field, $\phi^{(-)}(x)$ is the negative frequency part of the field and $\phi^{(+)}(x) = (\phi^{(-)}(x))^*$. Since the Schrödinger

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equation has no physical negative energy solutions, one must discard the negative frequency half of the spectrum when reducing the Klein–Gordon field to a Schrödinger field. This leads to well known expressions for the probability density. Starting with the probability

$$p = 2\mathrm{i}\hbar c \int \mathrm{d}\sigma_x (\phi \partial_0 \phi) \tag{13.6}$$

and letting

$$\phi^{(+)}(x) = \frac{\psi(x)}{\sqrt{2mc^2}} \quad , \qquad \phi^{(-)}(x) = \frac{\psi^*(x)}{\sqrt{2mc^2}}, \tag{13.7}$$

one obtains

$$p = \frac{\mathrm{i}\hbar}{2mc} \int \mathrm{d}\sigma_x (\psi + \psi^*) \partial_0 (\psi + \psi^*). \tag{13.8}$$

Assuming only that $\phi(x)$ may be expanded in a complete set of plane waves $\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \omega t)$, satisfying the free wave equation $\hbar^2\omega^2 = \hbar^2\mathbf{k}^2c^2 + m^2c^4$, then in the non-relativistic limit $\hbar^2\mathbf{k}^2 \ll m^2c^4$, we may make the effective replacement $\mathrm{i}\hbar\partial_0 \to mc$ to lowest order. Thus we have

$$p = \int \mathrm{d}\sigma_x \psi^*(x)\psi(x), \tag{13.9}$$

which is the familiar result for non-relativistic particles. It is easy to check that p is a dimensionless quantity using our conventions.

This observation prompts us to define the invariant inner product of two fields ϕ_A and ϕ_B by

$$(\phi_A, \phi_B) = \mathrm{i}\hbar c \int \mathrm{d}\sigma_x \frac{1}{2}(\phi_A^* \partial_0 \phi_B - (\partial_0 \phi_A^*)\phi_B). \tag{13.10}$$

The complex conjugate symbol is only a reminder here of how to take the non-relativistic limit, since ϕ_A is real. This product vanishes unless $A \neq B$, thus it must represent an amplitude to make a transition from ϕ_1 to ϕ_2 or vice versa. The non-relativistic limit of this expression is

$$(\phi_A, \phi_B) \to \frac{1}{2} \int \mathrm{d}\sigma_x \left[\psi_A^* \psi_B + \psi_B^* \psi_A\right]. \tag{13.11}$$

Since $\psi(x)$ is the field theoretical destruction operator and $\psi^*(x)$ is the creation operator, this is now manifestly a transition matrix, annihilating a lower state and creating an upper state or vice versa. The apparent A, B symmetry of eqn. (13.11) is a feature only of the lowest order term. Higher order corrections to this expression are proportional to $E_1 - E_2$, the energy difference between the two field levels, owing to the presence of ∂_0.

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Using the interaction term $\overline{P}$, we may compute the non-relativistic limit of the action in eqn. (13.1). This procedure is unambiguous only up to re-definitions of the origin for the arbitrary energy scale. Equivalently, we are free to define the mass used to scale the fields in any convenient way. The simplest procedure is to re-scale the fields by the true atomic mass, as in eqn. (13.7). In addition, we note that the non-relativistic energy operator $\mathrm{i}\hbar\partial_t$ is related to the non-relativistic energy operator $\mathrm{i}\hbar\tilde{\partial}_t$ by a shift with respect to the rest energy of particles:

$$\mathrm{i}\hbar\partial_t = mc^2 + \mathrm{i}\hbar\tilde{\partial}_t. \tag{13.12}$$

This is because the non-relativistic Hamiltonian does not include the rest energy of particles, its zero point is shifted so as to begin just about the rest energy. Integrating the kinetic term by parts so that $(\partial_\mu\phi)^2 \to \phi(-\Box)\phi$ and substituting eqn. (13.7) into eqn. (13.1) gives

$$S = \int \mathrm{d}\sigma_x \mathrm{d}t \frac{1}{2}(\psi + \psi^*)_A \left\{ \frac{\hbar^2\tilde{\partial}_t^2}{2mc^2} - \mathrm{i}\hbar\tilde{\partial}_t + \frac{E_A^2}{2mc^2} + E_A - \frac{\hbar^2}{2m}\nabla^2 \right\} (\psi + \psi^*)_A. \tag{13.13}$$

If we use the fact that $\psi_A(x)$ is composed of only positive plane wave frequencies, it follows that terms involving ψ^2 or $(\psi^*)^2$ vanish since they involve delta functions imposing a non-satisfiable condition on the energy $\delta(mc^2+\hbar\tilde{\omega})$, where both m and $\tilde{\omega}$ are greater than zero. This assumption ceases to be true only if there is an explicit time dependence in the action, indicating a non-equilibrium scenario, or if the mass of the atoms goes to zero (in which case the NR limit is unphysical). We are therefore left with

$$S_{\mathrm{NR}} = \lim_{c\to\infty} \int \mathrm{d}\sigma_x \mathrm{d}t \left\{ \frac{\mathrm{i}}{2}\left(\psi_A^*(\tilde{\partial}_t\psi_A) - (\tilde{\partial}_t\psi_A^*)\psi_A \right) - \psi_A^* H \psi_A \right\}, \tag{13.14}$$

where the differential operator H_A is defined by

$$H_A = -\frac{\nabla^2}{2m} + E_A + \frac{1}{2mc^2}(E_A^2 + \tilde{\partial}_t^2), \tag{13.15}$$

and we have re-defined the action by a sign in passing to a Euclideanized non-relativistic metric. It is now clear that, in the NR limit $c \to \infty$, the final two terms in H_A become negligible, leading to the field equation

$$H_A\psi_A(x) = \mathrm{i}\hbar\tilde{\partial}_t\psi_A(x), \tag{13.16}$$

which is the Schrödinger equation of a particle of mass m moving in a constant potential of energy E_A with a dipole interaction. The fact that it is possible to

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identify what is manifestly the Hamiltonian H in such an easy way is a special property of theories which are linear in the time derivative.

