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PAPER

Scott topology on Smyth power posets†

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Dedicated to Professor Dana Scott on the occasion of his 90th birthday

Abstract

For a T_0 space X, let $\mathsf{K}(X)$ be the poset of all nonempty compact saturated subsets of X endowed with the Smyth order $\sqsubseteq$. $(\mathsf{K}(X), \sqsubseteq)$ (shortly $\mathsf{K}(X)$) is called the Smyth power poset of X. In this paper, we mainly discuss some basic properties of the Scott topology on Smyth power posets. It is proved that for a well-filtered space X, its Smyth power poset $\mathsf{K}(X)$ with the Scott topology is still well-filtered, and a T_0 space Y is well-filtered iff the Smyth power poset $\mathsf{K}(Y)$ with the Scott topology is well-filtered and the upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(Y)$. A sober space Z is constructed for which the Smyth power poset $\mathsf{K}(Z)$ with the Scott topology is not sober. A few sufficient conditions are given for a T_0 space X under which its Smyth power poset $\mathsf{K}(X)$ with the Scott topology is sober. Some other properties, such as local compactness, first-countability, Rudin property and well-filtered determinedness, of Smyth power spaces, and the Scott topology on Smyth power posets, are also investigated.

Keywords: Scott topology; Smyth power poset; Smyth power space; sobriety; well-filteredness; local compactness; first-countability

1. Introduction

An important problem in domain theory is the modeling of non-deterministic features of programming languages and of parallel features treated in a non-deterministic way. If a non-deterministic program runs several times with the same input, it may produce different outputs. To describe this behavior, powerdomains were introduced by Plotkin (1976, 1982) and Smyth (1978) to give denotational semantics to non-deterministic choice in higher-order programming languages. The three main such powerdomains are the Smyth powerdomain for demonic non-determinism, the Hoare powerdomain for angelic non-determinism, and the Plotkin powerdomain for erratic non-determinism. This viewpoint traditionally stays with the category of dcpos, but is easily and profitably extended to general topological spaces (see, for example, Abramsky et al. 1994, Sections 6.2.3 and 6.2.4 and Schalk 1993).

A subset A of a T_0 space X is called *saturated* if A equals the intersection of all open sets containing it (or equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth order*

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$\sqsubseteq$, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. We call $(\mathsf{K}(X), \sqsubseteq)$ (shortly $\mathsf{K}(X)$) the *Smyth power poset* of X. The *upper Vietoris topology* on $\mathsf{K}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\mathcal{O}(X)$ is the set of all open subsets of X and $\Box U = \{K \in \mathsf{K}(X) : K \subseteq U\}$, and the resulting space is called the *Smyth power space* or *upper space* of X and is denoted by $P_S(X)$.

In domain theory and non-Hausdorff topology, we encounter numerous links between topology and order theory (cf. Gierz et al. 2003; Goubault 2013). For a poset, the Scott topology is probably the most important one among the intrinsic topologies on it. As pointed out by Goubault in Goubault (2012), there is naturally another prominent topology one can put on $\mathsf{K}(X)$, namely the Scott topology. It is well-known that when X is well-filtered, $\mathsf{K}(X)$ is a dcpo, with least upper bounds of directed families computed as filtered intersections, and the upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(X)$; when X is locally compact and well-filtered (equivalently, locally compact and sober), the two topologies coincide on $\mathsf{K}(X)$, and $\mathsf{K}(X)$ is then a continuous domain (see Schalk 1993, Proposition 7.25 and Lemma 7.26 and Xu et al. 2021*c*, Theorem 3.9).

In this paper, we mainly discuss some basic properties of the Scott topology on Smyth power posets. The paper is organized as follows:

In Section 2, some standard definitions and notations are introduced which will be used in the whole paper. A few basic properties of irreducible sets and compact saturated sets are listed.

In Section 3, we briefly recall the concepts of Scott topology and continuous domains and some fundamental results about them.

In Section 4, we list a few important results of d-spaces, well-filtered spaces, and sober spaces that will be used in other sections.

In Section 5, we recall some concepts and results about the topological Rudin Lemma, Rudin spaces, and well-filtered determined spaces that will be used in the next four sections.

In Section 6, we mainly investigate the well-filteredness of the Scott topology on Smyth power posets. It is proved that the Scott space $\Sigma\mathsf{K}(X)$ of a well-filtered space X is still well-filtered, and a T_0 space Y is well-filtered iff $\Sigma\mathsf{K}(Y)$ is well-filtered and the upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(Y)$.

In Section 7, a sober space X is constructed for which the Scott space $\Sigma\mathsf{K}(X)$ is not sober.

In Section 8, we study the question under what conditions the Scott space $\Sigma\mathsf{K}(X)$ of a sober space X is sober. This question is related to the investigation of conditions under which the upper Vietoris topology coincides with the Scott topology on $\mathsf{K}(X)$, and further it is closely related to the local compactness and first-countability of X.

In Section 9, the Rudin property and well-filtered determinedness of Smyth power spaces and the Scott topology on Smyth power posets are discussed.

2. Preliminaries

In this section, we briefly recall some standard definitions and notations that will be used in the paper. Some basic properties of irreducible sets and compact saturated sets are presented.

For a set X, $|X|$ will denote the cardinality of X. Let $\mathbb{N}$ denote the set of all natural numbers and $\omega = |\mathbb{N}|$. As a poset (indeed a chain), $\mathbb{N}$ is always endowed with the usual order if no other explanation is given. The set of all subsets of X is denoted by 2^X. Let $X^{(<\omega)} = \{F \subseteq X : F \text{ is a finite set}\}$ and $X^{(\leq\omega)} = \{F \subseteq X : F \text{ is a countable set}\}$.

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). Let $\mathbf{Fin}P = \{\uparrow F : F \in P^{(<\omega)}\}$. For a nonempty subset A of P, define $\min(A) = \{u \in A : u \text{ is a minimal element of } A\}$ and $\max(A) = \{v \in A : v \text{ is a maximal element of } A\}$.

