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A VARIATIONAL CHARACTERIZATION OF CONTACT METRIC MANIFOLDS WITH VANISHING TORSION

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ABSTRACT. Chern and Hamilton considered the integral of the Webster scalar curvature as a functional on the set of CR-structures on a compact 3-dimensional contact manifold. Critical points of this functional can be viewed as Riemannian metrics associated to the contact structure for which the characteristic vector field generates a 1-parameter group of isometries *i.e.* K-contact metrics. Tanno defined a higher dimensional generalization of the Webster scalar curvature, computed the critical point condition of the corresponding integral functional and found that it is not the K-contact condition. In this paper two other generalizations are given and the critical point conditions of the corresponding integral functionals are found. For the second of these, this is the K-contact condition, suggesting that it may be the proper generalization of the Webster scalar curvature.

1. **Introduction** In [6] Chern and Hamilton considered the integral of the Webster scalar curvature as a functional on the set of CR-structures on a compact 3-dimensional contact manifold. The critical points of this functional can be viewed as Riemannian metrics associated to the contact structure for which the characteristic vector field generates a 1-parameter group of isometries *i.e.* a K-contact structure, a structure which is also characterized by the vanishing of a torsion tensor introduced in [6]. Note that in dimensions > 3, the notion of a contact metric structure is wider than the notion of a strongly pseudo-convex (integrable) CR-structure. As a generalization of the Webster scalar curvature, Tanno [10] defined the generalized Tanaka-Webster scalar curvature, W_1, on a contact metric manifold and considered $E_1(g) = \int_M W_1\, dV$ as a functional on the set $\mathcal{A}$ of metrics associated to the underlying contact form on the compact contact manifold M. He computed the critical point condition for $E_1(g)$ but it is not the K-contact condition. The situation in dimension 3 is quite special and the Webster curvature can be written in more than one way suggesting other generalizations. We first give such a generalization to higher dimensions, W_2, and compute the critical point condition of $E_2(g) = \int_M W_2\, dV$ on $\mathcal{A}$. We observe that if a metric is critical for both E_1 and E_2 it is K-contact.

The main result of this paper is to define a third generalization of the Webster scalar curvature, W_3, as the average of W_1 and W_2 and to show that the critical point condition of $E_3(g) = \int_M W_3\, dV$ is precisely the K-contact condition, thus W_3 may be the proper generalization of the Webster scalar curvature.

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After giving some preliminaries in Section 2, we develop this theory in Section 3.

2. **Preliminaries** By a *contact manifold* we mean a $(2n+1)$-dimensional C^∞ manifold M together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Given a contact form η, it is well known that there exists a unique vector field ξ, called the *characteristic vector field*, such that $d\eta(\xi, X) = 0$ for all vector fields X and normalized by $\eta(\xi) = 1$. At each point $m \in M$, let $B_m = \{X \in T_mM \mid \eta(X) = 0\}$; then $B = \cup B_m$ is called the *contact subbundle* on M. Note that if M is 3-dimensional, each B_m is a plane and we can speak of its sectional curvature with respect to a Riemannian metric which we denote simply by $K(B)$.

A Riemannian metric g is said to be an *associated metric* if there exists a tensor field ϕ of type $(1,1)$ such that $d\eta(X, Y) = g(X, \phi Y)$, $\eta(X) = g(X, \xi)$ and $\phi^2 = -I + \eta \otimes \xi$ and we refer to M with this structure as a *contact metric manifold*. For a given form η, the set $\mathcal{A}$ of all such metrics is infinite dimensional. Moreover each associated metric has the same volume element, *viz.* $dV = \frac{(-1)^n}{2^n n!}\eta \wedge (d\eta)^n$.

