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# MAXIMUM IDEMPOTENTS IN NATURALLY ORDERED REGULAR SEMIGROUPS

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## 1. Naturally partially ordered regular semigroups.

We shall denote by $\omega$ the natural partial order on the idempotents $E=E(S)$ of a regular semigroup $S$, so that in $E$,

$$e\omega f \quad \text{if and only if } e=ef=fe.$$

A partially ordered semigroup $S(\leqq)$ is called *naturally partially ordered* [**9**] if the imposed partial order $\leqq$ extends $\omega$ in the sense that

$$(\forall\, e,f\in E) \quad e\omega f \text{ implies } e\leqq f.$$

No assumption is made about the reverse implication.

Nambooripad [**12**] has shown that on any regular semigroup $S$ it is possible to extend $\omega$ to $\leqq$ on all of $S$. One can show that Nambooripad's definition of $\leqq$ is equivalent to

$$(\forall\, a,b\in S)\ a\leqq b \text{ if and only if } a=aa'b=ba''a \text{ for some } a',a''\in V(a),$$

where as usual $V(x)$ denotes the set of inverses of $x\in S$ [**8**]. In general $\leqq$ is not compatible with multiplication on $S$, so that $S(\leqq)$ is not a partially ordered semigroup. McAlister [**9**] has shown that a regular semigroup $S$ can be naturally partially ordered by some partial order $\leqq$ if and only if $S$ is *locally inverse*; that is, each local submonoid $eSe$, $e\in E$, is an inverse semigroup. Nambooripad [**12**] uses the term pseudo-inverse to describe the locally inverse semigroups.

There is a substantial literature on the subject of partially ordered semigroups as such; a good account of it is given in [**6**]. Most of the results in that theory were obtained by concentrating on the partial order. More recently Blyth [**2, 3, 4, 5**] has studied the structure of several classes of regular Dubreil–Jacotin semigroups, and McAlister has shown [**9**] that each of these semigroups is in fact naturally partially ordered and contains an idempotent $u$ maximum with respect to the imposed order. (Of course $u$ is not necessarily maximum with respect to $\omega$, for this would imply that $u$ is the identity element of the locally inverse semigroup $S$ in question, and therefore $S=uSu$ would necessarily be inverse.)

Suppose then that $S$ is a naturally ordered regular semigroup containing an idempotent $u$ which is maximum in the imposed order. Two things can be said of $S$. The first is that $S$ is a locally isomorphic image of a Rees matrix semigroup over the naturally ordered inverse monoid $uSu$ [9]. The second is that $S$ is algebraically isomorphic to

$$W=\{(g,a,h)\in Eu\times uSu\times uE; g\mathscr{L}aa^{-1}, h\mathscr{R}a^{-1}a\},$$

where multiplication in $W$ is defined by

$$(g,a,h)(v,b,w)=(gahva^{-1}, ahvb, b^{-1}hvbw),$$

with a corresponding order isomorphism under certain amenability conditions on the imposed order [7].

Blyth and McFadden [7] proved their results by identifying two characteristics of the maximum idempotent $u$. These are:

(i) $u$ is *medial*, in the sense that $xux=x$ for each $x$ in the idempotent-generated part $IG(S)$ of $S$.
(ii) $u$ is *normal*, in the sense that $uIG(S)u$ is a semilattice.

The main result of this paper is to prove that a regular semigroup $S$ can be naturally partially ordered with a maximum idempotent $u$ if (and only) $S$ contains a medial normal idempotent $u$.

The following proposition is in [7]; we include it here for completeness.