A nonempty subset D of a poset P is *directed* if every two elements in D have an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. $I \subseteq P$ is called an *ideal* of P if I is a directed

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lower subset of P. Let Id(P) be the poset (with the order of set inclusion) of all ideals of P. Dually, we define the notion of *filters* and denote the poset of all filters of P by Filt(P). The poset P is called a *directed complete poset*, or *dcpo* for short, if for any $D \in \mathscr{D}(P)$, $\vee D$ exists in P.

The poset P is said to be *Noetherian* if it satisfies the *ascending chain condition* (ACC for short): every ascending chain has a greatest member. Clearly, P is Noetherian iff every directed set of P has a largest element or, equivalently, every ideal of P is principal (cf. Zhao et al. 2015).

As in Erné (2018), a topological space X is *locally hypercompact* if for each $x \in X$ and each open neighborhood U of x, there is $\uparrow F \in \mathbf{Fin}X$ such that $x \in \mathrm{int} \uparrow F \subseteq \uparrow F \subseteq U$. The space X is called a *c-space* if for each $x \in X$ and each open neighborhood U of x, there is $u \in X$ such that $x \in \mathrm{int} \uparrow u \subseteq \uparrow u \subseteq U$). A set K of X is called *supercompact* if for any family $\{U_i : i \in I\}$ of open sets of X, $K \subseteq \bigcup_{i \in I} U_i$ implies $K \subseteq U$ for some $i \in I$. It is easy to verify that the nonempty supercompact saturated sets of X are exactly the sets $\uparrow x$ with $x \in X$ (see Heckmann and Keimel 2013, Fact 2.2). It is well-known that X is a c-space iff $\mathscr{O}(X)$ is a *completely distributive* lattice (cf. Erné 2009).

The category of all T_0 spaces and continuous mappings is denoted by $\mathbf{Top}_0$. For $X \in ob(\mathbf{Top}_0)$, we use $\leq_X$ to denote the *specialization order* of X: $x \leq_X y$ iff $x \in \overline{\{y\}}$). In the following, when a T_0 space X is considered as a poset, the order always refers to the specialization order if no other explanation is given. Let $\mathscr{O}(X)$ (resp., $\mathscr{C}(X)$) be the set of all open subsets (resp., closed subsets) of X, and let $\mathscr{S}^u(X) = \{\uparrow x : x \in X\}$. Define $\mathscr{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathscr{D}_c(X) = \{\overline{D} : D \in \mathscr{D}(X)\}$.

It is straightforward to verify the following.

Remark 1. Let X be a topological space and $A, B \subseteq X$. Then,

(1) $\overline{A} = \overline{B}$ if and only if for any $U \in \mathscr{O}(X)$, $A \cap U \neq \emptyset$ iff $B \cap U \neq \emptyset$.
(2) If τ_1, τ_2 are two topologies on the set X and $\tau_1 \subseteq \tau_2$, then $\mathrm{cl}_{\tau_2} A = \mathrm{cl}_{\tau_2} B$ implies $\mathrm{cl}_{\tau_1} A = \mathrm{cl}_{\tau_1} B$.

For a T_0 space X and a nonempty subset A of X, A is *irreducible* if for any $\{F_1, F_2\} \subseteq \mathscr{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Clearly, every subset of X that is directed under $\leq_X$ is irreducible.

The following lemma is well-known and can be easily verified.

Lemma 2. *If* $f : X \longrightarrow Y$ *is continuous and* $A \in \mathsf{Irr}(X)$, *then* $f(A) \in \mathsf{Irr}(Y)$.

For any T_0 space X, the *lower Vietoris topology* on $\mathsf{Irr}_c(X)$ is the topology $\{\Diamond U : U \in \mathscr{O}(X)\}$, where $\Diamond U = \{A \in \mathsf{Irr}_c(X) : A \cap U \neq \emptyset\}$. The resulting space, denoted by X^s, with the canonical mapping $\eta_X : X \longrightarrow X^s, x \mapsto \overline{\{x\}}$, is the *sobrification* of X (cf. Gierz et al. 2003; Goubault 2013).

Remark 3. For a T_0 space X, $\eta_X : X \longrightarrow X^s$ is a dense topological embedding (cf. Gierz et al. 2003; Goubault 2013; Schalk 1993).

A subset A of a space X is called *saturated* if A equals the intersection of all open sets containing it (or equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth preorder* $\sqsubseteq$, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. The poset $(\mathsf{K}(X), \sqsubseteq)$ (shortly $\mathsf{K}(X)$) will be called the *Smyth power poset* of X. The *upper Vietoris topology* on $\mathsf{K}(X)$ is the topology that has $\{\Box U : U \in \mathscr{O}(X)\}$ as a base, where $\Box U = \{K \in \mathsf{K}(X) : K \subseteq U\}$, and the resulting space is called the *Smyth power space* or *upper space* of X and is denoted by $P_S(X)$ (cf. Heckmann 1992; Schalk 1993). For $A \subseteq X$, define $\Diamond A = \{K \in \mathsf{K}(X) : K \cap A \neq \emptyset\}$.

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Remark 4. Let X be a T_0 space.

(1) The specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \sqsubseteq$).
(2) The *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding (cf. Heckmann 1992; Heckmann and Keimel 2013; Schalk 1993).
(3) X is homeomorphic to the subspace $S^u(X)$ of $P_S(X)$ by means of ξ_X.