Given a contact metric structure (ϕ, ξ, η, g), define a tensor field h by $h = \frac{1}{2}\mathfrak{L}_\xi \phi$ where $\mathfrak{L}$ denotes Lie differentiation. h is a symmetric operator, $h\xi = 0$

$$\phi h + h\phi = 0, \tag{2.1}$$

and $h \equiv 0$ if and only if ξ is Killing, *i.e.* ξ generates a 1-parameter group of isometries. A contact metric structure for which ξ is Killing is called a *K-contact structure*. Moreover h is related to the covariant derivative of ξ by

$$\nabla_X \xi = -\phi X - \phi h X.$$

We also define a tensor field ℓ by $\ell X = R_{X\xi}\xi$, where R is the curvature tensor of g. Other formulas for a general contact metric structure that we will need are

$$(\nabla_k \phi_{ip})\phi_j{}^p = \phi_k{}^p(\nabla_p \phi_{ij}) + \eta_j \phi_{ki} - \eta_j h_{km}\phi^m{}_i + 2\phi_{jk}\eta_i \tag{2.2}$$

(see [8]),

$$\nabla_t \nabla_k \phi_j{}^t + \nabla_t \nabla_j \phi_k{}^t = R_{kt}\phi_j{}^t + R_{jt}\phi_k{}^t + 2n(h_{km}\phi^m{}_j + h_{jm}\phi^m{}_k) \tag{2.3}$$

(see [4]) and

$$\mathrm{Ric}(\xi) = 2n - \mathrm{tr}\, h^2 \tag{2.4}$$

On a contact metric manifold the $*$-*Ricci tensor* and $*$-*scalar curvature* are defined by

$$R^*_{ij} = R_{ik\ell t}\phi^{k\ell}\phi_j{}^t, \quad R^* = R^{*i}_i.$$

The idea behind the derivation of critical point conditions, is to differentiate the functional in question along a path of metrics in $\mathcal{A}$. Let $g(t)$ be a smooth curve in $\mathcal{A}$ and let

$$D_{ij} = \left.\frac{\partial g_{ij}}{\partial t}\right|_{t=0}$$

We also write D for the tensor field of type $(1, 1)$ corresponding to D_{ij} via $g = g(0)$ and let ϕ be the fundamental collineation as above corresponding to g. Then D is tangent to a path $g(t)$ in $\mathcal{A}$ at g if and only if

$$D\phi + \phi D = 0, \quad D\xi = 0 \tag{2.5}$$

as is shown in [1,2]. The following lemma is proved in [4].

LEMMA. *Let T be a second order symmetric tensor field on M. Then*

$$\int_M T^{ij} D_{ij}\, dV = 0$$

for all D satisfying (2.5) if and only if T and ϕ commute when restricted to B, i.e. $\phi T - T\phi = \eta \otimes \phi T\xi - (\eta \circ T\phi) \otimes \xi$ or equivalently

$$T_{ij} = T_{pq}\phi_i{}^p\phi_j{}^q + T_{jr}\xi^r\eta_i + T_{ir}\xi^r\eta_j - (T_{rs}\xi^r\xi^s)\eta_i\eta_j.$$

3. **Main results** On a 3-dimensional contact metric manifold the *Webster scalar curvature* W was defined by Chern and Hamilton [6], p. 284, as

$$W = \frac{1}{8}\big(\operatorname{Ric}(\xi) + 2K(B) + 4\big)$$

or since the scalar curvature $R = 2\operatorname{Ric}(\xi) + 2K(B)$

$$W = \frac{1}{8}\big(R - \operatorname{Ric}(\xi) + 4\big).$$

Tanno [10], not including the factor of $1/8$, defined the *generalized Tanaka-Webster curvature* W_1 by

$$W_1 = R - \operatorname{Ric}(\xi) + 4n.$$

We now state the theorem of Chern and Hamilton [6], an alternate proof of which was given in [9], and the theorem of Tanno [10] and sketch their proofs simultaneously.