**Proposition 1.1.** *If a regular semigroup $S$ contains a medial normal idempotent then $S$ is locally inverse.*

**Proof.** Suppose that $u\in S$ is a medial normal idempotent and let $e$, $f$, $g\in E$ with $f,g\in eSe$. Then

$$\begin{aligned} fg &= efege = euefeuegeue \text{ ($u$ is medial)} \\ &= e.uegeu.uefeu.e \\ &= gf. \qquad \square \end{aligned}$$

**Proposition 1.2.** *Let $S$ be a locally inverse semigroup containing an idempotent $u$ satisfying*

$$\text{(iii) } (\forall\, e\in E)\ e=eue.$$

*Then $I=Eu$ is a set of idempotent representatives of the $\mathscr{R}$-classes of $S$, and $\Lambda=uE$ is a set of idempotent representatives of the $\mathscr{L}$-classes of $S$. For each $(f,g)\in\Lambda\times I$ the element $fg$ is idempotent.*

This is the constant of Lemmas 3.1 and 3.2 of [9]. It was proved there under the assumptions that $S$ was naturally ordered and $u$ was the maximum idempotent of $S$, but in fact the proof in [9] used only the hypotheses of Proposition 1.2, and we omit it here.

## 2. Locally inverse semigroups with a maximum idempotent

Let $S$ be a regular semigroup, let $I$ and $\Lambda$ be sets, and let $P=(p_{\lambda i})$ be a $\Lambda \times I$ matrix over $S$. Then

$$\mathscr{M}(S; I, \Lambda; P) = I \times S \times \Lambda$$

is a semigroup under the operation defined by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu),$$

but in general it is not regular. But if we define $\mathscr{RM}(S; I, \Lambda; P)$ to be the set of regular elements of $\mathscr{M}(S; I, \Lambda; P)$ we have the following result, Lemma 2.1 of [9].

**Proposition 2.1.** *Let $S$ be a regular semigroup, $I$, $\Lambda$ sets and let $P$ be a $\Lambda \times I$ matrix over $S$. Then*

(i) $(i, a, \lambda) \in \mathscr{M}(S; I, \Lambda; P)$ *is regular if and only if*

$$V(a) \cap p_{\lambda j} S p_{\mu i} \neq \phi \quad \textit{for some} \quad j \in I, \mu \in \Lambda;$$

(ii) $\mathscr{RM}(S, I, \Lambda; P) = \{(i, a, \lambda): V(a) \cap p_{\lambda j} S p_{\mu i} \neq \phi \ \textit{for some} \ j \in I, \mu \in \Lambda\}$ *is a regular subsemigroup of* $\mathscr{M}$ $(S; I, \Lambda; P)$.

We call $\mathscr{RM}(S; I, \Lambda; P)$ a *regular Rees matrix semigroup* over $S$, *unital* if $S = S^1$ and 1 is an entry of $P$. For a good bibliography on this construction, see [11]; see also [14].

A homomorphism $\theta$ of a regular semigroup $T$ onto a regular semigroup $S$ is called a *local isomorphism* if $\theta$ maps each local submonoid $eTe$ of $T$ isomorphically into $S$, and $S$ is called a *locally isomorphic image* of $T$. McAlister has proved the following local structure theorem, which is an elucidation of a result of Allen [1]:

*If $S$ is a regular semigroup and $S = SeS$ for some idempotent $e$ of $S$, then $S$ is a locally isomorphic image of a unital Rees matrix semigroup $T$ over $eSe$.*

For our purposes we construct the covering semigroup $T$ as follows. Let $S$ be a locally inverse semigroup containing an idempotent $u$ satisfying (iii) above. Then obviously $S = SuS$, and if we take $I = Eu$, $\Lambda = uE$ and $p_{f,g} = fg$, then $p_{f,g} \in uE^2u \subseteq E$ and

$$\begin{aligned} T &= \mathscr{RM}(uSu; I, \Lambda; P) \\ &= \{(e, x, f) \in Eu \times uSu \times uE;\, x = uex = xfu\}, \end{aligned}$$

and $\theta$ defined by $(e, x, f)\theta = exf$ is an algebraic local isomorphism from $T$ onto $S$.