Lemma 5. *For a T_0 space X and $A \subseteq X$, $\mathrm{cl}_{\mathcal{O}(P_S(X))}\xi_X(A) = \Diamond\overline{A}$.*

Proof. Clearly, $\Diamond\overline{A} = \mathsf{K}(X) \setminus \Box(X \setminus \overline{A})$ is closed in $P_S(X)$, and hence, $\mathrm{cl}_{\mathcal{O}(P_S(X))}\xi_X(A) \subseteq \Diamond\overline{A}$. Since $\{\Diamond C : C \in \mathscr{C}(X)\}$ is a (closed) base of $P_S(X)$, there is a family $\{C_i : i \in I\} \subseteq \mathscr{C}(X)$ such that $\mathrm{cl}_{\mathcal{O}(P_S(X))}\xi_X(A) = \bigcap_{i \in I} \Diamond C_i$. Then for each $i \in I$, $\xi_X(A) \subseteq \Diamond C_i$, and consequently, $\uparrow a \cap C_i \neq \emptyset$ for each $a \in A$; whence, for each $a \in A$, $a \in C_i$ as $C_i = \downarrow C_i$. It follow that $\overline{A} \subseteq C_i$ for each $i \in I$, and hence, $\Diamond\overline{A} \subseteq \bigcap_{i \in I} \Diamond C_i = \mathrm{cl}_{\mathcal{O}(P_S(X))}\xi_X(A)$. Thus, $\mathrm{cl}_{\mathcal{O}(P_S(X))}\xi_X(A) = \Diamond\overline{A}$. □

Proposition 6. *(Xu 2021, Lemma 2.19) $P_S : \mathbf{Top}_0 \longrightarrow \mathbf{Top}_0$ is a covariant functor, where for any $f : X \longrightarrow Y$ in $\mathbf{Top}_0$, $P_S(f) : P_S(X) \longrightarrow P_S(Y)$ is defined by $P_S(f)(K) = \uparrow f(K)$ for all $K \in \mathsf{K}(X)$.*

Corollary 7. *Let X and Y be two T_0 spaces. If Y is a retract of X, then $P_S(Y)$ is a retract of $P_S(X)$.*

For a nonempty subset C of a T_0 space X, it is easy to see that C is compact iff $\uparrow C \in \mathsf{K}(X)$. Furthermore, we have the following useful result (see, e.g., Erné 2009, pp. 2068).

Lemma 8. *Let X be a T_0 space and $C \in \mathsf{K}(X)$. Then, $C = \uparrow \min(C)$ and $\min(C)$ is compact.*

Lemma 9. *Let X be a T_0 space. For any nonempty family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, $\bigvee_{i \in I} K_i$ exists in $\mathsf{K}(X)$ iff $\bigcap_{i \in I} K_i \in \mathsf{K}(X)$. In this case, $\bigvee_{i \in I} K_i = \bigcap_{i \in I} K_i$.*

Proof. Suppose that $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$ is a nonempty family and $\bigvee_{i \in I} K_i$ exists in $\mathsf{K}(X)$. Let $K = \bigvee_{i \in I} K_i$. Then, $K \subseteq K_i$ for all $i \in I$, and hence, $K \subseteq \bigcap_{i \in I} K_i$. For any $x \in \bigcap_{i \in I} K_i$, $\uparrow x$ is a upper bound of $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, whence $K \sqsubseteq \uparrow x$ or, equivalently, $\uparrow x \subseteq K$. Therefore, $\bigcap_{i \in I} K_i \subseteq K$. Thus, $\bigcap_{i \in I} K_i = K \in \mathsf{K}(X)$.

Conversely, if $\bigcap_{i \in I} K_i \in \mathsf{K}(X)$, then $\bigcap_{i \in I} K_i$ is an upper bound of $\{K_i : i \in I\}$ in $\mathsf{K}(X)$. Let $G \in \mathsf{K}(X)$ be another upper bound of $\{K_i : i \in I\}$, then $G \subseteq K_i$ for all $i \in I$, and hence, $G \subseteq \bigcap_{i \in I} K_i$, that is, $\bigcap_{i \in I} K_i \sqsubseteq G$, proving that $\bigvee_{i \in I} K_i = \bigcap_{i \in I} K_i$. □

Similarly, we have the following.

Lemma 10. *Let P be a poset. For any nonempty family $\{\uparrow F_i : i \in I\} \subseteq \mathbf{Fin}P$, $\bigvee_{i \in I} \uparrow F_i$ exists in $\mathbf{Fin}P$ iff $\bigcap_{i \in I} \uparrow F_i \in \mathbf{Fin}P$. In this case $\bigvee_{i \in I} \uparrow F_i = \bigcap_{i \in I} \uparrow F_i$.*

Lemma 11. *(Schalk 1993, Proposition 7.21) Let X be a T_0 space.*

(1) *If $\mathscr{K} \in \mathsf{K}(P_S(X))$, then $\bigcup \mathscr{K} \in \mathsf{K}(X)$.*
(2) *The mapping $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$, $\mathscr{K} \mapsto \bigcup \mathscr{K}$, is continuous.*

3. Scott Topology and Continuous Domains

For a poset P, a subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D with $\vee D$ existing, $\vee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, called the *Scott* *topology* on P and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P. For the chain $2 = \{0, 1\}$ (with the order $0 < 1$), we have $\sigma(2) = \{\emptyset, \{1\}, \{0, 1\}\}$. The space $\Sigma 2$ is well-known under the name of *Sierpiński space*. The *upper topology* on P, generated by the complements of the principal ideals of P, is denoted by $\upsilon(P)$. The upper sets of P form the (*upper*) *Alexandroff topology* $\alpha(P)$.

Lemma 12. *(Gierz et al. 2003, Proposition II-2.1) For posets P, Q, and $f : P \longrightarrow Q$, the following two conditions are equivalent:*

(1) $f : \Sigma P \longrightarrow \Sigma Q$ *is continuous.*
(2) *For each $D \in \mathcal{D}(P)$ for which $\vee D$ exists in P, $\vee f(D)$ exists in Q and $f(\vee D) = \vee f(D)$.*

A mapping $f : P \longrightarrow Q$ satisfying the equivalent conditions (1) and (2) of Lemma 12 is said to be *Scott continuous*. Let **DCPO** denote the category of all dcpos and Scott continuous mappings.