THEOREM (CHERN-HAMILTON). *Let M be a compact 3-dimensional contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $E_1(g) = \int_M W_1\, dV$ if and only if g is K-contact.*

THEOREM (TANNO). *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of E_1 if and only if*

$$(Q\phi - \phi Q) - (\ell\phi - \phi\ell) = 4\phi h - \eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi$$

where Q is the Ricci operator.

PROOFS. Clearly it is enough to consider $\int_M \{R - \mathrm{Ric}(\xi)\}\, dV$ and differentiate along a path $g(t) \in \mathcal{A}$, $g(0) = g$. Having differentiated R and $\mathrm{Ric}(\xi)$ separately in [4] and [2] respectively, we have

$$\frac{d}{dt}\int_M \{R - \mathrm{Ric}(\xi)\}\, dV|_{t=0} = \int_M (-R^{ki} + h^i_m h^{mk} + R^k{}_{rs}{}^i \xi^r \xi^s - 2h^{ik}) D_{ik}\, dV.$$

Thus by the Lemma and (2.4) we see that the critical point condition is

$$(Q\phi - \phi Q) - (\ell\phi - \phi\ell) = 4\phi h - \eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi. \tag{3.1}$$

Now in dimension 3, the Ricci operator determines the full curvature tensor, *i.e.*

$$\begin{aligned} R_{XY}Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ - \frac{R}{2}\big(g(Y,Z)X - g(X,Z)Y\big). \end{aligned} \tag{3.2}$$

Thus the operator ℓ is given by

$$\ell X = QX - \eta(X)Q\xi + g(Q\xi,\xi)X - g(QX,\xi)\xi - \frac{R}{2}\big(X - \eta(X)\xi\big)$$

from which

$$\ell\phi - \phi\ell = Q\phi - \phi Q + \eta \otimes \phi Q\xi - (\eta \circ Q\phi) \otimes \xi. \tag{3.3}$$

Combining (3.1) and (3.3) we have $4\phi h = 0$ and hence, since $h\xi = 0$, $h = 0$. ∎

Now on a general contact metric manifold Olszak [8] showed that

$$R - R^* - 4n^2 = -\frac{1}{2}|\nabla\phi|^2 + 2n - \mathrm{tr}\, h^2 \leq 0 \tag{3.4}$$

with equality if and only if the structure is Sasakian and from the form (3.2) of the curvature tensor in dimension 3

$$|\nabla\phi|^2 = 4 + 2\,\mathrm{tr}\, h^2.$$

Combining these with (2.4), in dimension 3, we have

$$R - R^* = 2\,\mathrm{Ric}(\xi).$$

Thus the Webster scalar curvature can be written as $\frac{1}{8}\big(R^* + \mathrm{Ric}(\xi) + 4\big) = \frac{1}{8}\big(R + \frac{1}{2}|\nabla\phi|^2\big)$ which in arbitrary dimension becomes $\frac{1}{8}\big(R^* + \mathrm{Ric}(\xi) + 4n^2\big)$. Thus we define another generalization of the Webster scalar curvature W_2 by

$$W_2 = R^* + \mathrm{Ric}(\xi) + 4n^2$$

THEOREM I. *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $E_2(g) = \int_M W_2\, dV$ if and only if*

$$(Q\phi - \phi Q) - (\ell\phi - \phi\ell) = -4(2n-1)\phi h + (\eta \circ Q\phi) \otimes \xi - \eta \otimes \phi Q\xi.$$

PROOF. We compute $\frac{dE_2}{dt}\big|_{t=0}$ for a path $g(t)$ in $\mathcal{A}$ with $g(0) = g$. In [4], R^* was differentiated along such a path and we indicate each of these by square brackets in the following integral formula

$$\frac{dE_2}{dt}\Big|_{t=0} = \int_M \{[-2nh^{j\ell} - \nabla_i(\phi^{k\ell}\nabla_k\phi^{ij}) - R^{*j\ell}] + [-h^j_m h^{m\ell} - R^{\ell}{}_{rs}{}^{j}\xi^r\xi^s + 2h^{j\ell}]\}D_{j\ell}\,dV.$$

