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Moduli Spaces of Polygons and Punctured Riemann Spheres

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Abstract. The purpose of this note is to give a simple combinatorial construction of the map from the canonically compactified moduli spaces of punctured complex projective lines to the moduli spaces $\mathcal{P}_r$ of polygons with fixed side lengths in the Euclidean space $\mathbb{E}^3$. The advantage of this construction is that one can obtain a complete set of linear relations among the cycles that generate homology of $\mathcal{P}_r$. We also classify moduli spaces of pentagons.

1 Introduction

Let $r = (r_1, \dots, r_n)$ be a collection of positive real numbers and let $\mathcal{P}_r$ be the moduli space of polygons with consecutive side lengths $r_1, \dots, r_n$ in Euclidean space $\mathbb{E}^3$. The study of $\mathcal{P}_r$ was originated by Hausmann [4], Klyachko [10], Kapovich and Millson [7], Hausmann and Knutson [5], and Hu [6]. The complex-analytic structure on $\mathcal{P}_r$ is defined using Deligne-Mostow weighted quotients of the projective line [2]. This structure depends only upon the relations between $\sum_{i\in I} r_i$ and $\sum_{j\in J} r_j$ for partitions $I \coprod J = \{1, \dots, n\}$. Each n-tuple $(r_1, \dots, r_n)$ can be perturbed a little so that all r_i become rational numbers and the complex-analytic structure of $\mathcal{P}_r$ doesn't change. We will always assume therefore that all r_i are rational numbers. In this case, for a generic choice of r the space $\mathcal{P}_r$ has the structure of a smooth complex projective variety. This structure is given by the identification of Deligne-Mostow quotients with Mumford's quotients.

Let $\mathcal{M}_n$ be a canonically compactified space $M_{0,n}$ of n distinct labeled points on $\mathbb{CP}^1$ as in Knudsen [11]. The projective manifold $\mathcal{M}_n$ is obtained from $M_{0,n}$ by adding a normal-crossing divisor. We give a simple construction of a surjective algebraic map $\phi\colon \mathcal{M}_n \to \mathcal{P}_r$ such that its restriction onto $M_{0,n}$ is an isomorphism. Hu in [6] realized $\mathcal{M}_n$ as an iterated sequence of blow-ups of $\mathcal{P}_r$ for a generic r.

There are certain algebraic cycles on $\mathcal{P}_r$ given by a set of conditions that several sides of a polygon from $\mathcal{P}_r$ go in the same direction. (This is a stronger condition than just being parallel.) Using the map ϕ and a theorem of Keel [9] one can easily see that these cycles span the homology groups of $\mathcal{P}_r$. For a non-smooth $\mathcal{P}_r$ we establish a similar result in intersection homology.

Kontsevich and Manin in [12] found a complete set of linear relations among algebraic cycles corresponding to isomorphism classes of trees on $\mathcal{M}_n$. We show how one can use their result and our map ϕ to get a complete set of linear relations among the algebraic cycles on $\mathcal{P}_r$.

When $\mathcal{P}_r$ is not smooth, its singularities are isolated. There exist natural resolutions of singularities $\mathcal{P}_s \to \mathcal{P}_r$ with $\mathcal{P}_s$ being smooth. These resolutions are small in many cases.

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We also find a complete classification of moduli spaces of pentagons. It turns out that such a moduli space is one of the following types: $\mathbb{CP}^2$ with 0, 1, 2, 3, or 4 points blown up, or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

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2 Moduli Spaces of Polygons

Here we mainly collect facts already known about moduli spaces of polygons in Euclidean space $\mathbb{E}^3$. Our major sources of knowledge here are Klyachko [10] and Kapovich and Millson [7].

Let us have an ordered collection of positive rational numbers $r = (r_1, \ldots, r_n)$ and let us consider the moduli space $\mathcal{P}_r$ of n-gons in $\mathbb{E}^3$ with consecutive side lengths $r_1, \ldots, r_n$. Two polygons obtained from one another by a motion of $\mathbb{E}^3$ preserving the numbering of sides are identified. A polygon $P \in \mathcal{P}_r$ is uniquely determined by a collection of unit vectors $v = (v_1, \ldots, v_n)$ satisfying $\sum r_i v_i = 0$ and conversely, each polygon from $\mathcal{P}_r$ gives rise to such a collection up to the action of $SO(3)$. The space $\mathcal{P}_r$ is non-empty if and only if for each $1 \leq j \leq n$ we have $2r_j \leq \sum r_i$. (Of course if we have an actual equality among those, the space $\mathcal{P}_r$ is just a point.) If all those inequalities hold then we call the collection r admissible. A polygon $P \in \mathcal{P}_r$ is called *degenerate* if it looks like a line segment, *i.e.*, if there exists a partition of $(1, \ldots, n)$ onto two non-intersecting subsets I and J such that $\sum_{i \in I} r_i = \sum_{j \in J} r_j$ and for any $i \in I$, $j \in J$ the vectors v_i and v_j have opposite directions.