For a dcpo P and $A, B \subseteq P$, we say A is *way below* B, written $A \ll B$, if for each $D \in \mathcal{D}(P)$, $\vee D \in \uparrow B$ implies $D \cap \uparrow A \neq \emptyset$. For $B = \{x\}$, a singleton, $A \ll B$ is written $A \ll x$ for short. For $x \in P$, let $w(x) = \{F \in P^{(<\omega)} : F \ll x\}$, $\Downarrow x = \{u \in P : u \ll x\}$ and $K(P) = \{k \in P : k \ll k\}$. Points in $K(P)$ are called *compact elements* of P.

For the following definition and related conceptions, please refer to Gierz et al. (2003).

Definition 13. *Let P be a dcpo and X a T_0 space.*

(1) *P is called a* continuous domain, *if for each $x \in P$, $\Downarrow x$ is directed and $x = \vee \Downarrow x$. When a complete lattice L is continuous, we call L a* continuous lattice.
(2) *P is called an* algebraic domain, *if for each $x \in P$, $\{k \in K(P) : k \leq x\}$ is directed and $x = \vee\{k \in K(P) : k \leq x\}$. When a complete lattice L is algebraic, we call L an* algebraic lattice.
(3) *P is called a* quasicontinuous domain, *if for each $x \in P$, $\{\uparrow F : F \in w(x)\}$ is filtered and $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$.*
(4) *X is called* core-compact *if $\mathcal{O}(X)$ is a* continuous lattice.

A topological space X is said to be a *Noetherian space* if every open subset is compact (see Goubault 2013, Definition 9.7.1) or, equivalently, if $U \ll U$ in $\mathcal{O}(X)$ for any open subset U of X. Clearly, if X is a Noetherian space, then $\mathcal{O}(X)$ is an algebraic lattice.

Remark 14. It is well-known that if a topological space X is locally compact, then it is core-compact (see, e.g., Gierz et al. 2003, Examples I-1.7). In Hofmann et al. (1978, Section 7) (see also Gierz et al. 2003, Exercise V-5.25), Hofmann and Lawson gave a second-countable core-compact T_0 space X in which every compact subset of X has empty interior, and hence, it is not locally compact.

The following result is well-known (see Gierz et al. 2003).

Theorem 15. *Let P be a dcpo.*

(1) *If P is algebraic, then it is continuous.*
(2) *If P is continuous, then it is quasicontinuous.*
(3) *P is continuous iff ΣP is a c-space.*
(4) *P is quasicontinuous iff ΣP is locally hypercompact.*

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4. d-spaces, Well-Filtered Spaces, and Sober Spaces

A T_0 space X is called a *d-space* (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. Gierz et al. 2003; Wyler 1981).

It is easy to verify the following result (cf. Gierz et al. 2003; Xu et al. 2020*b*).

Proposition 16. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a d-space.*
(2) *$\mathcal{D}_c(X) = \mathcal{S}_c(X)$.*
(3) *X is a dcpo, and $\overline{D} = \overline{\{\vee D\}}$ for any $D \in \mathcal{D}(X)$.*
(4) *For any $D \in \mathcal{D}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \subseteq U$ implies $\uparrow d \subseteq U$ (i.e., $d \in U$) for some $d \in D$.*

Lemma 17. *(Xu et al. 2021d, Lemma 2.1) Let X be a d-space. Then for any nonempty closed subset A of X, $A = \downarrow \max(A)$, and hence, $\max(A) \neq \emptyset$.*

A topological space X is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$. Hausdorff spaces are always sober (see, e.g., Goubault 2013, Proposition 8.2.12), and sober spaces are always T_0 since $\overline{\{x\}} = \overline{\{y\}}$ always implies $x = y$. The Sierpinski space $\Sigma 2$ is sober but not T_1, and an infinite set with the co-finite topology is T_1 but not sober (see Example 57).

The following conclusion is well-known (see, e.g., Gierz et al. 2003, 1983; Heckmann 1992).

Proposition 18. *For a quasicontinuous domain P, ΣP is sober.*

For the sobriety of the Smyth power spaces, we have the following well-known result.

Theorem 19. *(Heckmann-Keimel-Schalk Theorem) (Heckmann and Keimel 2013, Theorem 3.13) (Schalk 1993, Lemma 7.20) For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *For any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*
(3) *$P_S(X)$ is sober.*

A T_0 space X is called *well-filtered* if for any filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ and open set U, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. The full subcategory of $\mathbf{Top}_0$ of well-filtered spaces is denoted by $\mathbf{Top}_w$.

Remark 20. The following implications are well-known (which are irreversible) (cf. Gierz et al. 2003):

$$\text{sobriety} \Rightarrow \text{well-filteredness} \Rightarrow d\text{-space}.$$

In Xu et al. (2020*b*), Xu et al. obtained the following equational characterization of well-filtered spaces.

Proposition 21. *(Xu et al. 2020b, Theorem 5.1) Let X be a T_0 space and $\mathbf{K}$ a full subcategory of $\mathbf{Top}_0$ containing $\mathbf{Sob}$. Then, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *For every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap_{K \in \mathcal{K}} \uparrow f(K)$.*

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(3) *For every continuous mapping* $f : X \longrightarrow Y$ *from* X *to a well-filtered space* Y *and a filtered family* $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap_{K \in \mathcal{K}} \uparrow f(K)$.
(4) *For every continuous mapping* $f : X \longrightarrow Y$ *from* X *to a sober space* Y *and a filtered family* $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap_{K \in \mathcal{K}} \uparrow f(K)$.

In Xi et al. (2017), Hofmann et al. (1978), and Kou (2001), the following two useful results were given.

Proposition 22. *(Xi et al. 2017, Corollary 3.2) If a dcpo* P *endowed with the Lawson topology is compact (in particular,* P *is a complete lattice), then* ΣP *is well-filtered.*

Theorem 23. *(Hofmann et al. 1978, Corollary 4.6) (Kou 2001, Theorem 2.3) For a* T_0 *space* X*, the following conditions are equivalent:*

(1) X *locally compact and sober.*
(2) X *is locally compact and well-filtered.*
(3) X *is core-compact and sober.*

For the well-filteredness of topological spaces, a similar result to Theorem 19 was proved in Xu et al. (2021*b*) (see also Xu et al. 2020*b*).