Thus from the Lemma we see that the critical point condition is

$$\begin{aligned}
&2(1-n)h^{j\ell} - \frac{1}{2}\nabla_i(\phi^{k\ell}\nabla_k\phi^{ij} + \phi^{kj}\nabla_k\phi^{i\ell}) - \frac{1}{2}(R^{*j\ell} + R^{*\ell j}) - h^j_m h^{m\ell} - R^{\ell}{}_{rs}{}^{j}\xi^r\xi^s \\
&= 2(1-n)h^{pq}\phi^j{}_p\phi^\ell{}_q - \frac{1}{2}\big(\nabla_i(\phi^{kq}\nabla_k\phi^{ip} + \phi^{kp}\nabla_k\phi^{iq})\big)\phi^j{}_p\phi^\ell{}_q \\
&\quad - \frac{1}{2}(R^{*pq} + R^{*qp})\phi^j{}_p\phi^\ell{}_q - h^p_m h^{mp}\phi^j{}_p\phi^\ell{}_q - R^{q}{}_{rs}{}^{p}\xi^r\xi^s\phi^j{}_p\phi^\ell{}_q \\
&\quad + \xi^\ell\eta_r\Big[-\frac{1}{2}\nabla_i(\phi^{kr}\nabla_k\phi^{ij} + \phi^{kj}\nabla_k\phi^{ir}) - \frac{1}{2}R^{*rj}\Big] \\
&\quad + \xi^j\eta_r\Big[-\frac{1}{2}\nabla_i(\phi^{kr}\nabla_k\phi^{i\ell} + \phi^{k\ell}\nabla_k\phi^{ir}) - \frac{1}{2}R^{*r\ell}\Big] \\
&\quad - \xi^j\xi^\ell\Big[-\frac{1}{2}\nabla_i(\phi^{ks}\nabla_k\phi^{ir} + \phi^{kr}\nabla_k\phi^{is})\Big]\eta_r\eta_s
\end{aligned}$$

As in [4] it is easy to see from the definition of $R^*_{j\ell}$ that all terms involving the $*$-Ricci tensor vanish. Expanding the terms involving covariant derivatives of ϕ, the several terms containing products of first derivatives cancel as in [4] mainly by virtue of (2.2). Similarly a computation using (2.3) and also done in [4] yields

$$\begin{aligned}
-\frac{1}{2}\phi^{kp}(\nabla_i\nabla_k\phi^{iq})\phi^j{}_p\phi^\ell{}_q &= -\frac{1}{2}\phi^{k\ell}\nabla_i\nabla_k\phi^{ij} + \frac{1}{2}R^{pq}\phi^j{}_p\phi^\ell{}_q - \frac{1}{2}R^{j\ell} + \frac{1}{2}R^j_r\xi^r\xi^\ell \\
&\quad + \frac{1}{2}R^\ell_r\xi^r\xi^j - \frac{1}{2}\operatorname{Ric}(\xi)\xi^j\xi^\ell \\
&\quad + 2nh^{j\ell} + \frac{1}{2}\xi^j\eta_r\phi^{k\ell}\nabla_i\nabla_k\phi^{ir}.
\end{aligned}$$

Substituting this into the critical point condition, using $\phi h + h\phi = 0$ and simplifying, we have

$$\begin{aligned}
0 &= 4(2n-1)h^{j\ell} + R^{\ell}{}_{rs}{}^{j}\xi^r\xi^s - R^{q}{}_{rs}{}^{p}\xi^r\xi^s\phi^j{}_p\phi^\ell{}_q - R^{j\ell} + R^{pq}\phi^j{}_p\phi^\ell{}_q \\
&\quad + R^j_r\xi^r\xi_\ell + R^\ell_r\xi^r\xi^j - \operatorname{Ric}(\xi)\xi^j\xi^\ell.
\end{aligned}$$