The direct use of the action (a non-physical quantity) in this way requires some care, so it is useful to confirm the above derivation with an approach based on the field equations, which are physical. As an additional spice, we also choose to scale the two components of the field by a factor involving the effective mass m_A rather than the true atomic mass m. The two fields are then scaled differently. This illustrates another viewpoint, namely of the particles as two species with a truly different mass, as would be natural in particle physics. We show that the resulting field equations have the same form in the non-relativistic limit, up to a shift in the arbitrary zero point energy.

Starting from eqn. (13.3), we define new pseudo-canonical variables by

$$\begin{aligned} P_A &= \sqrt{\frac{\omega_A}{2}}\left(\phi_A + \frac{\mathrm{i}}{\omega_A}\dot{\phi}_A\right) \\ Q_A &= \frac{1}{\sqrt{2\omega_A}}\left(\phi_A - \frac{\mathrm{i}}{\omega_A}\dot{\phi}_A\right), \end{aligned} \tag{13.17}$$

where $\hbar\omega_A \to m_A c^2$ in the non-relativistic limit, and the time dependence of the fields is of the form of a plane wave $\exp(-\mathrm{i}\omega_A t)$, for $\omega_A > 0$. This is the same assumption that was made earlier. We note that, owing to this assumption, the field $P_A(x)$ becomes large compared with $Q_A(x)$ in this limit. Substituting this transformation into the field equation (13.3) and neglecting Q, one obtains

$$\mathrm{i}\hbar\partial_t P_A = -\frac{\hbar^2}{2m_A}\nabla^2 P_A + \frac{1}{2}m_A c^2 P_A. \tag{13.18}$$

These terms have a natural physical interpretation: the first term on the right hand side is the particle kinetic term for the excited and unexcited atoms in our system. The second term is the energy offset of the two levels in the atomic system.

Our new point of view now leads to a free particle kinetic term with a mass m_A, rather than the true atomic mass m. There is no contradiction here, since E_A is small compared to mc^2, so we can always expand the reciprocal mass to first order. Expanding these reciprocal masses m_A we obtain

$$m_A^{-1} = m^{-1} + \mathrm{O}\left(\frac{E_A}{m^2c^2} \to 0\right) \tag{13.19}$$

showing that a consistent NR limit requires us to drop the A-dependent pieces.

Eqn. (13.18) may then be compared with eqn. (13.16). It differs only by a shift in the energy. A shift by the average energy level $\frac{1}{2}(E_1 + E_2)$ makes these equations identical.

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13.2.2 Non-relativistic limit of $G_{\rm F}(x, x')$

As we have already indicated, the non-relativistic theory contains only positive energy solutions. We also noted in section 5.5 that the Schrödinger Green function $G_{\rm NR}(x, x')$ satisfied purely retarded boundary conditions. There was no Feynman Green function for the non-relativistic field. Formally, this is a direct result of the lack of negative energy solutions to the Schrödinger equation (or anti-particles, in the language of quantum field theory). We shall now show that object, which we refer to as the Feynman Green function, becomes the non-relativistic retarded Green function in the limit $c \to \infty$. The same argument applies to the relativistic retarded function, and it is clear from eqn. (5.74) that the reason is the vanishing of the negative frequency Wightman function in the non-relativistic limit.

We begin with eqn. (5.95) and reinstate c and $\hbar$:

$$G_{\rm F}(x, x') = c \int \frac{{\rm d}^{n+1}k}{(2\pi)^{n+1}} \frac{c}{2\hbar\omega_k} \frac{{\rm e}^{{\rm i}k\Delta x}}{\hbar^2 c} \left[\frac{1}{(c\hbar k_0 + \hbar\omega_k - {\rm i}\epsilon)} - \frac{1}{(c\hbar k_0 - \hbar\omega_k + {\rm i}\epsilon)} \right]. \tag{13.20}$$

In order to compare the relativistic and non-relativistic Green functions, we have to re-scale the relativistic function by the rest energy, as in eqn. (13.7), since the two objects have different dimensions. Let

$$2mc^2\, G_{\rm F}(x, x') \to G_{\rm F,NR}, \tag{13.21}$$

so that the dimensions of $G_{\rm F,NR}$ are the same as those for $G_{\rm NR}$:

$$\left(-\frac{\hbar^2}{2m}\Box + \frac{1}{2}mc^2 \right) G_{\rm F,NR} = \delta(\mathbf{x}, \mathbf{x}')\delta(t, t') = c\delta(x, x');$$
$$\left(-\frac{\hbar^2}{2m}\nabla^2 - {\rm i}\hbar\partial_t \right) G_{\rm NR} = \delta(\mathbf{x}, \mathbf{x}')\delta(t, t'). \tag{13.22}$$