Assuming that $r_1 \pm r_2 \pm \cdots \pm r_n \neq 0$ the space $\mathcal{P}_r$ has the structure of a compact complex manifold. It is identified [2] with the space of n (not necessarily distinct) stable weighted points on the projective line $\mathbb{CP}^1$ with weights $r_1, \ldots, r_n$ modulo projective automorphisms. When several points collide we just add up the weights. We recall that a configuration of m distinct points on $\mathbb{CP}^1$ with weights $s_1, \ldots, s_m$ is called (semi-)stable if for each j we have $2s_j < \sum s_i$ ($\leq$ for semistable).

We mentioned in Introduction that each n-tuple $r = (r_1, \ldots, r_n)$ of real numbers can be perturbed a little so that the isomorphism class of $\mathcal{P}_r$ does not change and all r_i become rational numbers. It follows from the fact that the isomorphism class of $\mathcal{P}_r$ depends only upon the set of (in)equalities where for each subset $I \subset \{1, \ldots, n\}$ the sum $\sum_{i \in I} r_i$ is compared with the half the perimeter.

If for some partition $I \coprod J = \{1, \ldots, n\}$ we have $\sum_{i \in I} r_i = \sum_{j \in J} r_j$, then the space $\mathcal{P}_r$ is singular with isolated singularities corresponding exactly to the degenerate polygons. At each such point P its neighbourhood is analytically isomorphic to a homogeneous quadratic cone in $\mathbb{C}^{pq}$ (where $p = \#(I) - 1$, and $q = \#(J) - 1$) given by $x_{ij}x_{kl} = x_{il}x_{kj}$. Here $\{x_{ij}\}$, $1 \leq i \leq p$, $1 \leq j \leq q$ is a coordinate system on $\mathbb{C}^{pq}$. We refer to [7] for details.

Let us have a semi-stable configuration of not necessarily distinct points $x_1, \ldots, x_n$ on $\mathbb{CP}^1$ with respective weights $r_1, \ldots, r_n$ which is not stable. It means that there is a point $z \in \mathbb{CP}^1$ such that the points $x_{i_1}, \ldots, x_{i_k}$ collide at z and the sum $\sum_{j=1}^{k} r_{i_j}$ is equal to the half of the total weight. This defines a partition of the set $\{1, \ldots, n\}$ onto two subsets: $\{i_1, \ldots, i_k\}$ and the rest. We shall call two semi-stable and not stable configurations equivalent if the corresponding partitions coincide. A degenerate polygon corresponds to an equivalence class of semi-stable and not stable configurations.

3 The Map $\phi\colon \mathcal{M}_n \to \mathcal{P}_r$

Let $M_{0,n}$ be the moduli space of n distinct labeled points on $\mathbb{CP}^1$. It is obtained from $(\mathbb{CP}^1)^n$ by throwing out divisors $x_i = x_j$ for $i \neq j$ and factoring out by the diagonal action of $\mathrm{PSL}(2,\mathbb{C})$. This space has a smooth canonical compactification $\mathcal{M}_n$ constructed by Deligne, Grothendieck, Mumford, and Knudsen [11] such that $M_{0,n}$ is a complement to a normal-crossing divisor D in $\mathcal{M}_n$. (See *e.g.* [8] for a simple geometric construction of $\mathcal{M}_n$.) The purpose of this section is to construct an algebraic map $\phi\colon \mathcal{M}_n \to \mathcal{P}_r$ which is an isomorphism between two Zariski open subsets.

The strata of the compactification $\mathcal{M}_n$ are labeled by the isomorphism classes of trees with n labeled legs. A point in $\mathcal{M}_n$ is a system

$$(C, x_1, \dots, x_n),$$

where C is a (possibly reducible) curve with at most nodal singularities of arithmetic genus 0 such that the intersection graph of C is a connected tree without loops. We also require that $x_i \in C$ is a smooth point and C has no infinitesimal automorphisms preserving all the x_i and the nodes. Such a curve C is called *stable*.

The space $\mathcal{M}_n$ has one stratum $M\big((T)\big)$ for each tree T with n labeled legs. Points in $M\big((T)\big)$ correspond to the stable curves with the graphs isomorphic to T. The graph of a stable curve has vertices corresponding to the components of the subvariety of smooth points, the edges are the double points, and the legs correspond to the marked points. Codimension of the stratum is equal to the number of edges (*cf.* [3]).

Now we shall define an operation of *contraction* which produces from such a curve $(C, x_1, \dots, x_n)$ a configuration of n not necessarily distinct points of corresponding weights $r_1, \dots, r_n$ on $\mathbb{CP}^1$. To do this we have to choose an irreducible component C' of C which we will call a stem and all the other irreducible components will be called branches then. Given a stem, we look at its nodes and replace each one of them by the collection of points $x_{i_1}, \dots, x_{i_k}$ which belong to the branches growing from this node. Each node y of the stem gets replaced by $x_{i_1}, \dots, x_{i_k}$ which are collided into the single point y. At each such new marked point the weight will be equal to the sum $\sum_j r_{i_j}$.

Lemma 3.1 *Given $(C, x_1, \dots, x_n)$ as above and an admissible collection of weights $r = (r_1, \dots, r_n)$ there exists an irreducible component C' of C such that if we make C' to be a stem and contract to it then the resulting configuration of n points on $\mathbb{CP}^1$ will be semi-stable. If $r_1 \pm \cdots \pm r_n \neq 0$ then a choice of C' is unique and the resulting configuration is actually stable. Otherwise, there could be no more than two choices of C' and they produce equivalent semi-stable configurations.*

Proof Let us pick an irreducible component C' of C such that the contraction to C' minimizes the maximum value of weight assigned to a single point in C'. Let $y \in C'$ be this point with a maximum weight r'. Let us assume that $2r' > \sum r_i$ and arrive later to a contradiction.

