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ON THE INTERSECTION OF A FAMILY OF MAXIMAL SUBGROUPS CONTAINING THE SYLOW SUBGROUPS OF A FINITE GROUP

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1. Introduction and statement of results. Given a finite group G, the Frattini subgroup of G, $\Phi(G)$ is defined to be the intersection of all the maximal subgroups of G. Of late there have been several attempts to consider generalizations of $\Phi(G)$. For example, Gaschütz [**7**] and Rose [**13**] have investigated the intersection of all non-normal, maximal subgroups of a finite group. Deskins [**6**] has discussed the intersection of the family of maximal subgroups of a finite group whose indices are co-prime to a given prime. In [**4-5**, **12**] we have considered the investigation of the family $\mathscr{I}$ of all maximal subgroups of a finite group whose indices are composite and co-prime to a given prime. We have obtained several results about the family $\mathscr{I}$. In this paper which is a sequel to [**4**] we prove some further results about this family indicating the interesting role it plays especially when G is solvable or p-solvable. First we recall the main definition from [**4**].

Definition. Let G be a finite group and p be any prime. Let

$$\mathscr{I} := \{M <\cdot\, G{:}[G{:}M] \text{ is composite, } [G{:}M]_p = 1\}$$

where $[G{:}M]_p$ denotes the "p-part" of $[G{:}M]$ and the notation $M <\cdot\, G$ is used to denote that M is a maximal subgroup of G. Define

$$S_p(G) := \cap\{M{:}M \in \mathscr{I}\}$$

if $\mathscr{I}$ is nonempty, otherwise we let $S_p(G) = G$.

It is clear from the definition that $S_p(G)$ is a characteristic subgroup of G and moreover $S_p(G)$ contains $\Phi(G)$. We prove

THEOREM 1.1. *Let G be a finite group and p be any prime. Then $S_p(G)$ contains $Z(G)$, the center of G and also $S_p(G)$ contains $H(G)$, the hypercenter of G.*

(We recall the definition of $H(G)$: Let $1 \leqq Z_1(G) \leqq Z_2(G) \leqq \ldots$ be a tower of subgroups where $Z_1(G) = Z(G)$ and $Z_i(G)$ is defined by

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$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

Then the *hypercenter*

$$H(G) := \bigcup_i Z_i(G).$$

For several equivalent definitions and properties of $H(G)$, see [**1**].)

It is a well known property of the Frattini subgroup $\Phi(G)$ that a group G is solvable if and only if $G/\Phi(G)$ is solvable. The following result is a generalisation of this and it shows how $S_p(G)$ controls the solvability of the group G.

THEOREM 1.2. *Assume that either G is p-solvable or p is the largest prime dividing the order of G. Then G is solvable if and only if $G/S_p(G)$ is solvable.*

We recall the definition that a group G is called a *Sylow tower group of supersolvable type* if the following conditions hold:

(i) $p_1 > p_2 > \ldots > p_r$ are all the primes dividing the order of G,

(ii) $P_1P_2 \ldots P_k \lhd G$, $1 \leqq k \leqq r$ where P_k is a Sylow p_k-subgroup of G.

In [**4**, Theorem 1.4] we proved that if $G = S_p(G)$ then G is a Sylow tower group of supersolvable type. We now generalise this result.

THEOREM 1.3. *Let G be a p-solvable group where p is the largest prime dividing the order of G. Then $S_p(G)$ is a Sylow tower group of supersolvable type.*

Since every group which is a Sylow tower group of supersolvable type is not necessarily supersolvable, it is natural to ask: is $S_p(G)$ supersolvable under the hypothesis of Theorem 1.3? The following example shows that this is however not always the case.

Example 1.4. Let G be a group with the presentation:

$$G = \langle a, b, x : a^3 = 1, ab = ba, a^x = b, b^x = a^2 \rangle.$$

The group G is of order 36 and is a split extension of the elementary abelian group $\langle a, b \rangle$ of order 9 by an automorphism of order 4. Further, it is easy to see that every maximal subgroup whose index is prime to 3 is of index 2. Therefore, by taking $p = 3$, it follows that $S_3(G) = G$. However, G is not supersolvable since the normal, elementary abelian subgroup $\langle a, b \rangle$ of order 9 does not contain any normal subgroup of G of order 3.

We now prove that a group G of least order for which $S_p(G)$ is not supersolvable is of the type illustrated in Example 1.4.