Next, we must express the relativistic energy $\hbar\omega$ in terms of the non-relativistic energy $\hbar\tilde{\omega}$ and examine the definition of ω_k with c reinstated,

$$ck_0 = -\omega = -\left(\tilde{\omega} + \frac{mc^2}{\hbar} \right)$$
$$\hbar\omega_k = \sqrt{\hbar^2 c^2 \mathbf{k}^2 + m^2 c^4}. \tag{13.23}$$

The change of $k_0 \to -\omega/c$, both in the integral limits and the measure, means that we effectively replace ${\rm d}k_0 \to {\rm d}\tilde{\omega}/c$. In the non-relativistic limit of large c, the square-root in the preceding equation can be expanded using the binomial theorem,

$$\hbar\omega_k = mc^2 + \frac{\hbar^2\mathbf{k}^2}{2m} + {\rm O}\left(\frac{1}{c^2} \right). \tag{13.24}$$

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Substituting these results into eqn. (13.20), we have for the partial fractions

$$\begin{aligned}\frac{1}{c\hbar k_0 + \hbar\omega_k - \mathrm{i}\epsilon} &= \frac{1}{\frac{\hbar^2\mathbf{k}^2}{2m} - \hbar\tilde{\omega} - \mathrm{i}\epsilon} \\ \frac{1}{c\hbar k_0 - \hbar\omega_k + \mathrm{i}\epsilon} &= \frac{1}{-\frac{\hbar^2\mathbf{k}^2}{2m} - \hbar\tilde{\omega} - 2mc^2 + \mathrm{i}\epsilon},\end{aligned} \tag{13.25}$$

while the pre-factor becomes

$$\mathrm{d}\tilde{\omega}\frac{2mc^2}{2\hbar\omega_k} = \left(1 + \frac{\hbar^2\mathbf{k}^2}{2m^2c^2} + \mathrm{O}\left(\frac{1}{c^4}\right)\right)^{-1}. \tag{13.26}$$

Taking the limit $c \to \infty$ in these expressions causes the second partial fraction in eqn. (13.25) to vanish. This is what removes the negative energy solutions from the non-relativistic theory. The remainder may now be written as

$$G_{\mathrm{F,NR}}(x, x') = \int \frac{\mathrm{d}^n\mathbf{k}}{(2\pi)^n}\frac{\mathrm{d}\tilde{\omega}}{2\pi}\left(\frac{\hbar^2\mathbf{k}^2}{2m} - \tilde{\omega} - \mathrm{i}\epsilon\right)^{-1}. \tag{13.27}$$

We see that this is precisely the expression obtained in eqn. (5.140). It has poles in the lower half-plane for positive frequencies. It is therefore a retarded Green function and satisfies a Kramers–Kronig relation.

13.3 Dirac field

The non-relativistic limit of the Dirac equation is more subtle than that for scalar particles since the fields are spinors and the γ-matrices imply a constraint on the components of the spinors. There are several derivations of this limit in the literature, all of them at the level of the field equations. Here we base our approach, as usual, on the action and avoid introducing specific solutions or making assumptions about their normalization.

13.3.1 The free Dirac field

The Dirac action may be written

$$S_{\mathrm{D}} = \int (\mathrm{d}x)\overline{\psi}\left(-\frac{1}{2}\mathrm{i}\hbar c(\gamma^\mu \overrightarrow{\partial}_\mu - \gamma^\mu \overleftarrow{\partial}_\mu^\dagger) + mc^2\right)\psi. \tag{13.28}$$

We begin by re-writing this in terms of the two-component spinors χ (see chapter 20) and with non-symmetrical derivatives for simplicity. The latter

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choice is of no consequence and only aids notational simplicity:

$$\begin{aligned} S_{\mathrm{D}} &= \int (\mathrm{d}x)\psi^{\dagger}\gamma^{0}(-\mathrm{i}\hbar c\gamma^{\mu}\partial_{\mu} + mc^{2})\psi \\ &= \int (\mathrm{d}x)(\chi_1^{\dagger}\chi_2^{\dagger}) \begin{pmatrix} -\mathrm{i}\hbar\partial_t - mc^2 & -\mathrm{i}\hbar c\sigma^i\partial_i \\ -\mathrm{i}\hbar c\sigma^i\partial_i & -\mathrm{i}\hbar\partial_t + mc^2 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \end{aligned} \tag{13.29}$$

This block matrix can be diagonalized by a unitary transformation. The eigenvalue equation is

$$(-\mathrm{i}\hbar\partial_t - mc^2 - \lambda)(-\mathrm{i}\hbar\partial_t + mc^2 - \lambda) + \hbar^2c^2\sigma^i\sigma^j\partial_{\mathrm{i}}\partial_j = 0. \tag{13.30}$$

Noting that

$$\sigma^i\sigma^j\partial_{\mathrm{i}}\partial_j = \partial^{\mathrm{i}}\partial_i + \mathrm{i}\epsilon^{ijk}\partial_{\mathrm{i}}\partial_j\sigma_k, \tag{13.31}$$

the eigenvalues may be written as

$$\lambda_{\pm} = -\mathrm{i}\hbar\partial_t \pm \sqrt{m^2c^4 - \hbar^2c^2(\partial^{\mathrm{i}}\partial_i + \mathrm{i}\epsilon^{ijk}\partial_{\mathrm{i}}\partial_j\sigma_k)}. \tag{13.32}$$