Theorem 24. *(Xu et al. 2020b, Theorem 5.3) (Xu et al. 2021b, Theorem 4) For a* T_0 *space, the following conditions are equivalent:*

(1) X *is well-filtered.*
(2) $P_S(X)$ *is a d-space.*
(3) $P_S(X)$ *is well-filtered.*

Corollary 25. *For a well-filtered space (especially, a sober space)* X, $\mathsf{K}(X)$ *(with the Smyth order) is a dcpo and the upper Vietoris topology is coarser than the Scott topology on* $\mathsf{K}(X)$.

By Theorem 24 and Corollary 25, we know that for a T_0 space X, if $P_S(X)$ is a d-space (equivalently, X is a well-filtered space), then $\Sigma\mathsf{K}(X)$ is a d-space. Example 68 below shows that $\Sigma\mathsf{K}(X)$ is a sober space does not imply that X is well-filtered (i.e., $P_S(X)$ is a d-space) in general.

The following example shows that there is a T_0 space X such that $\mathsf{K}(X)$ (with the Smyth order) is a dcpo but X is not well-filtered.

Example 26. (Johnstone's dcpo adding a top element) Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with ordering defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \infty$ and $k \leq m$. $\mathbb{J}$ is a well-known dcpo constructed by Johnstone in Johnstone (1981) (see Fig. 1).
The set $\mathbb{J}_{max} = \{(n, \infty) : n \in \mathbb{N}\}$ is the set of all maximal elements of $\mathbb{J}$. Adding top $\top$ to $\mathbb{J}$ yields a dcpo $\mathbb{J}_\top = \mathbb{J} \cup \{\top\}$ ($x \leq \top$ for any $x \in \mathbb{J}$). Then, $\top$ is the largest element of $\mathbb{J}_\top$ and $\{\top\} \in \sigma(\mathbb{J}_\top)$. The following three conclusions about $\Sigma\mathbb{J}$ are known (see, e.g., Lu et al. 2017, Example 3.1 and Miao et al. 2021, Lemma 3.1):

(i) $\mathsf{Irr}_c(\Sigma\mathbb{J}) = \{\overline{\{x\}} = \downarrow_{\mathbb{J}} x : x \in \mathbb{J}\} \cup \{\mathbb{J}\}$.
(ii) $\mathsf{K}(\Sigma\mathbb{J}) = (2^{\mathbb{J}_{max}} \setminus \{\emptyset\}) \bigcup \mathbf{Fin}\mathbb{J}$.
(iii) $\Sigma\mathbb{J}$ is not well-filtered.

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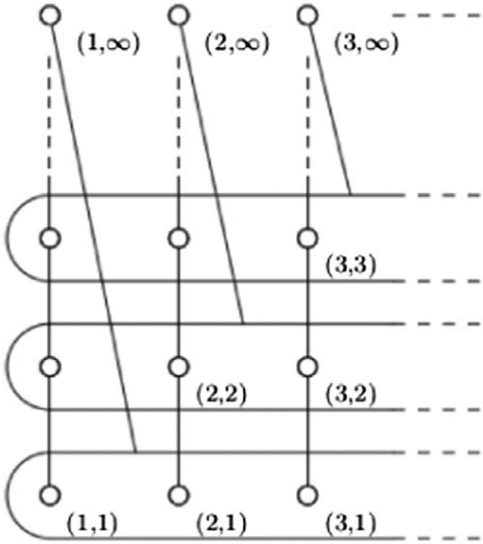

Figure 1. Johnstone's dcpo $\mathbb{J}$.