Since we deal with an admissible collection $r = (r_1, \dots, r_n)$, the point y is a node at C connecting irreducible components C' and C''. If the contraction to C'' were performed then the new marked points in C'' would bear weights less than r' each. This is clear for the new marked points of C'' other than y since the total weight of points on branches growing from y is initially equal to r' and the fact that C'' has at least three nodes and marked points. At the point $y \in C''$ the new weight is equal to the total weight minus r' which is less than r' by our assumption. Therefore the branch C'' would solve our minimization problem. This contradicts to our choice of C' and the existence is established. This argument also shows the uniqueness of the choice of C' if the resulting configuration is stable.

Let us assume that $r_1 \pm \cdots \pm r_n = 0$ and that after contracting to C' the weight of the point $y \in C'$ defined above equals half the total weight. Let C'' be the irreducible component which meets C' at y. Then one sees that only C' and C'' satisfy our minimization problem and the partitions for contractions to C' and C'' clearly coincide. ∎

The above result is now used to construct a map $\phi\colon \mathcal{M}_n \to \mathcal{P}_r$ for each admissible $r = (r_1, \dots, r_n)$ in an obvious way. The map ϕ is holomorphic and bimeromorphic (it is clearly biholomorphic on a Zariski open set). We notice that $\mathcal{P}_r$ has a Zariski open set U (given by the conditions that v_i does not have the same direction as v_j for $i \neq j$) such that $\phi_{|M_{0,n}}\colon M_{0,n} \to U$ is an isomorphism. It is hardly worth mentioning that ϕ is proper and surjective.

After rescaling by the common denominator of $r_1, \dots, r_n$ we can assume that all r_i are positive integer numbers. The following is now obvious:

Proposition 3.2 *For each n there exists a constant $C(n)$ such that each isomorphism class of $\mathcal{P}_r$ can be realized with positive integer numbers r_i and the perimeter not exceeding $C(n)$.*

From now on we always assume that r is an n-tuple of positive integer numbers.

Deligne and Mostow in [2, (4.6)] relate the quotient $\mathcal{P}_r$ of weighted points on the projective line $\mathbb{CP}^1$ to Mumford's GIT quotients as follows. Let L be the line bundle on $\mathbb{CP}^1$ dual to the tautological line bundle, so *e.g.* $T^*\mathbb{CP}^1 = L^{\otimes -2}$. Let p_i be the projection from $(\mathbb{CP}^1)^n$ to the i-th factor. Then on $(\mathbb{CP}^1)^n$ we define the line bundle

$$\mathcal{L} = \bigotimes_i p_i^*(L^{\otimes 2r_i}).$$

The line bundle $\mathcal{L}$ has a natural $\mathrm{PSL}(2,\mathbb{C})$ action induced by the one on $T^*\mathbb{CP}^1$. The space $\mathcal{P}_r$ is now just the Mumford's quotient $\big((\mathbb{CP}^1)^n / \mathrm{PSL}(2,\mathbb{C})\big)_{\mathcal{L}}$.

Kapranov in [8] showed that the Chow quotient $(\mathbb{CP}^1)^n // \mathrm{PSL}(2,\mathbb{C})$ is isomorphic to $\mathcal{M}_n$. He also constructed a regular birational morphism from the Chow quotient to Mumford's quotients (Theorem 0.4.3). Using the previous paragraph and verifying that our construction is compatible with that in [8] one can conclude that our map ϕ is a specific case of this result of Kapranov. As a consequence, we have

Proposition 3.3 *The map ϕ is regular birational.*

We must mention an important paper by Hu [6] where for a generic r he explicitly realizes $\mathcal{M}_n$ as an iterated sequence of blow-ups of $\mathcal{P}_r$. We briefly review his construction.

For each proper subset $J \subset \{1, \ldots, n\}$ such that $\#(J) > 2$ we consider $\mathcal{P}_{r_J}$ - the moduli space of $(\#J+1)$-gons with side lengths r_j, $j \in J$ and $\sum_{j \in J} r_j - 1$. A pair $(P, P') \in \mathcal{P}_r \times \mathcal{P}_{r_J}$ is called a bubble pair if P has only edges numbered by J parallel. Consider a subset $\tilde{\mathcal{M}}_r$ of $P_r \times \prod_J \mathcal{P}_{r_J}$ consisting of $(P, P_1, \ldots, P_m)$ such that whenever $J_i \subset J_k$ then (P_i, P_s) is a bubble pair and if P_i does not have a bubble then it is generic (does not have parallel edges). Then it is claimed that $\tilde{\mathcal{M}}_r$ is isomorphic to $\mathcal{M}_n$, because each

$$P \in \tilde{\mathcal{M}}_r \subset P_r \times \prod_J \mathcal{P}_{r_J}$$

gives rise to a stable curve of genus zero: each irreducible component corresponds to a projection to a factor and nodes correspond to the longest edges in P_{r_J}. We refer to [6] for details. Our map ϕ is just the projection of $\tilde{\mathcal{M}}_r \simeq \mathcal{M}_n$ onto the first factor.