Thus, the action takes on a block-diagonal form

$$\begin{aligned} S_{\mathrm{D}} &= \int (\mathrm{d}x)\overline{\psi}^{\dagger}\gamma^{0}(-\mathrm{i}\hbar c\gamma^{\mu}\partial_{\mu} + mc^{2})\psi \\ &= \int (\mathrm{d}x)(\chi_1^{\dagger}\ \chi_2^{\dagger}) \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \end{aligned} \tag{13.33}$$

In the non-relativistic limit, $c \to \infty$, we may expand the square-root in the eigenvalues

$$\lambda_{\pm} = -\mathrm{i}\hbar\partial_t \pm mc^2\left(1 - \frac{\hbar^2(\partial^{\mathrm{i}}\partial_i + \mathrm{i}\epsilon^{ijk}\partial_{\mathrm{i}}\partial_j\sigma_k)}{2m^2c^2} + \mathrm{O}(c^{-4}) + \cdots\right). \tag{13.34}$$

The final step is to re-define the energy operator by the rest energy of the field, for consistency with the non-relativistic definitions:

$$\lambda_{\pm} = -\mathrm{i}\hbar\tilde{\partial}_t - mc^2 \pm mc^2\left(1 - \frac{\hbar^2\nabla^2}{2m^2c^2} + \mathrm{O}(c^{-4}) + \cdots\right). \tag{13.35}$$

Thus, in the limit, $c \to \infty$, the two eigenvalues, corresponding to positive and negative energy, give

$$\begin{aligned} \lambda_+ &= -\mathrm{i}\hbar\tilde{\partial}_t - \frac{\hbar^2\nabla^2}{2m} \\ \lambda_- &= \infty. \end{aligned} \tag{13.36}$$

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Apart from an infinite contribution to the zero point energy which may be re-defined (renormalized) away, and making an overall change of sign as in the Klein–Gordon case, the non-relativistic action is

$$S_{\mathrm{D}} \to \int (\mathrm{d}x) \left\{ \chi^{\dagger} \left(\mathrm{i}\hbar\tilde{\partial}_t + \frac{\hbar^2\nabla^2}{2m} \right) \chi \right\}. \tag{13.37}$$

13.3.2 The Dirac Green function

The non-relativistic limit of the Dirac Green function may be inferred quite straightforwardly from the Green function for the scalar field. The Dirac Green function $S(x, x')$ satisfies the relation

$$(-\mathrm{i}\hbar c\gamma^{\mu}\partial_{\mu} + mc^2)S(x, x') = c\delta(x, x'). \tag{13.38}$$

We also know that the squared operator in this equation leads to a Klein–Gordon operator, thus

$$(\mathrm{i}\hbar c\gamma^{\mu}\partial_{\mu} + mc^2)S(x, x') = G(x, x'), \tag{13.39}$$

so operating on eqn. (13.38) with this conjugate operator leaves us with

$$(-\hbar^2c^2\Box + m^2c^4)G(x, x') = c\delta(x, x'). \tag{13.40}$$

Both sides of this equation are proportional to a spinor identity matrix, which therefore cancels, leaving a scalar equation. Since we know the limiting properties of $G(x, x')$ from section 13.2.2, we may take the limit by introducing unity in the form $2mc^2/2mc^2$, such that $2mc^2G(x, x') = G_{\mathrm{NR}}(x, x')$ and the operator in front is divided by $2mc^2$. After re-defining the energy operator, as in eqn. (13.12), the limit of $c \to \infty$ causes the quadratic time derivative to vanish, leaving

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \mathrm{i}\hbar\tilde{\partial}_t \right) G_{\mathrm{NR}}(x, x') = \delta(\mathbf{x}, \mathbf{x}')\delta(t, t'). \tag{13.41}$$

This is the scalar Schrödinger Green function relation. To get the Green function for the two-component spinors found in the preceding section, it may be multiplied by a two-component identity matrix.

13.3.3 Spinor electrodynamics

The interaction between electrons and radiation complicates the simple procedure outlined in the previous section. The minimal coupling to radiation via the gauge potential $A_{\mu}(x)$ involves x-dependence, which means that the derivatives do not automatically commute with the diagonalization procedure. We must

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therefore modify the discussion to account for this, in particular taking more care with time reversal invariance. In addition, we must consider the reaction of the electronic matter to the presence of an electromagnetic field. This leads to a polarization of the field, or effective refractive index (see section 21.2 for a simple discussion of classical polarization). The action for electrodynamics is thus

$$S_{\rm QED} = \int ({\rm d}x) \left\{ \overline{\psi} \left(-\frac{1}{2} {\rm i}\hbar c (\gamma^\mu \overrightarrow{D}_\mu - \gamma^\mu \overleftarrow{D}_\mu^\dagger) + mc^2 \right) \psi + \frac{1}{4\mu_0} F^{\mu\nu} G_{\mu\nu} \right\}, \tag{13.42}$$

where $G_{\mu\nu}$ is the covariant displacement field, defined in eqn. (21.62). We proceed once again by re-writing this in terms of the two-component spinors χ. We consider the matter and radiation terms separately. The matter action is given by

$$\begin{aligned} S_{\rm D} &= \int ({\rm d}x) \psi^\dagger \gamma^0 (-{\rm i}\hbar c \gamma^\mu D_\mu + mc^2) \psi \\ &= \int ({\rm d}x) (\chi_1^\dagger \chi_2^\dagger) \begin{pmatrix} -{\rm i}\frac{\hbar}{2} \overleftrightarrow{D}_t - mc^2 & -{\rm i}\hbar c \sigma^i D_i \\ -{\rm i}\hbar c \sigma^i D_i & -{\rm i}\frac{\hbar}{2} \overleftrightarrow{D}_t + mc^2 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \end{aligned} \tag{13.43}$$