THEOREM 1.5. *Among all the groups V for which $S_p(V)$ is not supersolvable, let G be a group of minimal order. Assume that G is p-solvable. Then we have that $G = S_p(G) = NT$ where N is the supersolvable residual of G and is also the unique minimal normal subgroup of G. The order of N is p^s for some $s \geqq 1$ and T is a supersolvable projector of $S_p(G)$.*

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For a group G, the subgroup $S_p(G)$ need not be always solvable. For example, consider $G = PSL(2,7)$ and let $p = 2$. It is well known that any maximal subgroup $PSL(2,7)$ has index 7 or 8. So we have that $S_2(G) = G$ which is a simple group. The following result gives a characterisation of a minimal element in the family of groups G for which $S_p(G)$ is not solvable.

THEOREM 1.6. *Let G be a group of least order for which $S_p(G)$ is not solvable. Then G has a unique minimal normal subgroup N and a core-free maximal subgroup M such that* (i) $G = MN$, $[G:M] = r$ *where r is the largest prime dividing the order of G and r divides the order of $S_p(G)$,* (ii) *N is simple, $N \subseteq S_p(G)$ but $N \not\subseteq \Phi_p(G)$ where $\Phi_p(G)$ denotes the intersection of all maximal subgroups M of G such that $[G:M]_p = 1$.*

By extending the proof of Theorem 1.5 we get

THEOREM 1.7. *Let G be a p-solvable group. Then*

$$S_p(S_p(G)) = S_p(G).$$

We use standard notation as in [**8**] and [**9**]. In addition, we use the notation $M <\cdot\, G$ to denote that M is a maximal subgroup of G. If H is a subgroup of G, then $[G:H]_p$ denotes the "p-part" of the index $[G:H]$. All the groups considered here are finite. It is assumed that the reader is familiar with the well known properties of solvable, supersolvable groups and the Frattini subgroup (see [**9**] for an exhaustive treatment).

2. Preliminary results. For convenience we list here some results used in proving the theorems described in Section 1. For more details see [**4**, **12**]. The first result given below is used extensively in induction arguments.

(2.1) [**12**, Proposition 3]. *Let $H \lhd G$. Then*

$$S_p(G)H/H \subseteq S_p(G/H).$$

In particular if $H \subseteq S_p(G)$ then

$$S_p(G/H) = S_p(G)/H.$$

It is a well known result of B. Huppert that a group G is supersolvable if and only if $G/\Phi(G)$ is supersolvable. We shall use the following generalisation.

(2.2) [**12**, Theorem 9]. *Let H be a normal subgroup of a group G such that H contains $\Phi(G)$. Then H is supersolvable if and only if $H/\Phi(G)$ is supersolvable.*

(2.3) [**12**, Proposition 5]. *Let p be the largest prime dividing the order of G. Then*

(i) *if p divides the order of $S_p(G)$ and P is a Sylow p-subgroup of $S_p(G)$, then $P \lhd G$.*

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(ii) *if p does not divide the order of $S_p(G)$ then for the largest prime divisor q of the order of $S_p(G)$, a Sylow q-subgroup Q of $S_p(G)$ is normal in G.*

(2.4) [**12**, Theorem 8]. *Let p be the prime taken in the definition of $S_p(G)$. Then*

(i) *if p is the largest prime dividing the order of G, we have that $S_p(G)$ is solvable.*

(ii) *if G is p-solvable, then $S_p(G)$ is solvable.*

In [**6**], Deskins has considered the subgroup $\Phi_p(G)$ which is the intersection of all maximal subgroups M of G such that $[G:M]_p = 1$. Clearly, $\Phi_p(G)$ is contained in $S_p(G)$. We shall use the following result:

(2.5) [**6**, **12**]. *$\Phi_p(G)$ is solvable.*

In [**3**], Bhatia has introduced another characteristic subgroup $L(G)$ which is the intersection of all maximal subgroups M of G such that $[G:M]$ is composite. We shall use the following result (a proof is given in [**5**])

(2.6) [**3**]. *For a group G we have that $L(G)$ is supersolvable.*

3. Proofs. We now give the proofs of the results stated in Section 1.

Proof of Theorem 1.1. Let M be a maximal subgroup of G. If $Z(G) \nleqq M$, then M is normal and hence of prime index. Therefore it follows that $Z(G) \leqq S_p(G)$. The fact that $H(G) \leqq S_p(G)$ follows by (2.1) and induction.

Theorem 1.2 is a direct consequence of (2.4) and we omit the details.

Proof of Theorem 1.3. We distinguish two cases. We use induction on the order of G.