Hence, we have

(a) $\mathsf{Irr}_c(\Sigma\mathbb{J}_\top) = \{\overline{\{x\}} = \downarrow_{\mathbb{J}_\top} x : x \in \mathbb{J}_\top\} \cup \{\mathbb{J}\}$ by (i).
(b) $\mathsf{K}(\Sigma\mathbb{J}_\top) = \{\uparrow_{\mathbb{J}_\top} G : G \text{ is nonempty and } G \subseteq \mathbb{J}_{max} \cup \{\top\}\} \bigcup \mathbf{Fin}\mathbb{J}_\top$ by (ii).
(c) $\mathsf{K}(\Sigma\mathbb{J})$ is not a dcpo.
Let $\mathcal{G} = \{\mathbb{J}_{max} \setminus F : F \in (\mathbb{J}_{max})^{(<\omega)}\}$. Then by (ii), $\mathcal{G} \subseteq \mathsf{K}(\Sigma\mathbb{J}_\top)$ is a filtered family and $\bigcap \mathcal{G} = \bigcap_{F \in (\mathbb{J}_{max})^{(<\omega)}} (\mathbb{J}_{max} \setminus F) = \mathbb{J}_{max} \setminus \bigcup (\mathbb{J}_{max})^{(<\omega)} = \emptyset$, whence by Lemma 9 $\mathcal{G}$ has no least upper bound in $\mathsf{K}(X)$. Thus $\mathsf{K}(\Sigma\mathbb{J})$ is not a dcpo.
(d) $\mathsf{K}(\Sigma\mathbb{J}_\top)$ is a dcpo.
Suppose that $\{K_d : d \in D\}$ is directed in $\mathsf{K}(\Sigma\mathbb{J}_\top)$ (with the Smyth order). Then, $\top \in \bigcap_{d \in D} K_d$, and hence, $\bigcap_{d \in D} K_d \neq \emptyset$. Now we show that $\bigcap_{d \in D} K_d \in \mathsf{K}(\Sigma\mathbb{J}_\top)$. If $\bigcap_{d \in D} K_d = \{\top\}$, then obviously $\bigcap_{d \in D} K_d \in \mathsf{K}(\Sigma\mathbb{J}_\top)$. Now we assume $\bigcap_{d \in D} K_d \neq \{\top\}$ and $\{V_i : i \in I\} \subseteq \sigma(\mathbb{J}_\top))$ is an open cover of $\bigcap_{d \in D} K_d$. For each $d \in D$ and $i \in I$, let $H_d = K_d \setminus \{\top\}$ and $U_i = V_i \setminus \{\top\}$. Then, $H_d \in \mathsf{K}(\Sigma\mathbb{J})$ $(d \in D)$, $U_i \in \sigma(\mathbb{J})$ $(i \in I)$ and $\emptyset \neq \bigcap_{d \in D} H_d = \bigcap_{d \in D} K_{d_0} \setminus \{\top\} \subseteq \bigcup_{i \in I} V_i \setminus \{\top\} = \bigcup_{i \in I} U_i$. By Lu et al. (2017, Example 3.1), there is $d_0 \in D$ such that $H_{d_0} \in \bigcup_{i \in I} U_i$, and consequently, there is $J \in I^{(<\omega)}$ such that $H_d \subseteq \bigcup_{i \in J} U_i$. It follows that $\bigcap_{d \in D} K_d \subseteq K_{d_0} \subseteq \bigcup_{i \in J} V_i$. Thus , $\bigcap_{d \in D} K_d \in \mathsf{K}(\Sigma\mathbb{J}_\top)$. By Lemma 9, $\mathsf{K}(\Sigma\mathbb{J}_\top)$ is a dcpo.
(e) $\Sigma\mathbb{J}_\top$ is not well-filtered.
Indeed, let $\mathcal{K} = \{\uparrow_{\mathbb{J}_\top}(\mathbb{J}_{max} \setminus F) : F \in (\mathbb{J}_{max})^{(<\omega)}\}$. Then by (b), $\mathcal{K} \subseteq \mathsf{K}(\Sigma\mathbb{J}_\top)$ is a filtered family and $\bigcap \mathcal{K} = \bigcap_{F \in (\mathbb{J}_{max})^{(<\omega)}} \uparrow_{\mathbb{J}_\top}(\mathbb{J}_{max} \setminus F) = \bigcap_{F \in (\mathbb{J}_{max})^{(<\omega)}} ((\mathbb{J}_{max} \setminus F) \cup \{\top\}) = \{\top\} \cup (\mathbb{J}_{max} \setminus \bigcup (\mathbb{J}_{max})^{(<\omega)}) = \{\top\} \in \sigma(\mathbb{J}_\top)$, but there is no $F \in (\mathbb{J}_{max})^{(<\omega)}$ with $\uparrow_{\mathbb{J}_\top}(\mathbb{J}_{max} \setminus F) \subseteq \{\top\}$. Therefore, $\Sigma\mathbb{J}_\top$ is not well-filtered.

5. Topological Rudin Lemma, Rudin Spaces, and Well-Filtered Determined Spaces

In Section 5, we recall some concepts and results about the topological Rudin Lemma, Rudin spaces, ω-Rudin spaces, well-filtered determined spaces, and ω-well-filtered determined spaces that will be used in the next four sections.

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Rudin's Lemma is a useful tool in non-Hausdorff topology and plays a crucial role in domain theory (see Gierz et al. 2003, 1983; Heckmann 1992). Rudin (1980) proved her lemma by transfinite methods, using the Axiom of Choice. Heckmann and Keimel (2013) presented the following topological variant of Rudin's Lemma.

Lemma 27. *(Topological Rudin Lemma) Let X be a topological space and $\mathscr{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then, every closed set $C \subseteq X$ that meets all members of $\mathscr{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathscr{A}$.*

Applying Lemma 27 to the Alexandroff topology on a poset P, one obtains the original Rudin's Lemma.

Corollary 28. *(Rudin's Lemma) Let P be a poset, C a nonempty lower subset of P, and $\mathscr{F} \in \mathbf{Fin}P$ a filtered family with $\mathscr{F} \subseteq \Diamond C$. Then, there exists a directed subset D of C such that $\mathscr{F} \subseteq \Diamond{\downarrow}D$.*

For a T_0 space X and $\mathscr{K} \subseteq \mathsf{K}(X)$, let $M(\mathscr{K}) = \{A \in \mathscr{C}(X) : K \bigcap A \neq \emptyset \text{ for all } K \in \mathscr{K}\}$ (that is, $\mathscr{K} \subseteq \Diamond A$) and $m(\mathscr{K}) = \{A \in \mathscr{C}(X) : A \text{ is a minimal member of } M(\mathscr{K})\}$.

In Shen et al. (2019) and Xu et al. (2020*b*), based on topological Rudin's Lemma, Rudin spaces and well-filtered determined spaces (WD spaces for short) were introduced and investigated. These two spaces are closely related to sober spaces and well-filtered spaces (see Shen et al. 2019; Xu et al. 2020*b*).

Definition 29. *(Shen et al. 2019; Xu et al. 2020b) Let X be a T_0 space.*

(1) *A nonempty subset A of X is said to have the* Rudin property*, if there exists a filtered family $\mathscr{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathscr{K})$ (i.e., $\overline{A}$ is a minimal closed set that intersects all members of $\mathscr{K}$). Let $\mathsf{RD}(X) = \{A \in \mathscr{C}(X) : A$ has Rudin property$\}$. The sets in $\mathsf{RD}(X)$ will also be called* Rudin sets.
(2) *X is called a* Rudin space, RD space *for short, if $\mathsf{Irr}_c(X) = \mathsf{RD}(X)$, that is, all irreducible closed sets of X are Rudin sets.*

The Rudin property is called the *compactly filtered property* in Shen et al. (2019). In order to emphasize its origin from (topological) Rudin's Lemma, such a property was called the Rudin property in Xu et al. (2020*b*). Clearly, A has Rudin property iff $\overline{A}$ has Rudin property (that is, $\overline{A}$ is a Rudin set).

Proposition 30. *(Shen et al. 2019; Xu et al. 2020b) Let X be a T_0 space and Y a well-filtered space. If $f : X \longrightarrow Y$ is continuous and $A \subseteq X$ has Rudin property, then there exists a unique $y_A \in X$ such that $\overline{f(A)} = \overline{\{y_A\}}$.*

Motivated by Proposition 30, the following concept was introduced in Xu et al. (2020*b*).