In electrodynamics, the covariant derivative is $D_\mu = \partial_\mu + {\rm i}\frac{e}{\hbar} A_\mu$, from which it follows that

$$[D_\mu, D_\nu] = {\rm i}\frac{e}{\hbar} F_{\mu\nu}. \tag{13.44}$$

The block matrix in eqn. (13.43) can be diagonalized by a unitary transformation. The symmetrized eigenvalue equation is

$$\left(-{\rm i}\frac{\hbar}{2} \overleftrightarrow{D}_t - mc^2 - \lambda \right) \left(-{\rm i}\frac{\hbar}{2} \overleftrightarrow{D}_t + mc^2 - \lambda \right) + \hbar^2 c^2 \sigma^i \sigma^j D_i D_j = 0, \tag{13.45}$$

or

$$\lambda^2 + 2{\rm i}\hbar\lambda D_t + \hbar^2 c^2 \, \sigma^i \sigma^j \, D_i D_j - \hbar^2 D_t^2 - m^2 c^4 - {\rm i}\frac{\hbar}{2} (\overleftrightarrow{\partial_t} \lambda) = 0, \tag{13.46}$$

where the last term arises from the fact that the eigenvalues themselves depend on x due to the gauge field. It is important that this eigenvalue equation be time-symmetrical, as indicated by the arrows. We may write this in the form

$$\lambda = -{\rm i}\frac{\hbar}{2} \overleftrightarrow{D}_t \pm \sqrt{m^2 c^4 - \hbar^2 c^2 \sigma^i \sigma^j D_i D_j + {\rm i}\frac{\hbar}{2} (\overleftrightarrow{\partial_t} \lambda)} \tag{13.47}$$

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and we now have an implicit equation for the positive and negative energy roots of the operator λ. The fact that the derivative term $\partial_t\lambda$ is a factor of c^2 smaller than the other terms in the square-root means that this contribution will always be smaller than the others. In the strict non-relativistic limit $c \to \infty$ it is completely negligible. Since the square-root contains operators, we represent it by its binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots, \tag{13.48}$$

after extracting an overall factor of mc^2, thus:

$$\lambda = -\mathrm{i}\frac{\hbar}{2}\overleftrightarrow{D}_t \pm \Bigg[mc^2 - \frac{\hbar^2\sigma^i\sigma^j D_i D_j}{2m} - \frac{\hbar^4\left(\sigma^i\sigma^j D_i D_j\right)^2}{8m^3c^2} + \mathrm{i}\frac{\hbar}{4mc^2}\left(\overleftrightarrow{\partial_t}\lambda\right) + \cdots \Bigg]. \tag{13.49}$$

The final term, ∂_t, can be evaluated to first order by iterating this expression. Symmetrizing over time derivatives, the first order derivative of eqn. (13.49) is

$$\begin{aligned}\left(\overleftrightarrow{\partial_t}\lambda\right)^{(1)} &= \mp\frac{\hbar^2}{2m}\sigma^i\sigma^j\left(D_i\overleftrightarrow{\partial_t}D_j\right)\\ &= \mp\frac{\mathrm{i}e\hbar}{2m}\sigma^i\sigma^j\left(D_iE_j - E_iD_j\right)\end{aligned} \tag{13.50}$$

since we may add and subtract $\partial_i A_t$ with impunity. To go to next order, we must substitute this result back into eqn. (13.49) and take the time derivative again. This gives a further correction

$$\left(\overleftrightarrow{\partial_t}\lambda\right)^{(2)} = \mp\mathrm{i}\frac{\hbar}{4mc^2}\partial_t\left[\frac{\mathrm{i}\hbar}{4mc^2}\left(\frac{\mathrm{i}e\hbar}{2m}\sigma^i\sigma^j\left(D_iE_j - E_iD_j\right)\right)\right] \tag{13.51}$$

Noting the energy shift $-\mathrm{i}\hbar\partial_t \to -\mathrm{i}\hbar\tilde{\partial}_t - mc^2$ and taking the positive square-root, we obtain the non-relativistic limit for the positive half of the solutions:

$$\begin{aligned}S_{\mathrm{D}} \to \int(\mathrm{d}x)\Bigg\{\chi^\dagger\Big(&\mathrm{i}\hbar\tilde{D}_t + \frac{\hbar^2 D^iD_i}{2m} - \frac{e\hbar B^i\sigma_i}{2m}\\ &- \frac{e\hbar^2}{8m^2c^2}\sigma^i\sigma^j\left(D_iE_j - E_iD_j\right)\\ &- \frac{\hbar^4}{8m^3c^2}\left(\left(D^iD_i\right) - \frac{e}{\hbar}(\sigma^iB_i)\right)^2\\ &- \mathrm{i}\frac{e\hbar^3}{32m^3c^4}\sigma^i\sigma^j\overleftrightarrow{\partial_t}\left(D_iE_j - E_iD_j\right) + \cdots\Big)\chi\Bigg\},\end{aligned} \tag{13.52}$$

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where $B_k = \frac{1}{2}\epsilon_{ijk}F_{jk}$ and the overall sign has been changed to conform to the usual conventions. The negative root in eqn. (13.47) gives a similar result for anti-particles. The fifth term contains $\partial^i E_i$, which is called the Darwin term. It corresponds to a correction to the point charge interaction due to the fact that Dirac 'particles' are spread out over a region with radius of the order of the Compton wavelength $\hbar/mc$. In older texts, this is referred to as the *Zitterbewegung*, since, if one insists on a particle interpretation, it is necessary to imagine the particles jittering around their average position in a kind of random walk. The $\sigma^i B_i$ term is a Zeeman splitting term due to the interaction of the magnetic field with particle trajectories.