Now we assume that we deal with the case of equal weights and consider $\phi \colon \mathcal{M}_n \to \mathcal{P}_n$. Here by $\mathcal{P}_n$ we understand the moduli space of equal-side n-gons. The projective variety $\mathcal{P}_n$ is smooth for $n = 2m + 1$ and has $\frac{1}{2}\binom{n}{m}$ isolated singularities for $n = 2m$, $m > 2$. Obviously the map ϕ is equivariant with respect to the action of the symmetric group $\mathbb{S}_n$. (On the right hand side the group $\mathbb{S}_n$ acts by renumbering the vectors v_i.) We also notice that for $n = 3, 4, 5$ this map is an isomorphism. Therefore, using [10] we may conclude that $\mathcal{M}_5$ is the del Pezzo surface of degree 5 obtained from $\mathbb{CP}^2$ by blowing up 4 generic points.

Let $P \in \mathcal{P}_n$ be a non-singular point such that the corresponding configuration on $\mathbb{CP}^1$ is of m different points with weights $s_1, \ldots, s_m$, $\sum s_i = n$. The dimension of the fiber $\phi^{-1}(P)$ is equal to $\sum_{i=1}^{m} (s_i - 2)_+$. (By definition, for $y \in \mathbb{R}$ we have $y_+ = (y + |y|)/2$.) The fact that P is non-singular for $n > 4$ can be rewritten as $2s_i < n$ and this amounts to the observation that the fiber $\phi^{-1}(P)$ is of dimension at most $n - 5$, and thus has codimension at least 2 in $\mathcal{M}_n$. But the fiber at a singular point is of dimension $n - 4$ and is of codimension 1 in $\mathcal{M}_n$.

We now shall describe two algebraic maps. First, we will define $\psi \colon \mathcal{M}_n \to \mathcal{M}_{n-1}$ for $n > 3$. Let us have a curve $(C, x_1, \ldots, x_n)$ representing a point in $\mathcal{M}_n$. Consider the irreducible component C'containing x_n. There are three possibilities:

1. C' has more than three special points (such as nodes and x_is). In this case we just throw x_n away.
2. If C' has x_n and two nodes connecting it to components C_1 and C_2 at the points y_1 and y_2 then we throw away the whole C' and connect the rest by a node identifying y_1 and y_2.
3. If C' has x_n, x_l and a node connecting it to C_1 at a point y then we throw C' away and replace y by x_l.

In fact, the map ψ also appears in a work of Knudsen [11].

The second map $\chi \colon \mathcal{P}_n \to P_{n-1}$ is defined for $n = 2m+1$ and goes as follows. If we have a stable configuration of points on $\mathbb{CP}^1$ then we just throw x_n away. We will clearly end up with a semi-stable configuration. It remains to notice that for an odd n each semi-stable configuration of equally weighted points is actually stable.

Lemma 3.4 *For $n = 2m$ the following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{M}_n & \xrightarrow{\psi} & \mathcal{M}_{n-1} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathcal{P}_n & \xrightarrow{\chi} & \mathcal{P}_{n-1}. \end{array}$$

Proof Straightforward. ■

Our next goal is to describe a section $\theta\colon \mathcal{P}_r \to \mathcal{M}_n$ such that $\phi \circ \theta = \mathrm{Id}_{|\mathcal{P}_r}$. We start with a semi-stable configuration of n points $(x_1, \dots, x_n)$ on $\mathbb{CP}^1$ such that the weight of x_i is equal to r_i. We assume that we have a partition $I_1 \coprod \cdots \coprod I_m = \{1, \dots, n\}$. We also assume that physically we have m points $y_1, \dots, y_m$ on $C' := \mathbb{CP}^1$ such that the points $\{x_i\}_{i \in I_k}$ collide at y_k and the weight of y_k is equal to $\sum_{i \in I_k} r_i$. What we do is replace each y_j with $m := \#(I_j) > 1$ by a tree growing from y_j with branches $C_1, \dots, C_{m-1}$. Each $C_i \simeq \mathbb{CP}^1$; C_1 is connected by a node with C' at y_j, and C_i and C_{i-1} are connected by a node and there are no more singularities. Let $I_j := \{i_1, \dots, i_m\}$ and $i_1 < \cdots < i_m$. We place x_{i_1} at a smooth point on C_1, x_{i_2} at a smooth point on $C_2, \dots, x_{i_{m-2}}$ at a smooth point on C_{m-2}, and finally we place $x_{i_{m-1}}$ and x_{i_m} at two distinct smooth points at C_{m-1}. The curve C of arithmetic genus zero obtained in this fashion with marked points $x_1, \dots, x_n$ defines a point in $\mathcal{M}_n$ and we see that the choice of this point depends only upon the initial location of the points $x_1, \dots, x_n$ on $\mathbb{CP}^1$. Therefore we get a section $\theta\colon \mathcal{P}_r \to \mathcal{M}_n$ and it is straightforward that $\phi \circ \theta$ is the identity map on $\mathcal{P}_r$. Unfortunately, the section θ is not algebraic (nor even continuous), but it is algebraic on each relatively closed stratum corresponding to a combinatorial choice of locations of weighted points on $\mathbb{CP}^1$.