Applying ϕ to this we have

$$0 = 4(2n-1)\phi h + \phi\ell - \ell\phi - \phi Q + Q\phi - (\eta\circ Q\phi)\otimes\xi + \eta\otimes\phi Q\xi$$

completing the proof. ∎

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We remark that if g is a critical point of both E_1 and E_2 then g is a K-contact metric. Our goal is to seek a single functional whose critical points are the K-contact metrics. To this end we define a third generalization of the Webster scalar curvature which in view of the result may be the proper generalization. We define W_3 to be the average of W_1 and W_2, *i.e.*

$$W_3 = \frac{1}{2}\big(R + R^* + 4n(n+1)\big).$$

THEOREM II. *Let M be a compact contact manifold and $\mathcal{A}$ the set of metrics associated to the contact form. Then $g \in \mathcal{A}$ is a critical point of $E_3(g) = \int_M W_3\, dV$ if and only if g is K-contact.*

PROOF. Clearly it is enough to consider $\int_M \{R + R^*\}\, dV$. Again having differentiated the terms separately in [4], we have

$$\frac{d}{dt}\int_M \{R + R^*\}\, dV|_{t=0} = \int_M \{[-R^{j\ell}] + [-2nh^{j\ell} - \nabla_i(\phi^{k\ell}\nabla_k\phi^{ij}) - R^{*j\ell}]\} D_{j\ell}\, dV$$

and hence the critical point condition is

$$\begin{aligned}
-R^{j\ell} - 2nh^{j\ell} - \frac{1}{2}\nabla_i(\phi^{k\ell}\nabla_k\phi^{ij} + \phi^{kj}\nabla_k\phi^{i\ell}) - \frac{1}{2}(R^{*j\ell} + R^{*\ell j})& \\
= -R^{pq}\phi^j{}_p\phi^\ell{}_q - 2nh^{pq}\phi^j{}_p\phi^\ell{}_q - \frac{1}{2}\big(\nabla_i(\phi^{kq}\nabla_k\phi^{ip} + \phi^{kp}\nabla_k\phi^{iq})\big)\phi^j{}_p\phi^\ell{}_q& \\
- \frac{1}{2}(R^{*pq} + R^{*qp})\phi^j{}_p\phi^\ell{}_q& \\
+ \xi^\ell\eta_r\Big(-R^{jr} - \frac{1}{2}\nabla_i(\phi^{kr}\nabla_k\phi^{ij} + \phi^{kj}\nabla_k\phi^{ir}) - \frac{1}{2}R^{*rj}\Big)& \\
+ \xi^j\eta_r\Big(-R^{\ell r} - \frac{1}{2}\nabla_i(\phi^{kr}\nabla_k\phi^{i\ell} + \phi^{k\ell}\nabla_k\phi^{ir}) - \frac{1}{2}R^{*r\ell}\Big)& \\
- \xi^j\xi^\ell\Big[-R^{rs} - \frac{1}{2}\nabla_i(\phi^{ks}\nabla_k\phi^{ir} + \phi^{kr}\nabla_k\phi^{is})\Big]\eta_r\eta_s&
\end{aligned}$$

Terms involving the $*$-Ricci tensor and products of first derivatives of ϕ cancel as in the previous theorem. Terms involving the second derivatives are also treated as in the previous theorem. The critical point condition then reduces to $-2nh^{j\ell} = -2nh^{pq}\phi^j{}_q\phi^\ell{}_q + 2nh^{j\ell} + 2nh^{\ell j}$ which since $\phi h + h\phi = 0$ yields $h = 0$ as desired. ∎

REMARK 1. A contact manifold is said to be *regular* if every point has a neighborhood such that any integral curve of ξ passing through the neighborhood passes through only once. The celebrated Boothby-Wang Theorem [5] states that a compact regular contact manifold is a principal circle bundle over a symplectic manifold of integral class. In [2,3], it was shown for a compact regular contact manifold, $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M \mathrm{Ric}(\xi)\, dV$ if and only if g is K-contact, but that without the regularity a counterexample can be given. In particular the standard contact metric structure on the tangent sphere bundle of a compact surface of constant curvature -1 is a critical point of L but is not K-contact.