To relate $T$ to a possible partial order on $S$ we must first define a partial order on $T$. This is done by coordinates, and for the second coordinate we make the obvious choice of $\omega$ on $uSu$; its identity element $u$ is then the maximum idempotent of the naturally partially ordered inverse monoid $uSu$. For $I$ we define

$$(\forall e \in I) \quad e \leqq u,$$

and similarly for $\Lambda$, using the same symbol $\leqq$,

$$(\forall f \in \Lambda) \quad f \leqq u.$$

These are the finest partial orders on $I$, $\Lambda$ which make $u$ the maximum element of each; since $I$ and $\Lambda$ are for the present considered simply as sets there is no question of compatibility at this stage. Finally, we use the cartesian ordering on $T$:

$$(e, x, f) \leqq (g, y, h) \quad \text{if and only if} \quad e \leqq g, x\omega y, f \leqq h.$$

**Lemma 2.2.** *$T = \mathscr{RM}(uSu; I, \Lambda; P)$ is a partially ordered semigroup under the cartesian ordering.*

**Proof.** Clearly it is enough to prove that $P$ is an isotone map from $\Lambda \times I$ (under the cartesian ordering) to $uSu$. Suppose therefore that $(f, e) \leqq (h, g)$ in $\Lambda \times I$. There is nothing to prove if $f = h$ and $e = g$. For the other three cases, suppose first that $h = u = g$; then $p_{f,e} = fe\omega u^2 = hg = p_{h,g}$. Next let $h = u \geqq f, g = e$. Then $p_{h,g} = ug, p_{f,e} = fe = fg$, and so $p_{f,e}p_{h,g} = fgug = fg$ (using (iii)) $= p_{f,e}$; *since* $p_{h,g}$ and $p_{f,e}$ are in $E(uSu)$ it follows that $p_{f,e}\omega p_{h,g}$. The last case, $h = f$ and $g = u$, is similar, so $P$ is isotone. □

**Lemma 2.3.** *Under the cartesian ordering $T = \mathscr{RM}$ $(uSu; I, \Lambda; P)$ is a naturally partially ordered semigroup with maximum idempotent $(u, u, u)$.*

**Proof.** If $(e, x, f) \in E(T)$ then $x = x.fe.x\omega x^2$, since $fe \in E(uSu)$; therefore $x = x^2$. It follows that if $(e, x, f)\omega(g, y, h)$ in $E(T)$ then $e = g$, $f = h$ and $x = xfey = yfex$; consequently $e = g$, $f = h$ and $x\omega y$, that is, $(e, x, f) \leqq (g, y, h)$ in $T$. Thus $T$ is naturally ordered; obviously $(u, u, u)$ is its maximum idempotent. □

Since $\theta$: $T \rightarrow S$ defined by $(e, x, f)\theta = exf$ is an epimorphism it will follow that $S$ can be partially ordered if the congruence relation $\rho$ associated with $\theta$ is regular in the following sense [**6**]. An equivalence relation $\rho$ on a partially ordered set $A(\leqq)$ is *regular* if $A/\rho$ can be partially ordered in such a way that the natural map $A \rightarrow A/\rho$ is isotone. A *closed bracelet* modulo $\rho$ is a finite subset consisting of $2n$ ($n$ a positive integer) elements $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ of $A$ satisfying

$$a_1 \equiv b_1 \leqq a_2 \equiv b_2 \leqq \cdots \leqq a_{n-1} \equiv b_{n-1} \leqq a_n \equiv b_n \leqq a_1$$

where $\equiv$ denotes equivalence modulo $\rho$. An *open bracelet* modulo $\rho$ is a finite subset of the form

$$x \leqq a_1 \equiv b_1 \leqq a_2 \equiv b_2 \leqq \cdots \leqq a_n \equiv b_n \leqq y.$$

In this $x$ is called the *initial clasp* and $y$ the *terminal clasp*. By Theorem 6.1 of [**6**], $\rho$ is regular on $A$ if and only if for every closed bracelet modulo $\rho$ all the elements belong to the same $\rho$-class. In this case $A/\rho$ can be ordered by

$\rho_x \leqq \rho_y$ *if and only if there is an open bracelet with initial clasp x and terminal clasp y.*

When $A$ is a semigroup and $\rho$ the congruence associated with a homomorphism of $A$, this ordering clearly makes $A/\rho$ a partially ordered semigroup, the isotone epimorphic image of $A$ under the canonical map $A \to A/\rho$.