4 Homology of $\mathcal{P}_r$

In this section unless specified otherwise we work over the coefficient ring $\mathbb{Z}$.

The space $\mathcal{P}_n$ is singular with isolated singularities for $n = 2m > 4$.

Proposition 4.1 *Let $n = 2m$ and let $\tilde{\mathcal{P}}_n$ be the moduli space of n-gons with $r = (2, 1, \dots, 1)$. There exists a small algebraic map $\xi\colon \tilde{\mathcal{P}}_n \to \mathcal{P}_n$ which resolves the singularities of $\mathcal{P}_n$.*

Proof It is clear that each semi-stable configuration of points on $\mathbb{CP}^1$ corresponding to a point in $\tilde{\mathcal{P}}_n$ is in fact stable (as a consequence of the fact that n is even). Moreover if we reduce the weight of x_1 from 2 to 1 then we end up with a (semi-)stable configuration of points of equal weights. Thus $\tilde{\mathcal{P}}_n$ is smooth and the algebraic map $\xi : \tilde{\mathcal{P}}_n \to \mathcal{P}_n$ is well-defined. The surjectivity of ξ is not hard to notice. Besides, if $P \in \mathcal{P}_n$ is a singular point then the fiber $\xi^{-1}(P)$ is of dimension $m-2$. This fiber corresponds to the locus in $\tilde{\mathcal{P}}_n$ where m labeled points of weight 1 have collided. ■

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As a consequence we immediately obtain:

Proposition 4.2 *The above resolution gives rise to the following isomorphism:*

$$IC^{\bullet}_{\mathcal{P}_n} \simeq R\xi_{\bullet}\mathbb{C}_{\tilde{\mathcal{P}}_n}.$$

In particular,

$$IH^i(\mathcal{P}_n, \mathbb{C}) \simeq H^i(\tilde{\mathcal{P}}_n, \mathbb{C}).$$

The last equality also holds over $\mathbb{Z}$.

One can also notice that for $n = 2m$ there is a natural inclusion $\iota\colon \tilde{\mathcal{P}}_n \to \mathcal{P}_{n+1}$ which is given by the treating the side corresponding to v_1 which is of length 2 as two sides x_{n+1} and x_1 of lengths 1 of the new $n+1$-gon. Moreover,

Lemma 4.3 *The diagram*

$$\begin{tikzcd} & \tilde{\mathcal{P}}_n \arrow[dl, "\iota"'] \arrow[dr, "\xi"] & \\ \mathcal{P}_{n+1} \arrow[rr, "\chi"] & & \mathcal{P}_n \end{tikzcd}$$

is commutative.

The map ξ has its counterpart in the spaces $\mathcal{M}_n$ which works as follows. Let $(C, x_1, \ldots, x_n) \in \mathcal{M}_n$. We add to C another irreducible component C' by replacing the point x_1 by a node which connects C with C' and we place x_1 and x_{n+1} on C' after that. Let us call $\tilde{\mathcal{M}}_n$ the new space (which is a subspace of $\mathcal{M}_{n+1}$) obtained in such a fashion and let β be the isomorphism $\mathcal{M}_n \to \tilde{\mathcal{M}}_n$ so constructed. From definitions one has

Lemma 4.4

$$\psi \circ \beta = \mathrm{Id}_{\mathcal{M}_n}.$$

One can also resolve the singularities for any moduli space $\mathcal{P}_r$ with $r = (r_1, \ldots, r_n)$ such that for a partition $I \coprod J = \{1, \ldots, n\}$ one has $\sum_{i\in I} r_i = \sum_{j \in J} r_j$. Let us consider a new n-tuple $s = (s_1, \ldots, s_n)$ constructed as follows. Let k be such that $r_k = \min(r_i)$ and we put $s_l = r_l$ for $l \neq k$ and $s_k = r_k + 1$. Now the resolution $\xi\colon \mathcal{P}_s \to \mathcal{P}_r$ is given by the obvious reduction of weight of the k-th point on $\mathbb{CP}^1$ from s_k to r_k. Let $P \in \mathcal{P}_r$ be a degenerate n-gon corresponding to the above partition $I \cup J = \{1, \ldots, n\}$, $I \cap J = \varnothing$. A similar resolution appears in [6]. Let us assume that $k \in J$. The fiber $\xi^{-1}(P)$ is isomorphic to the moduli space of $\big(\#(J) + 1\big)$-gons with side lengths equal to $\{s_j, j \in J$, and $\sum_{i \in I} s_i\}$. The dimension of the fiber $\xi^{-1}(P)$ is equal to $\#(J) - 2$ and in general ξ is not a small map.