Definition 31. *(Xu et al. 2020b) Let X be a T_0 space.*

(1) *A subset A of X is called a* well-filtered determined set, WD set *for short, if for any continuous mapping $f : X \longrightarrow Y$ to a well-filtered space Y, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}(X)$ the set of all closed well-filtered determined subsets of X.*
(2) *X is called a* well-filtered determined *space,* WD space *for short, if all irreducible closed subsets of X are well-filtered determined, that is, $\mathsf{Irr}_c(X) = \mathsf{WD}(X)$.*

Obviously, a subset A of a space X is well-filtered determined iff $\overline{A}$ is well-filtered determined.

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Proposition 32. *(Shen et al. 2019; Xu et al. 2020b) Let X be a T_0 space. Then, $\mathscr{S}_c(X) \subseteq \mathscr{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$.*

Definition 33. *(Xu et al. 2020b) A T_0 space X is called a* directed closure space, DC space *for short, if $\mathsf{Irr}_c(X) = \mathscr{D}_c(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a directed subset of X such that $A = \overline{D}$.*

Corollary 34. *(Xu et al. 2020b, Corollary 6.3) Sober $\Rightarrow$ DC $\Rightarrow$ RD $\Rightarrow$ WD.*

Proposition 35. *(Xu et al. 2020b, Corollary 7.11) For a T_0 space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) $\mathsf{RD}(X) = \mathscr{S}_c(X)$.
(3) $\mathsf{WD}(X) = \mathscr{S}_c(X)$.

Theorem 36. *(Xu et al. 2020b, Theorem 6.6) For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *X is a DC d-space.*
(3) *X is a well-filtered DC space.*
(4) *X is a well-filtered Rudin space.*
(5) *X is a well-filtered WD space.*

Proposition 37. *(Erné 2018, Proposition 3.2) Let X be a locally hypercompact T_0 space and $A \in \mathsf{Irr}(X)$. Then, there exists a directed subset $D \subseteq \downarrow A$ such that $\overline{A} = \overline{D}$. Therefore, X is a DC space.*

Proposition 38. *(Xu et al. 2020b, Theorems 6.10 and 6.15) Let X be a T_0 space.*

(1) *If X is locally compact, then X is a Rudin space.*
(2) *If X is core-compact, then X is a WD space.*

It is still not known whether every core-compact T_0 space is a Rudin space (see Xu et al. 2021*d*, Question 5.14).

Question 39. For a core-compact T_0 space X, is the Smyth power space $P_S(X)$ a WD space? Is the Scott space $\Sigma\mathsf{K}(X)$ a WD space?

From Theorem 36 and Proposition 38, one can immediately get the following result, which was first proved by Lawson et al. (2020) using a different method.

Corollary 40. *(Lawson et al. 2020; Xu et al. 2020b) Every core-compact well-filtered space is sober.*

By Corollary 40, Theorem 23 can be strengthened into the following one.

Theorem 41. *For a T_0 space X, the following conditions are equivalent:*

(1) *X locally compact and sober.*
(2) *X is locally compact and well-filtered.*
(3) *X is core-compact and sober.*
(4) *X is core-compact and well-filtered.*

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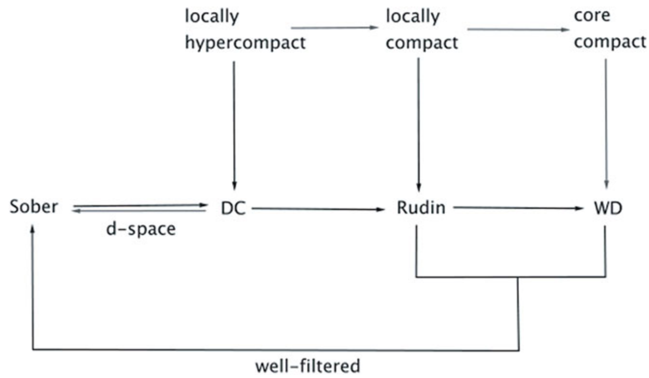

Figure 2. Certain relations among some kinds of spaces.

Fig. 2 shows certain relations among some kinds of spaces.
In order to emphasize the Scott topology, we introduce the following notions.

Definition 42. *A poset P is called a* sober dcpo *(resp., a* well-filtered dcpo*) if ΣP is a sober space (resp., well-filtered space).*

Clearly, a sober dcpo is a well-filtered dcpo. For Isbell's lattice L constructed in Isbell (1982), ΣL is non-sober, namely L is not a sober dcpo, and by Proposition 22, L is well-filtered. The Johnstone's dcpo $\mathbb{J}$ (see Example 26) is not well-filtered.

Definition 43. *Let P be a poset.*

(1) *P is said to be a* DC poset *if ΣP is a* DC *space.*
(2) *P is said to be a* Rudin poset *if ΣP is a Rudin space.*
(3) *P is said to be a* well-filtered determined poset*, a* WD poset *for short, if ΣP is a well-filtered determined space.*
(4) *When a dcpo P is a Rudin poset (resp., a well-filtered determined poset), we will call P a* Rudin dcpo *(resp., a* well-filtered determined dcpo*).*

The following corollary follows directly from Theorem 36.

Corollary 44. *For a poset P, the following conditions are equivalent:*

(1) *P is a sober dcpo.*
(2) *P is a* DC *dcpo.*
(3) *P is a* DC *well-filtered dcpo.*
(4) *P is a Rudin well-filtered dcpo.*
(5) *P is a* WD *well-filtered dcpo.*

In Xu et al. (2020*a*), the following countable versions of Rudin spaces and WD spaces were introduced and studied.