Case 1. p divides the order of $S_p(G)$. Let P be a Sylow p-subgroup of $S_p(G)$. Then by (2.3) we have that $P \lhd G$. Consider G/P. By (2.1) we have that

$$S_p(G/P) = S_p(G)/P.$$

If P is not a Sylow p-subgroup of G, then G/P has p as the largest prime dividing its order. Further G/P is p-solvable. Thus in this situation the theorem follows by induction. Now, suppose that P is a Sylow p-subgroup of G. Then p does not divide the order of G/P. Now if M is a maximal subgroup of G such that $[G:M]_p = 1$ and $[G:M]$ is composite then it is easy to see that P is contained in M. Further we have that

$$[G/P:M/P]_p = 1$$

and $[G/P:M/P]$ is composite. Again, if R/P is any maximal subgroup of G/P which is of composite index, then R is a maximal subgroup of G, $[G:R]_p = 1$ and $[G:R]$ is composite. Thus we have that

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$$S_p(G/P) = S_p(G)/P = L(G/P),$$

refer to the definition of $L(G)$ given immediately before (2.6). Now using Bhatia's result (2.6) we have that $L(G/P)$ is supersolvable. So $S_p(G)/P$ is supersolvable and consequently $S_p(G)$ is a Sylow tower group of supersolvable type.

Case 2. p does not divide the order of $S_p(G)$. Let q be the largest prime divisor of the order of $S_p(G)$ and Q be a Sylow q-subgroup of $S_p(G)$. By (2.3) we have that $Q \lhd G$. Consider G/Q. By (2.1) we have that

$$S_p(G/Q) = S_p(G)/Q.$$

By induction the result now follows by arguments similar to Case 1.

Proof of Theorem 1.6. Let N be a minimal normal subgroup of G contained in $S_p(G)$. Using the minimality property of G, we have that $S_p(G/N)$ is solvable. By (2.1),

$$S_p(G/N) = S_p(G)/N.$$

Now if W is another minimal normal subgroup of G contained in $S_p(G)$ then again as before $S_p(G)/W$ is solvable. So we get that

$$S_p(G)/(W \cap N) \simeq S_p(G)$$

is solvable, proving the result. Thus we may now assume that N is the unique minimal normal subgroup of G contained in $S_p(G)$. Further, let B be another minimal normal subgroup G. Let T/B be the intersection of all maximal subgroups of G/B which have composite indices and which are co-prime to p. Then using the minimality of G we have that T/B is solvable. However,

$$S_p(G)B/B \subseteq TB/B.$$

Therefore

$$S_p(G)B/B \simeq S_p(G)/S_p(G) \cap B \simeq S_p(G)$$

is solvable which is a contradiction to our hypothesis. Therefore we may assume that N is the unique minimal normal subgroup of G. Now, we claim that N is not contained in $\Phi_p(G)$. For, suppose if possible, that N is contained in $\Phi_p(G)$. Then we have that N is solvable using the fact (2.5) that $\Phi_p(G)$ is solvable. This now implies that $S_p(G)$ is solvable since we have already shown that $S_p(G)/N$ is solvable. However, it contradicts the hypothesis that $S_p(G)$ is not solvable. Therefore N is not contained in $\Phi_p(G)$. So, there must exist a maximal subgroup M of G such that $[G:M]_p = 1$ and N is not contained in M. Then $G = MN$. Let $[G:M] = r$. Now, r cannot be a composite number, since if r is composite, then $S_p(G) \subseteq M$ implying that $N \subseteq M$ and so we get $G = MN = M$, a contradiction. Again we have that M must be core-free as otherwise N,

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being the unique minimal normal subgroup of G, will be contained in M which would be a contradiction.

Now consider the permutation representation of G on the r cosets of M (as shown above, r is a prime). It follows that the order of G divides $r!$. Since r divides the order of G, we have that r must be the largest prime dividing the order of G. Since $G = MN$ and $[G:M] = r$, it is clear that r divides the order of N. Consequently, r divides the order of $S_p(G)$ since $S_p(G)$ contains N. This completes the proof.

Proof of Theorem 1.5. Let N be a minimal normal subgroup of G contained in $S_p(G)$. By arguing as in the proof of Theorem 1.6 given above, it follows that N is the unique minimal normal subgroup of G. Since G is p-solvable, by 2.4 (ii) we get that $S_p(G)$ is solvable. So N is a p-group or a p'-group. We distinguish two cases.