We have two maps $\phi_r\colon \mathcal{M}_n \to \mathcal{P}_r$ and $\phi_s\colon \mathcal{M}_n \to \mathcal{P}_s$. The following result is now clear:

Lemma 4.5

$$\xi \circ \phi_s = \phi_r.$$

If we denote $p = \#(I)-1$ and $q = \#(J)-1$ then a neighbourhood of P in $\mathcal{P}_r$ is analytically isomorphic to the variety V of complex $p \times q$ matrices of rank not more than 1. This easily follows from results of Kapovich and Millson [7] discussed in our Section 2. Without any loss of generality we shall assume that $p \geq q$. Let us denote $V^* \subset V$ the variety of $p \times q$ matrices of rank exactly equal to 1. If we consider V as sitting inside $\mathbb{C}^{pq}$ in an obvious fashion, then V^* is obtained from V by throwing out the origin. It follows that V^* is a homogeneous space under $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ and a stabilizer of each point is isomorphic to $\mathrm{GL}(p-1,\mathbb{C}) \times \mathrm{GL}(q-1,\mathbb{C}) \times \mathbb{C}^*$. Therefore, topologically, this neighbourhood of P is homeomorphic to $(S^{2p-1} \times S^{2q-1})/S^1$. If we assume that the sphere S^{2m-1} standardly sits in $\mathbb{C}^m$, then S^1 acts as the multiplication by $e^{\sqrt{-1}\alpha}$ on the first multiple and by $e^{-\sqrt{-1}\alpha}$ on the second. Therefore, it can be represented as the following fibration

$$\begin{array}{ccc} S^{2p-1} & \longrightarrow & (S^{2p-1} \times S^{2q-1})/S^1 \\ & & \downarrow \\ & & \mathbb{CP}^{q-1}. \end{array}$$

It is easy to see that the spectral sequence corresponding to this fibration degenerates at E_2 and thus we obtain

Proposition 4.6 *The local intersection homology stalk at P is a truncation in degrees $\leq p+q-2$ of $H_\bullet(S^{2p-1}) \otimes H_\bullet(\mathbb{CP}^{q-1})$.*

Next we shall have the result analogous to a theorem of Keel [9] (see also Theorem 3.3 in [3]). We start with $\mathcal{P}_r$ such that $r = (r_1, \ldots, r_n)$ is an admissible n-tuple of positive numbers. Let us have a partition of $\{1, \ldots, n\}$ onto l non-intersecting subsets $I_1, \ldots, I_l$ such that the l-tuple $t = (t_1, \ldots, t_l)$ given by

$$t_j = \sum_{i \in I_j} r_i$$

is an admissible l-tuple. In fact, this defines an inclusion $\mathcal{P}_t \hookrightarrow \mathcal{P}_r$ and $\mathcal{P}_t$ can be considered as an algebraic cycle in $\mathcal{P}_r$ denoted by $[\mathcal{P}_t]$. In terms of weighted points on $\mathbb{CP}^1$ this cycle corresponds to the collision of the points $\{x_i\}_{i \in I_j}$ separately for each j.

Theorem 4.7 *If $\mathcal{P}_r$ is non-singular then the cycles $[\mathcal{P}_t]$ span $H_\bullet(\mathcal{P}_r)$.*

Proof First of all we claim that the map $\phi_*\colon H_\bullet(\mathcal{M}_n) \to H_\bullet(\mathcal{P}_r)$ is surjective. This happens because one can see that $\phi_*\phi^! = \mathrm{Id}$ on $H_\bullet(\mathcal{P}_r)$. (Since we deal with a continuous map of degree 1 between two compact manifolds.) We also notice that the pre-image of $\mathcal{P}_t$ consists of $\overline{\mathcal{M}\big((T)\big)}$ for some collection $T \in \mathcal{T}\big((n)\big)$. (This notation was introduced in Section 3.)

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A result of Keel [9] (see also Theorem 3.3 in [3]) implies that the cycles $\big[\overline{\mathcal{M}\big((T)\big)}\big]$ span $H_\bullet(\mathcal{M}_n)$. The theorem now follows from the surjectivity of ϕ_*. ■

In particular, from this statement one easily obtains Corollary 2.2.2 i) of [10].

If $\mathcal{P}_r$ is singular then its singularities are isolated and the algebraic cycles $\mathcal{P}_t$ respect the natural stratification of $\mathcal{P}_r$. In fact, it is possible to define the class $[\mathcal{P}_t]$ of the cycle $\mathcal{P}_t$ of dimension d in the intersection homology group $IH_{2d}(\mathcal{P}_r, \mathbb{Q})$. First, we define a Zariski open subset $\mathcal{P}_t^o$ as the complement in $\mathcal{P}_t$ of all $\mathcal{P}_t \cap \mathcal{P}_{t'}$ for $t' \neq t$. Recall the section θ from Section 3. Then we consider the closure $\overline{\theta(\mathcal{P}_t^o)}$ in $\mathcal{M}_n$. It is an algebraic cycle in $\mathcal{M}_n$ and clearly it is one of $\overline{\mathcal{M}\big((T)\big)}$. Now we apply the decomposition theorem 6.2.5 of [1] which tells us that (in general, non-canonically) the complex $R\phi_* IC^\bullet_{\mathcal{M}_n}$ has $IC^\bullet_{\mathcal{P}_r}(\phi_*\mathbb{Q}_{M_{0,n}})$ as a direct summand. Since $\mathcal{M}_n$ is smooth, we conclude that there is a surjective map $H_\bullet(\mathcal{M}_n, \mathbb{Q}) \to IH_\bullet(\mathcal{P}_r, \mathbb{Q})$ which (abusing notation) we denote ϕ_*. We define

$$[\mathcal{P}_t] := \phi_*[\overline{\theta(P_t^o)}].$$

We mentioned above that the cycles $\overline{\mathcal{M}\big((T)\big)}$ span $H_\bullet(\mathcal{M}_n)$ and thus we obtain