**Theorem 2.4.** *Let S be a locally inverse semigroup which contains an idempotent u such that $e = eue$ for each idempotent e of S. Then the congruence relation $\rho$ associated with the local epimorphism $\theta: T = \mathscr{RM}(uSu; I, \Lambda; P) \to S$ where $p_{f,e} = fe$, given by $(e, x, f)\theta = exf$, is regular.*

**Proof.** Consider a closed bracelet

$$a_1 \equiv b_1 \leqq a_2 \equiv b_2 \leqq \cdots \leqq a_{n-1} \equiv b_{n-1} \leqq a_n \equiv b_n \leqq a_1 \quad \text{in} \quad T.$$

Clearly we can assume without loss of generality that $b_i \neq a_{i+1}$ for any $i$; and if $a_i \equiv a_{i+1}$ we can replace $a_i \equiv b_i \leqq a_{i+1} \equiv b_{i+1}$ by $a_i \equiv b_{i+1}$ so we can also assume $a_i \not\equiv a_{i+1}$ for any $i$, with obvious modifications if $i = n$.

By definition of $\theta$, two elements $(e, x, f)$ and $(g, y, h)$ of $T$ are $\rho$-equivalent if and only if $exf = gyh$, and by definition of $T$ this is so if and only if $ex = gy$, $xf = yh$ and $x = y$. It follows from the construction of the order on $T$ that all the elements in a closed bracelet have their second components equal.

Consider a closed bracelet with $n$ $\rho$-links:

$$a_i = (e_{2i-1}, x, f_{2i-1}) \equiv (e_{2i}, x, f_{2i}) = b_i, 1 \leqq i \leqq n > 1.$$

There are four cases to be considered in the comparison $b_n \leqq a_1$ and we are assuming that one of them, $b_n = a_1$, does not occur.

*Case 1.* $e_1 = u = f_1$. Here $e_1 x f_1 = x = e_2 x f_2$ while $e_2 \leqq e_3$ and $f_2 \leqq f_3$. Assuming $b_1 \neq a_2$ there are only three sub-cases. One of these is $e_3 = u = f_3$, and this immediately implies

$$e_2 x f_2 = e_1 x f_1 = uxu = e_3 x f_3,$$

so that $a_1 \equiv a_2$, contrary to assumption.

The second is $e_2 = e_3$ and $f_3 = u$, which leads to $e_3 x f_3 = e_2 x u = e_2 x = e_1 x$ (since $a_1 \equiv b_1$) $= e_3 x$, while $x f_1 = xu = x f_3$. But $e_1 x = e_3 x$ and $x f_1 = x f_3$ is also equivalent to $a_1 \equiv a_2$.

The last subcase $e_3 = u$, $f_2 = f_3$ is completely analogous to the preceding.

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*Case 2.* $e_1 = u, f_1 = f_{2n}$. Here $xf_1 = xf_{2n}$; we also have the relations

$$\begin{array}{rl} x = e_1 x = e_2 x & e_2 \leqq e_3 \\ e_3 x = e_4 x & e_4 \leqq e_5 \\ \vdots & \vdots \\ e_{2n-3} x = e_{2n-2} x & e_{2n-2} \leqq e_{2n-1} \\ e_{2n-1} x = e_{2n} x & e_{2n} \leqq e_1 \end{array}$$

There are precisely two ways in which $e_2 \leqq e_3$, namely (a) $e_2 = e_3$ or (b) $e_3 = u$.

(a) If $e_2 = e_3$ then $x = e_1 x = e_2 x = e_3 x = e_4 x$.
(b) If $e_3 = u$ then $x = e_1 x = e_2 x$ and $e_3 x = ux = x = e_4 x$,

so $x = e_1 x = e_2 x = e_3 x = e_4 x$ in either case.