REMARK 2. We note that the average of W_1 and W_2 is the best linear combination of W_1 and W_2 to take for the purpose of achieving a functional whose critical points are the K-contact metrics. In fact the critical point condition for $\int_M(aW_1 + bW_2)\,dV$, a, b constants, not both zero, is

$$\big(-8nb+4(b-a)\big)\phi h-(b-a)(Q\phi-\phi Q)=(b-a)[\eta\otimes\phi Q\xi-(\eta\circ Q\phi)\otimes\xi-(\ell\phi-\phi\ell)].$$

Now since $h = 0$ implies $\ell = I - \eta \otimes \xi$, we see that if $h = 0$, then either $a = b$ or $Q\phi - \phi Q = -\eta \otimes \phi Q\xi + (\eta \circ Q\phi) \otimes \xi$ and in general one would not want to restrict oneself to the latter alternative from the outset.

If g is a Sasakian metric, then it is a critical point of the functional

$$E(g) = \int_M (aW_1 + bW_2)\,dV,$$

for all a and b; in fact g Sasakian implies that $h = 0$ and that $Q\phi - \phi Q = 0$. The converse implication is an open question. On the other hand there are K-contact manifolds which are not Sasakian. To see this let N be a compact symplectic manifold with symplectic form Ω (*i.e.* $\Omega^n \neq 0$ and $d\Omega = 0$) such that $[\Omega] \in H^2(N, \mathbb{Z})$, then there is a compact regular contact manifold M which is an S^1-bundle over N by the Boothby-Wang fibration ([5]). Since N admits an almost Kähler structure (J, G) with Ω as its fundamental 2-form, this almost Kähler structure induces on M a K-contact structure which is Sasakian if and only if (J, G) is Kählerian. Since there exist compact almost Kähler manifolds whose fundamental 2-forms, Ω, determine an integral cohomology class and which are not Kähler (see *e.g.* [7,12]), we conclude that there exist K-contact manifolds which are not Sasakian.

REMARK 3. By a *B-homothetic deformation* (often called a *D-homothetic deformation*) [11] we mean a change of structure tensors of the form

$$\bar\eta = a\eta, \quad \bar\xi = \frac{1}{a}\xi, \quad \bar\phi = \phi, \quad \bar g = ag + a(a-1)\eta\otimes\eta$$

where a is a positive constant. It is well known and easy to see that $(\bar\phi, \bar\xi, \bar\eta, \bar g)$ is a contact metric structure. By direct computation one shows that, R, $\mathrm{Ric}(\xi)$ and R^* transform in the following manner.

$$\bar R = \frac{1}{a}R + \frac{1-a}{a^2}\mathrm{Ric}(\xi) - 2n\Big(\frac{a-1}{a}\Big)^2$$

$$\overline{\mathrm{Ric}}(\bar\xi) = \frac{1}{a^2}\big(\mathrm{Ric}(\xi) + 2n(a^2-1)\big)$$

$$\bar R^* = \frac{1}{a}R^* + \frac{a-1}{a^2}\mathrm{Ric}(\xi) + 2n\Big(2n\Big(\frac{1-a}{a}\Big) + \frac{1-a^2}{a^2}\Big)$$

From these we see that $\overline{W_i} = \frac{1}{a}W_i$, $i = 1, 2, 3$. In particular this also justifies the choice of constants depending on dimension in the definitions of the W_i's.

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REMARK 4. From (2.4) and (3.4) we note that $W_i \geq R + 2n$, $i = 1, 2, 3$. For W_1 equality holds if and only if the structure is K-contact and for W_i, $i = 2, 3$, equality holds if and only if the structure is Sasakian.

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