Theorem 4.8 *The cycles $[\mathcal{P}_t]$ span $IH_\bullet(\mathcal{P}_r, \mathbb{Q})$.*

Since the singularities of $\mathcal{P}_r$ are isolated, in case when $\dim(\mathcal{P}_t)$ is more than $(n-3)/2$ (half the dimension of $\mathcal{P}_r$) then the class $[\mathcal{P}_t]$ is defined completely canonically since for $2i > n-3$ all the $2i$-chains are allowable. (We deal with the middle perversity.) Thus $IH_{2i}(\mathcal{P}_r) \simeq H_{2i}(\mathcal{P}_r)$. If the defined above resolution $\xi\colon \mathcal{P}_s \to \mathcal{P}_r$ is small, then $H_\bullet(\mathcal{P}_s) \simeq IH_\bullet(\mathcal{P}_r)$ and we can define the classes $[\mathcal{P}_t]$ using this isomorphism. Now we apply our Theorem 4.7 for the smooth $\mathcal{P}_s$ to conclude that Theorem 4.8 is valid for $\mathcal{P}_r$ even if we replace $\mathbb{Q}$ by $\mathbb{Z}$.

We shall now address the question of finding a complete set of linear relations among the cycles $[\mathcal{P}_t]$ in $H_\bullet(\mathcal{P}_r)$ for a smooth $\mathcal{P}_r$. Essentially one can use the complete set of linear relations found by Kontsevich and Manin [12] (see also [3]) for the cycles generating the homology of $\mathcal{M}_n$ and our map ϕ. Let us recall these linear relations.

As we know, each tree T with n labeled legs determines an algebraic cycle in $\mathcal{M}_n$ and thus a homology class $[T] \in H_\bullet(\mathcal{M}_n)$. These classes were shown by Keel [9] to span $H_\bullet(\mathcal{M}_n)$. Let us have a collection $R = (T, i, j, k, l, v)$, where T is an n-tree, $1 \leq i, j, k, l, \leq n$ are pairwise distinct, and v is a vertex of T such that the paths leading from v to the legs labeled by i, j, k, and l start with pairwise distinct edges e_i, e_j, e_k, and e_l respectively. Next, consider an n-tree T' such that contraction of one of its internal edges results in T and that the lifts of two pairs e_i, e_j, and e_k, e_l are incident to two different vertices of T'. We will denote such a tree by $\{ijT'kl\}$. The Kontsevich-Manin set of linear relations is now

$$\sum_{\{ijT'kl\}} [T'] = \sum_{\{ikT''jl\}} [T''].$$

The number of these relations depends on the number non-isomorphic collections R and the number of the internal edges in T satisfying the above condition.

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These linear relations translate to linear relations in $H_\bullet(\mathcal{P}_r)$ via ϕ_*. Although these relations seem cumbersome, in each particular situation due to the explicit nature of the recipe the answer is easily attainable. We will give an example of such a computation in the next section.

As to the multiplicative structure, it was described by Hausmann-Knutson in [5]. There is an obvious observation that one should consider the cycles $\mathcal{P}_{ij}$ of co-dimension 1 given by the equation $v_i = v_j$, $i \neq j$. A cycle $\mathcal{P}_{ij}$ exists if and only if the $(n-1)$-tuple $(r_i + r_j, r_1, r_2, \ldots, \hat{r}_i, \ldots, \hat{r}_j, \ldots, r_n)$ satisfies the polygon inequalities, *i.e.*, is admissible. It is clear that all other cycles $\mathcal{P}_t$ in question are obtained by intersecting some of $\mathcal{P}_{ij}$.

In the equilateral situation when we deal with $\mathcal{P}_n$ we have an action of $\mathbb{S}_n$ by permuting v_i. The character of the symmetric group action on homology was computed in [10]. In [3] the character of the $\mathbb{S}_n$-module $H_i(\mathcal{M}_n)$ was computed together with the equivariant Poincaré polynomials of $\mathcal{M}_n$. Using our map ϕ these results can be easily translated to $\mathcal{P}_n$. In the smooth case, an embedding of $\mathcal{P}_r$ into a toric variety was used in [5] to address these and other questions related to the cohomology ring of $\mathcal{P}_r$.

5 Moduli of Pentagons

In this section we will classify moduli spaces of pentagons. As usual, we exclude the degenerate cases when one side is equal to the sum of all others. (The moduli space of triangles with fixed side lengths is just a point and the moduli space of 4-gons is $\mathbb{CP}^1$.) One important remark is that each moduli space of pentagons can be identified with a smooth projective manifold. It follows from the fact that at each candidate for a singular point in Proposition 4.6 (and the preceding discussion) we are forced to put $p = 2$ and $q = 1$, and observe that the origin is a smooth point at the space of 2×1 matrices of rank ≤ 1. Let $r = (r_1, r_2, r_3, r_4, r_5)$ be an admissible 5-tuple of positive integer numbers and without any loss of generality we assume that $r_i \leq r_j$ for $i \leq j$. Let R stand for half the perimeter: $2R = \sum_i r_i$. Essentially, the problem of classification boils down to comparing $\sum_{j \in J} r_j$ with R for all proper subsets $J \subset \{1, 2, 3, 4, 5\}$. It follows that isomorphism classes of $\mathcal{P}_r$ can be presented by the following 12 5-tuples completely exhausting the combinatorics among r_1, r_2, r_3, r_4, and r_5:

$$
\begin{array}{cccc}
(1,1,1,1,1) & (1,1,1,2,2) & (1,1,1,1,3) & (1,2,2,2,3) \\
(1,1,2,2,3) & (1,1,3,3,3) & (1,1,2,2,2) & (1,1,2,3,3) \\
(1,1,1,1,2) & (2,2,2,3,3) & (1,1,1,2,3) & (1,2,2,3,4)
\end{array}
$$