Note that our diagonalization of the Dirac action leads to no coupling between the positive and negative energy solutions. One might expect that interactions with A_μ which couple indiscriminately with both positive and negative energy parts of the field would lead to an implicit coupling between positive and negative energy parts. This is not the case classically, however, since the vector potential A_μ leads to no non-linearities with respect to ψ.

Radiative corrections (fluctuation corrections) in the relativistic fields give rise to back-reaction terms both in the fermion sector and in the electromagnetic sector. The effect of photon $D_{\mu\nu}$ exchange leads to an effective quartic interaction

$$S_\Pi = \int (\mathrm{d}x)(\mathrm{d}x')\ (\overline{\psi}(x')\gamma^\mu\psi(x'))\ D_{\mu\nu}(x,x')\ (\overline{\psi}(x)\gamma^\nu\psi(x)). \quad (13.53)$$

The photon propagator is clearly a non-local and gauge-dependent quantity. Non-locality is a feature of the full theory, and reflects the fact that the finite speed of light disallows an instantaneous response in the field during collisions (there is an intrinsic non-elasticity in relativistic particle scattering). Working to a limited order in $1/c$ makes the effective Lagrangian effectively local, however, since the non-local derivative expansion is truncated. The gauge dependence of the Lagrangian is more subtle. In order to obtain a physically meaningful result, one requires an effective Lagrangian which produces gauge-fixing independent results. This does not necessarily mean that the Lagrangian needs to be gauge-independent, however. The reason is that the Lagrangian is no longer covariant with respect to the necessary symmetries to make this apparent.

Gauge invariance is related to a conformal/Lorentz symmetry of the relativistic gauge field, so one would expect a loss of Lorentz invariance to result in a breakdown of invariance under choice of gauge-fixing condition. In fact, a non-relativistic effective Lagrangian is not unique: its form is indeed gauge-dependent. Physical results cannot be gauge-dependent, however, provided one works to consistent order in the expansion of the original covariant theory. Thus, the gauge condition independence of the theory will be secured by working to consistent order in the smallness parameters, regardless of the actual gauge

chosen. Propagators and Lagrangian are gauge-dependent, but in just the right way to provide total gauge independence.

Turning to the photon sector, we seek to account for the effects of vacuum and medium polarization in leading order kinetic terms for the photon. To obtain the non-relativistic limit of the radiation terms, it is advantageous to have a model of the dielectric medium in which photons propagate. Nevertheless, some progress can be made on the basis of a generic linear response. We therefore use linear response theory and assume a constitutive relation for the polarization of the form

$$G^{\mu\nu} = F^{\mu\nu} + \int (\mathrm{d}x)\ \chi(x, x') F^{\mu\nu}. \tag{13.54}$$

The second term is a correction to the local field, which is proportional to the field itself. Perhaps surprisingly, this relation plays a role even in the vacuum, since quantum field theory predicts that the field ψ may be polarized by the back-reaction of field fluctuations in A_μ. Since the susceptibility $\chi(x, x')$ depends on the dynamics of the non-relativistic matter field, one expects this polarization to break the Lorentz invariance of the radiation term. This occurs because, at non-relativistic speeds, the interaction between matter and radiation splits into electric and magnetic parts which behave quite differently. From classical polarization theory, we find that the momentum space expression for the susceptibility takes the general form

$$\chi(\omega) \sim \frac{Ne^2\omega^2/\epsilon_0 m}{\omega_0^2 - \mathrm{i}\gamma\omega + \omega^2}. \tag{13.55}$$

In an electron plasma, where there are no atoms which introduce interactions over and above the ones we are considering above, the natural frequency of oscillations can only be $\omega_0 \sim mc^2/\hbar$. These are the only scales from which to construct a frequency. The significance of this value arises from the correlations of the fields on the order of the Compton wavelength which lead to an elastic property of the field. This is related to the Casimir effect and to the Zitterbewegung mentioned earlier. It is sufficient to note that such a system has an ultra-violet resonance, where $\omega_0 \gg \omega$ in the non-relativistic limit. This means that $\chi(\omega)$ can be expanded in powers of ω/ω_0. From the equations of motion, $\hbar\omega \sim \hbar^2\mathbf{k}^2/2m$; thus, the expansion is in powers of the quantity

$$\frac{\omega}{\omega_0} \sim \frac{\hbar\mathbf{k}^2/2m}{\hbar(mc^2/\hbar)} = \frac{\hbar\mathbf{k}^2}{m^2c^2}. \tag{13.56}$$

It follows that the action for the radiation may be written in the generic form

$$\begin{aligned} S_{\mathrm{M}} = \int (\mathrm{d}x) \Bigg\{ & \frac{C_{\mathrm{E}}}{2} A_0 \left[\nabla^2 \left(\frac{-\hbar\nabla^2}{m^2c^2} + \cdots \right) A_0 \right] \\ & + \frac{C_B}{2} A_i \left[(-\nabla^2 g_{ij} + \partial_{\mathrm{i}}\partial_j) \left(\frac{-\hbar\nabla^2}{m^2c^2} + \cdots \right) \right] A_j \Bigg\}. \end{aligned} \tag{13.57}$$

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This form expresses only the symmetries of the field and dimensional scales of the system. In order to evaluate the constants $C_{\rm E}$ and C_B in this expression, it would necessary to be much more specific about the nature of the polarization. For a plasma in a vacuum, the constants are equal to unity in the classical approximation. The same form for the action would apply in the case of electrons in an ambient polarizable medium, below resonance. Again, to determine the constants in that case, one would have to introduce a model for the ambient matter or input effective values for the constants by hand.