Case 1. N is a p'-group. Let $o(N) = r^\lambda$ where r is a prime different from p and λ is an integer $\geqq 1$. Let K be any maximal subgroup of G. Assume, first, that N is not contained in K. Then we have that $G = KN$. Since N is a p'-group, $[G:K]_p = 1$. If $[G:K]$ is composite then it will follow that $S_p(G) \subseteq K$ and so $G = K$, a contradiction. Therefore, $[G:K]$ must be a prime number which is thus equal to r. Consequently the order of N is equal to r and N is cyclic. By using inductive argument we get that $S_p(G)/N$ is supersolvable. Since N is cyclic we then get that $S_p(G)$ is supersolvable, a contradiction. Thus we have that N is contained in every maximal subgroup of G and so $N \subseteq \Phi(G)$. Now

$$S_p(G)/\Phi(G) \simeq (S_p(G)/N)/(\Phi(G)/N).$$

So we get that $S_p(G)/\Phi(G)$ is supersolvable. By using (2.2) we now get that $S_p(G)$ is supersolvable, a contradiction to our hypothesis. Therefore we conclude that Case 1 cannot arise.

Case 2. N is a p-group. Let S_F be the supersolvable residual of $S_p(G)$. Then S_F is characteristic in $S_p(G)$. Since N is the unique minimal normal subgroup of G, it follows that $N = S_F$. Now by (2.4), $S_p(G)$ is solvable and so N is elementary abelian. Therefore, by [**9**, Satz 7.15, p. 703] we have that $S_p(G) = NT$ where T is a supersolvable projector in $S_p(G)$ and $N \cap T = \langle 1 \rangle$. Now, let X be a maximal subgroup of $S_p(G)$ whose index in $S_p(G)$ is prime to p. Clearly X contains N and so $N(T \cap X) = X$ and $T \cap X$ contains a Sylow p-subgroup of T. If $T \cap X$ is not maximal in T, then $T \cap X \subseteq H$ for some maximal subgroup H of T. This gives that X is properly contained in NH and so the maximality of X is violated. Hence $T \cap X$ is a maximal subgroup of T. Since T is supersolvable every maximal subgroup of T will have prime index by using a well known result of B. Huppert. Thus we have that

$$[S_p(G):X] = o(T)/o(T \cap X)$$

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is a prime number. It follows now that the family of all maximal subgroups of $S_p(G)$ whose index is composite and also not divisible by p, is empty. Therefore by definition,

$$S_p(S_p(G)) = S_p(G).$$

So if $S_p(G) \neq G$, then $S_p(G)$ is supersolvable because of the minimality of G. However this is a contradiction to the fact that $S_p(G)$ is not supersolvable. Hence we get that $G = S_p(G) = NT$, $N \cap T = \langle 1 \rangle$ where N is the supersolvable residual and T is a supersolvable projector.

Proof of Theorem 1.7. We use induction on the order of G. We remark that for any supersolvable group G we have that $S_p(G) = G$ since every maximal subgroup of G has prime index. We consider two cases:

Case 1. p does not divide the order of $S_p(G)$. Let N be a minimal normal subgroup of G contained in $S_p(G)$. By (2.4) $S_p(G)$ is solvable. So N is elementary abelian and $o(N) = r^a$ where r is a prime different from p and a is an integer $\geqq 1$. Now by the induction hypothesis, $S_p(G)/N$ is supersolvable. Let M be a maximal subgroup of G. If N is not contained in M, then $G = MN$. It is now easy to see that $M \cap N = \langle 1 \rangle$. So $[G:M] = o(N)$. Now $[G:M]_p = 1$ since N is a p'-group. Further $[G:M]$ must be a prime since otherwise $S_p(G)$ is contained in M and so $G = M$, a contradiction. It now follows that $[G:M] = o(N) = r$. Thus N is cyclic and this together with the fact that $S_p(G)/N$ is supersolvable, now gives $S_p(G)$ is supersolvable and so we obtain that

$$S_p(S_p(G)) = S_p(G)$$

proving the result. Thus we now assume that N is contained in every maximal subgroup of G, that is, $N \subseteq \Phi(G)$. Consequently,

$$S_p(G)/\Phi(G) \simeq (S_p(G)/N)/(\Phi(G)/N)$$

is supersolvable since $S_p(G)/N$ is supersolvable. By (2.2) $S_p(G)/\Phi(G)$ is supersolvable implies that $S_p(G)$ is supersolvable. Therefore we get that

$$S_p(S_p(G)) = S_p(G)$$

proving the result.

Case 2. p divides the order of $S_p(G)$. Let N be a minimal normal subgroup of G contained in $S_p(G)$. As in Case 1, N is elementary abelian. Now N is either a p-group or a p'-group. If N is a p'-group the result will follow by arguing as in Case 1. If N is a p-group, then by repeating the argument of the Case 2 of the proof of Theorem 1.5, it is now easy to complete the proof.

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