Similarly $e_4 \leqq e_5$ implies $x = e_i x$, $1 \leqq i \leqq 6$, and it follows by a simple inductive argument that $x = e_i x$, $1 \leqq i \leqq 2n$. Finally, $xf_1 = xf_{2n}$ and $e_1 x = e_{2n} x$ are equivalent to $b_n \equiv a_1$, again contrary to assumption.

The third case, $e_1 = e_{2n}$ and $f_1 = u$, is completely analogous to the second, and the result follows. □

**Theorem 2.5.** *Let $S$ be a locally inverse semigroup which contains an idempotent $u$ such that $e = eue$ for each idempotent $e$ of $S$. Then $S$ can be naturally partially ordered in such a way that $u$ is the maximum idempotent of $S$.*

**Proof.** We know that $S$ is a locally isomorphic image of $T = \mathscr{RM}(uSu: I, \Lambda; P)$, $p_{f,e} = fe$, and by Lemmas 2.2 and 2.3, that $T$ is a naturally ordered regular semigroup with maximum idempotent $u$. Each pair $s, t$ of elements of $S$ can be expressed in the form $s = exf$, $t = gyh$ for $(e, x, f)$, $(g, y, h) \in T$, and if we then order $S$ by

> $s \leqq t$ *if and only if there exist* $a_i = (e_{2i-1}, x_i, f_{2i-1})$ *and* $b_i = (e_{2i}, x_i, f_{2i})$, $1 \leqq i \leqq n$ *for some positive integer* $n$, *such that* $(e, x, f) \leqq a_1 \equiv b_1 \leqq a_2 \dots \leqq a_n \equiv b_n \leqq (g, y, h)$,

it follows from Theorem 2.4 that the partially ordered semigroup $S(\leqq)$ is an isotone homomorphic image of $T$. The result now follows from Corollary 3.4 of **[9]**, that any isotone homomorphic image of a naturally partially ordered regular semigroup is itself naturally ordered, together with the obvious fact that the image $u$ of $(u, u, u)$ is the maximum idempotent of $S$. □

Given that $S$ is locally inverse, the only property of the element $u$ used to prove Theorem 2.5 was (iii), and since every naturally partially ordered regular semigroup with a maximum idempotent $u$ is locally inverse and $u$ satisfies (i) [7], it follows that (i) and (iii) coincide for locally inverse semigroups. This can of course be proved directly; it is also inherent in Theorem 5.5 of **[10]**, which yields the result when applied to $IG(S)$. All the idempotents satisfying (iii) in a locally inverse semigroup $S$, if there are any at all, are contained in a single $\mathscr{J}$-class because for each one, say $u$, $SuS = S$. Actually they are all $\mathscr{D}$-equivalent because for any pair $u$, $v$ of them, $u=uvu$ and $v=vuv$. Finally, they form a rectangular band, because if $e^2=e\in S$ then $euve=euv.eue=eu.v.eu.e=eue=e$.

When a naturally partially ordered regular semigroup $S$ contains a maximum idempotent $u$ then $S$ is orthodox if and only if $u$ is a middle unit in the sense that $(\forall\, x,y\in S)\ xy=xuy$ [9].

Naturally partially ordered orthodox semigroups are liberally supplied with subsemigroups which have a maximum idempotent. For an idempotent $e$ in a naturally partially ordered regular semigroup $S$, let

$$S_e=\{x\in S:\exists\, x'\in V(x)\quad\text{with}\quad xx'\leqq e, x'x\leqq e\}.$$

Then $S_e$ is a subsemigroup of $S$, for if $x,y\in S_e$ and $x'\in V(x), y'\in V(y)$ with $xx'\leqq e$, $x'x\leqq e$, $yy'\leqq e$, $y'y\leqq e$, let $g\in S(x'x,yy')$ [13]; then $y'gx'\in V(xy)$ and $xyy'gx'=xgx'\leqq xx'\leqq e$, and similarly $y'gx'xy\leqq e$. Obviously $S_e$ is regular and naturally ordered, and if $f^2=f\in S_e$ and $f'\in V(f)$ satisfies $ff'\leqq e$, $f'f\leqq e$, then if $S$ is orthodox, $f'\in E$, and so $f=ff'f=ff'f'f\leqq e^2=e$.