13.4 Thermal and Euclidean Green functions

There are two common formulations of thermal Green functions. At thermal equilibrium, where time is an irrelevant variable on average, one can rotate to a Euclidean, imaginary time formulation, as in eqn. (6.46), where the imaginary part of time places the role of an inverse temperature β. Alternatively one can use a real-time formulation as in eqn. (6.61).

The non-relativistic limit of Euclideanized field theory is essentially no different from the limit in Minkowski spacetime, except that there is no direct concept of retarded or advanced boundary conditions in terms of poles in the propagator. There is nevertheless still a duplicity in the solutions with positive and negative, imaginary energy. This duplicity disappears in the non-relativistic limit, as before, since half of the spectrum is suppressed. The relativistic, Euclidean Green function, closely related to the zero-temperature Feynman Green function, is given by

$$G_\beta(x,x') = \int \frac{{\rm d}\omega}{2\pi} \frac{{\rm d}^n\mathbf{k}}{(2\pi)^n} \frac{{\rm e}^{{\rm i}k(x-x')}}{p_\beta^2 c^2 + m^2c^4}, \tag{13.58}$$

where the zeroth component of the momentum is given by the Matsubara frequencies $p_\beta^0 = 2n\pi/\beta\hbar c$:

$$2mc^2\, G_\beta(x,x') = \int \frac{{\rm d}\omega}{2\pi} \frac{{\rm d}^n\mathbf{k}}{(2\pi)^n} \frac{{\rm e}^{{\rm i}k(x-x')}}{\frac{p_\beta^2}{2m} + \frac{1}{2}mc^2}. \tag{13.59}$$

Shifting the energy ${\rm i}p_\beta^0 \rightarrow mc^2 + {\rm i}\tilde{p}_\beta^0$ leaves us with

$$G_{{\rm NR}\beta}(x,x') = \int \frac{{\rm d}\omega}{2\pi} \frac{{\rm d}^n\mathbf{k}}{(2\pi)^n} \frac{{\rm e}^{{\rm i}k(x-x')}}{\frac{\mathbf{p}^2}{2m} - {\rm i}\hbar\tilde{\omega}}, \tag{13.60}$$

which is the Green function for the Euclidean action

$$S = \int ({\rm d}x)\chi^\dagger \left[\frac{\hbar^2\nabla^2}{2m} + \hbar\tilde{\partial}_\tau\right]\chi. \tag{13.61}$$

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In the real-time formulation, in which we retain the auxiliary time dependence, the thermal character of the Green functions is secured through the momentum space boundary condition in eqn. (6.54), known in quantum theory as the Kubo–Martin–Schwinger relation. Considering the boundary terms in eqn. (6.61) and following the procedure in section 13.2.2, one has

$$\begin{aligned} 2mc^2\, 2\pi \mathrm{i}\, f(k)\theta(k_0)\delta(p^2c^2 + m^2c^4) \rightarrow \\ 2mc^2\, 2\pi \mathrm{i}\, f(k)\theta(k_0)\frac{\delta(p_0c - \hbar\tilde{\omega}_k)}{2\hbar\omega_k}. \end{aligned} \tag{13.62}$$

In the large c limit, $\hbar\omega_k \rightarrow mc^2$, thus the $c \rightarrow \infty$ limit of this term is simply

$$2\pi \mathrm{i} f(\tilde{\omega}_k), \tag{13.63}$$

where $\hbar\omega_k = mc^2 + \hbar\tilde{\omega}_k$.

Intimately connected to this form is the Kubo–Martin–Schwinger (KMS) relation. We looked at this relation in section 6.1.5, and used it to derive the form of the relativistic Green functions. Notice that the zero-temperature, negative frequency parts of the Wightman functions do not contribute to the derivation of this relation in eqn. (6.56). For this reason, the form of the relationship in eqn. (6.54) is unchanged,

$$-G^{(+)}(\tilde{\omega}) = \mathrm{e}^{\beta\tilde{\omega}} G^{(-)}(\tilde{\omega}). \tag{13.64}$$

This use of the non-relativistic energy in both the relativistic and non-relativistic cases is important and leads to a subtlety in the Euclidean formulation. From the simplistic viewpoint of a Euclidean imaginary-time theory, the meaning of a thermal distribution is different in the relativistic and non-relativistic cases. The Boltzmann factor changes from

$$\mathrm{e}^{-\beta(\hbar\tilde{\omega} + mc^2)} \rightarrow \mathrm{e}^{-\beta\hbar\tilde{\omega}}. \tag{13.65}$$

This change is reflected also in a change in the time dependence of wave modes,

$$\mathrm{e}^{+\mathrm{i}(\tilde{\omega} + mc^2/\hbar)\tau} \rightarrow \mathrm{e}^{+\mathrm{i}\tilde{\omega}\tau}. \tag{13.66}$$

The shift is necessary to reflect the change in dynamical constraints posed by the equations of motion. However, the Boltzmann condition applies (by convention) to the non-relativistic energy. It is this energy scale which defines the temperature we know.