Further we denote by *e.g.* $\mathcal{P}_{11233}$ the moduli space of pentagons with side lengths $(1, 1, 2, 3, 3)$. We already saw that $\mathcal{P}_{11111}$ is a blow-up of $\mathbb{CP}^2$ at 4 generic points; it was shown in [2, (4.5.1)] that $\mathcal{P}_{11112}$ is $\mathbb{CP}^2$. Now we complete the classification:

- $\mathcal{P}_{11123} \simeq \mathcal{P}_{11113} \simeq \mathcal{P}_{11112} \simeq \mathbb{CP}^2$,
- $\mathcal{P}_{11122} \simeq \mathcal{P}_{22233}$ is a blow-up of $\mathbb{CP}^2$ at 3 points,
- $\mathcal{P}_{12223} \simeq \mathcal{P}_{12234}$ is a blow-up of $\mathbb{CP}^2$ at one point,
- $\mathcal{P}_{11223}$ is a blow-up of $\mathbb{CP}^2$ at 2 points,
- $\mathcal{P}_{11222} \simeq \mathcal{P}_{11233} \simeq \mathcal{P}_{11333} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$.

In determining the types of these varieties, we used two essential techniques which we illustrate by the following two examples:

1. Assume that we know that $\mathcal{P}_{11123}$ is $\mathbb{CP}^2$ and consider a map $\mathcal{P}_{11223} \to \mathcal{P}_{11123}$ given by the reduction of the weight of the third point from 2 to 1. One can see that it is a regular birational morphism. The fiber at each point of $\mathcal{P}_{11123}$ except for two points corresponding to $v_1 = v_5$ and $v_2 = v_5$ is just a point. The fibers over $v_1 = v_5$ and $v_2 = v_5$ are $\mathbb{CP}^1$ corresponding to moduli space of 4-gons with side lengths $(1, 2, 2, 4)$. It follows that $\mathcal{P}_{11223}$ is a two point blow-up of $\mathbb{CP}^2$.

2. Consider $\mathcal{P}_{11333}$ and consider a $\mathrm{PSL}(2, \mathbb{C})$ equivariant map

$$(p_1, p_2)\colon (\mathbb{CP}^1)^5 \to \mathbb{CP}^1 \times \mathbb{CP}^1$$

given by

$$p_1 = c(v_1, v_3, v_4, v_5), \quad p_2 = c(v_2, v_3, v_4, v_5).$$

Here

$$c(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_3 - z_2}\frac{z_4 - z_2}{z_1 - z_4}$$

is the cross-ratio. One can easily verify that (p_1, p_2) induces an isomorphism on the quotient: $\mathcal{P}_{11333} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$.

We also would like to sketch an example which demonstrates how the relations of Kontsevich-Manin [12] and our map ϕ give linear relations for the cycles on $\mathcal{P}_{12223}$. We will work with 1-cycles. Any tree T with 5 legs and one internal edge we denote as $[ij - klm]$. For instance, if edges at one vertex are labeled by 1 and 2 and at the other by 3, 4, and 5, then we denote this tree (as well as the 1-cycle $\mathcal{M}\big((T)\big)$ corresponding to T) by $[12 - 345]$. The complete set of Kontsevich-Manin relations among 1-cycles now reads as

$$[12 - 345] + [34 - 125] = [24 - 135] + [13 - 245] = [23 - 145] + [14 - 235],$$

plus 4 similar equalities given by the action of the cyclic group in 5 elements on the set $\{1, 2, 3, 4, 5\}$. This set of generators and relations amounts to 5 independent generators, *e.g.* $[12 - 345]$, $[23 - 145]$, $[34 - 125]$, $[45 - 123]$, and $[15 - 234]$. Now if we apply the map ϕ_* to these cycles, we notice that $\phi_*[25 - 134] = \phi_*[35 - 124] = \phi_*[45 - 123]$ and is equal to a point (so they will not contribute to 1-cycles on $\mathcal{P}_{12223}$). Therefore, on the set of one cycles $\phi_*[ij - klm]$ we will have two independent generators, *e.g.* $\phi_*[12 - 345]$ and $\phi_*[34 - 125]$. Moreover,

$$\phi_*[23 - 145] = \phi_*[34 - 125], \quad \phi_*[15 - 234] = \phi_*[12 - 345] - \phi_*[34 - 125].$$

This explicit computation is in total agreement with our knowledge that $\mathcal{P}_{12223}$ is a one-point blow-up of $\mathbb{CP}^2$.

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