Orthodox semigroups are not the only ones for which each $S_e$ has a maximum idempotent. If $S$ is any completely 0-simple semigroup then it is naturally partially ordered under $\omega$ as the imposed order and $S_e$ is just the maximal subgroup of $S$ containing $e$. In fact a naturally partially ordered regular semigroup $S$ may coincide with $S_e$ for one of its idempotents $e$ without being orthodox. For example, let $G$ be a partially ordered group and let $x<1$ in $G$. Let $P$ be the $2\times 2$ matrix over $G$ with $p_{11}=x$, $p_{12}=p_{21}=p_{22}=1$, and let $S=\mathscr{M}(G;2,2;P)$ be partially ordered by the cartesian order. Then $P$ is an isotone map $\{1,2\}\times\{1,2\}\to G$ and $S$ is a partially ordered completely simple semigroup. It is therefore a naturally partially ordered regular semigroup. Its idempotents are:

$$u=(2,1,2),(1,1,2),(2,1,1),(1,x^{-1},1)$$

(note that $x^{-1}>1$) and under the imposed order $u>(1,1,2)$ and $u>(2,1,1)$. Let $y=(i,z,j)\in S$; then $y'=(2,z^{-1},2)\in V(y)$ and $yy'=(i,1,2)\leqq u$, $y'y=(2,1,j)\leqq u$, so $S=S_u$. But $S$ has no maximum idempotent because $u\not>(1,x^{-1},1)$.

## REFERENCES


**1.** D. Allen, Jr. A generalization of the Rees theorem to a class of regular semigroups, *Semigroup Forum* **2** (1971), 321–331.

**2.** T. S. Blyth, Dubreil–Jacotin inverse semigroups, *Proc. Roy. Soc. Edinburgh Sect. A* **71** (1974), 345–360.

**3.** T. S. Blyth, The structure of certain ordered regular semigroups, *Proc. Roy. Soc. Edinburgh Sect. A* **75** (1976), 235–257.

**4.** T. S. Blyth, On a class of Dubreil–Jacotin semigroups and a construction of Yamada, *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977), 145–150.

**5.** T. S. Blyth, Perfect Dubreil–Jacotin semigroups, *Proc. Roy. Soc. Edinburgh Sect A* **78** (1977), 101–104.

6. T. S. Blyth and M. F. Janowitz, *Residuation Theory* (International Series of Monographs on Pure and Applied Mathematics, Vol. 102, Pergamon, London/New York, 1972).

7. T. S. Blyth and R. McFadden, On the construction of a class of regular semigroups, *Journal of Algebra*, 1983, to appear.

8. J. M. Howie, *An Introduction to Semigroup Theory*, (Academic Press, London, 1976).

9. D. B. McAlister, Regular Rees matrix semigroups and regular Dubreil–Jacotin semigroups, *J. Austral. Math. Soc.* (*Series A*) **31** (1981), 325–336.

10. D. B. McAlister and T. S. Blyth, Split orthodox semigroups, *J. Algebra* **51** (1978), 491–525.

11. J. Meakin, The Rees construction in regular semigroups, submitted.

12. K. S. S. Nambooripad, The natural partial order on a regular semigroup, *Proc. Edinburgh Math. Soc.* **23**, (1980), 249–260.

13. K. S. S. Nambooripad, Structure of regular semigroups I, *Mem. Amer. Math. Soc.* **22**, Number 224 (1979).

14. F. J. Pastijn, Structure theorems for pseudo inverse semigroups, *Proceedings of a Symposium on Regular Semigroups* (Northern Illinois University, 1979).

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