Another way of looking at the change in the Boltzmann distribution is from the viewpoint of fluctuations. Thermal fluctuations give rise to the Boltzmann factor, and these must have a special causal symmetry: emission followed by absorption. These processes are mediated by the Green functions, which reflect the equations of motion and are therefore unambiguously defined. As we

take the non-relativistic limit, the meaning of the thermal distribution changes character subtly. The positive frequency/energy condition changes from being $\theta(\omega) = \theta(\tilde{\omega}+mc^2/\hbar)$ to $\theta(\tilde{\omega})$ owing to the re-definition of the zero point energy. Looking back at eqn. (6.56), we derived the Bose–Einstein distribution using the fact that

$$\theta(-\omega)\mathrm{e}^{\beta\omega} = 0. \tag{13.67}$$

But one could equally choose the zero point energy elsewhere and write

$$\theta(-(\omega + \Delta\omega))\mathrm{e}^{\beta'(\omega+\Delta\omega)} = 0. \tag{13.68}$$

As long as the Green functions are free of interactions which couple the energy scale to a third party, we can re-label the energy freely by shifting the variable of integration in eqn. (5.64). In an interacting theory, the meaning of such a re-labelling is less clear.

In a system which is already in thermal equilibrium, one might argue that the interactions are not relevant. Interactions are only important in the approach to equilibrium and to the final temperature. With a new definition of the energy, a temperature has the same role as before, but the temperature scale β' is modified.

This might seem slightly paradoxical, but the meaning it clear. The KMS condition expressed by eqn. (6.54) simply indicates that the fluctuations mediated by given Green functions should be in thermal balance. The same condition may be applied to any virtual process, based on any equilibrium value or zero point energy. If we change the Green functions, we change the condition and the physics underpinning it. In each case, one obtains an equilibrium distribution of the same general form, but the meaning depends on the original Green functions. In order to end up with equivalent temperature scales, one must use equivalent energy scales. Relativistic energies and non-relativistic energies are not equivalent, and neither are the thermal distributions obtained from these. In the non-relativistic case, thermal fluctuations comprise kinetic fluctuations in particle motion. In the relativistic case, the energy of the particles themselves is included.

Two thermal distributions

$$\mathrm{e}^{\hbar\beta\omega} = \mathrm{e}^{\hbar(\beta+\Delta\beta)(\omega+\Delta\beta)} \tag{13.69}$$

are equivalent if

$$\frac{\beta + \Delta\beta}{\beta} = \frac{\omega}{\omega + \Delta\omega}. \tag{13.70}$$

These two viewpoints are related by a renormalization of the energy or chemical potential; the reason why such a renormalization is required is precisely because of the change in energy conventions which affects the Euclidean formulation.

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13.5 Energy conservation

The speed of light is built into the covariant notation of the conservation law

$$\partial_\mu \theta^{\mu\nu} = 0. \tag{13.71}$$

We must therefore ascertain whether the conservation law is altered by the limit $c \to \infty$. From eqns. (11.5) and (11.44), one may write

$$\begin{aligned} \partial_\mu \theta^{\mu\nu} &= \frac{1}{c}\partial_t \theta^{0\mu} + \partial_i \theta^{i\mu} \\ &= \partial_t \theta^{t\mu} + \partial_i \theta^{i\mu}. \end{aligned} \tag{13.72}$$

It is apparent from eqn. (11.5) that, as $c \to \infty$,

$$\begin{aligned} \theta_{0i} &\to \infty \\ \theta_{i0} &\to 0. \end{aligned} \tag{13.73}$$

Splitting μ into space and time components, we have, for the time component,

$$\begin{aligned} \partial_\mu \theta^{\mu 0} &= \frac{1}{c}\partial_\mu \theta^{\mu t} \\ &= \frac{1}{c}\left[\partial_t \theta^{tt} + \partial_i \theta^{it}\right] \\ &= \frac{1}{c}\left[\partial_t H\right] = 0. \end{aligned} \tag{13.74}$$

Because of the limit, this equation is ambiguous, but the result is sensible if we interpret the contents of the brackets as being zero. For the space components one has

$$\begin{aligned} \partial_t \theta^{ti} + \partial_j \theta^{ji} &= 0 \\ \partial_t \overline{p} + \partial_j \sigma^{ji} &= 0, \end{aligned} \tag{13.75}$$

where σ_{ij} is the stress tensor. Thus, energy conservation is preserved but it becomes divided into two separate statements, one about the time independence of the total Hamiltonian, and another expressing Newton's law that the rate of change of momentum is equal to the applied force.

13.6 Residual curvature and constraints

The non-relativistic limit does not always commute with the limit of zero curvature, nor with that of dimensional reduction, such as projection in order to determine the effective dynamics on a lower-dimensional constraint surface [15]. Such a reduction is performed by derivative expansion, in which every derivative seeks out orders of the curvature of the embedded surface. Since the

non-relativistic limit is also an expansion in terms of small derivatives, there is an obvious connection between these. In particular, the shape of a constraint surface can have specific implications for the consistency of the non-relativistic limit [29, 30, 80, 94, 95]. Caution should be always exercised in taking limits, to avoid premature loss of information.