Definition 45. *(Xu et al. 2020a, Definition 5.1) Let X be a T_0 space and A a nonempty subset of X.*

(a) *The set A is said to be an ω-*Rudin set*, if there exists a countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\mathsf{RD}_\omega(X)$ denote the set of all closed ω-Rudin sets of X.*

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(b) *The space X is called* ω-Rudin space, *if* $\mathsf{Irr}_c(X) = \mathsf{RD}_\omega(X)$ *or, equivalently, all irreducible (closed) subsets of X are ω-Rudin sets.*

Definition 46. *(Xu et al. 2020a, Definition 3.9) A T_0 space X is called* ω-well-filtered, *if for any countable filtered family $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$, it holds that*

$$\bigcap_{n<\omega} K_n \subseteq U \Rightarrow \exists n_0 < \omega, K_{n_0} \subseteq U.$$

Definition 47. *(Xu et al. 2020a, Definition 5.4) Let X be a T_0 space and A a nonempty subset of X.*

(a) *The set A is called an* ω-well-filtered determined set, ω-WD set *for short, if for any continuous mapping $f : X \longrightarrow Y$ to an ω-well-filtered space Y, there exists a (unique) $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}_\omega(X)$ the set of all closed ω-well-filtered determined subsets of X.*
(b) *The space X is called* ω-well-filtered determined, ω-WD space *for short, if* $\mathsf{Irr}_c(X) = \mathsf{WD}_\omega(X)$ *or, equivalently, all irreducible (closed) subsets of X are ω-well-filtered determined.*

For a T_0 space X, it was proved in Xu et al. (2020a, Proposition 5.5) that $\mathcal{S}_c(X) \subseteq \mathsf{RD}_\omega(X) \subseteq \mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$. Therefore, every ω-Rudin space is ω-well-filtered determined.

The following result is a countable version of Theorem 36.

Proposition 48. *(Xu et al. 2020a, Theorem 5.11) For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *X is an ω-Rudin and ω-well-filtered space.*
(3) *X is an ω-well-filtered determined and ω-well-filtered space.*

Theorem 49. *(Xu et al. 2021a, Theorems 5.6 and 6.12) Let X be a T_0 space.*

(1) *If the sobrification X^s of X is first-countable, then X is an ω-Rudin space.*
(2) *If X is first-countable, then X is a* WD *space.*

From Theorems 36 and 49, we immediately deduce the following result.

Corollary 50. *(Xu et al. 2020a, Theorem 4.2) Every first-countable well-filtered T_0 space is sober.*

It is still not known whether a first-countable T_0 space is a Rudin space (see Xu et al. 2021a, Problem 6.15). Since the first-countability is a hereditary property, from Remark 4 and Theorem 49 we know that if the Smyth power space $P_S(X)$ of a T_0 space X is first-countable, then X is a WD space.

So naturally, we ask the following question.

Question 51. Is a T_0 space with a first-countable Smyth power space a Rudin space?

In Example 67, a T_0 space X is given for which the Scott space $\Sigma\mathsf{K}(X)$ is a first-countable sober c-space but X is not a WD space (and hence not a Rudin space).

By Proposition 48 and Theorem 49, we have the following result.

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Corollary 52. *(Xu et al. 2021a, Theorem 5.9) Every ω-well-filtered space with a first-countable sobrification is sober.*

In Theorem 49 and Corollary 52, the first-countability of X^s cannot be weakened to that of X as shown in the following example. It also shows that the first-countability of a T_0 space X does not imply the first-countability of X^s in general.

Example 53. Let ω_1 be the first uncountable ordinal number and $P = [0, \omega_1)$. Then,

(a) $\mathscr{C}(\Sigma P) = \{\downarrow t : t \in P\} \cup \{\emptyset, P\}$.
(b) ΣP is first-countable and compact (since P has a least element 0).
(c) $(\Sigma P)^s$ is not first-countable.
In fact, it is easy to verify that $(\Sigma P)^s$ is homeomorphic to $\Sigma[0, \omega_1]$. Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, $\Sigma[0, \omega_1]$ has no countable base at the point ω_1.
(d) $\mathsf{K}(\Sigma P) = \{\uparrow x : x \in P\}$ and ΣP is not an ω-Rudin space.
For $K \in \mathsf{K}(\Sigma P)$, we have $\inf K \in K$, and hence $K = \uparrow \inf K$. So $\mathsf{K}(\Sigma P) = \{\uparrow x : x \in P\}$. Now we show that the irreducible closed set P is not an ω-Rudin set. For any countable filtered family $\{\uparrow \alpha_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(\Sigma P)$, let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then, β is still a countable ordinal number. Clearly, $\downarrow \beta \in M(\{\uparrow \alpha_n : n \in \mathbb{N}\})$ and $P \neq \downarrow \beta$. Therefore, $P \notin m(\{\uparrow \alpha_n : n \in \mathbb{N}\})$. Thus, P is not an ω-Rudin set, and hence, ΣP is not an ω-Rudin space.
(e) ΣP is a Rudin space.
It is easy to check that $\mathsf{Irr}_c(\Sigma P) = \{\downarrow x : x \in P\} \cup \{P\}$. Clearly, $\downarrow x$ is a Rudin set for each $x \in P$. Now we show that P is a Rudin set. First, $\{\uparrow s : s \in P\}$ is filtered. Second, $P \in M(\{\uparrow s : s \in P\})$. For a closed subset B of ΣP, if $B \neq P$, then $B = \downarrow t$ for some $t \in P$, and hence, $\uparrow(t+1) \cap \downarrow t = \emptyset$. Thus, $B \notin M(\{\uparrow s : s \in P\})$, proving that P is a Rudin set.
(f) P is not a dcpo (note that P is directed and $\vee P$ does not exist). So ΣP is not a d-space, and hence, ΣP is neither well-filtered nor sober (see Remark 20).
(g) ΣP is ω-well-filtered.
If $\{\uparrow x_n : n \in \mathbb{N}\} \subseteq \mathsf{K}(\Sigma P)$ is countable filtered family and $U \in \sigma(P)$ with $\bigcap_{n\in\mathbb{N}} \uparrow x_n \subseteq U$, then $\{x_n : i \in \mathbb{N}\}$ is a countable subset of $P = [0, \omega_1)$. Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, we have $\beta = \sup\{x_n : n \in \mathbb{N}\} \in P$, and hence, $\uparrow \beta = \bigcap_{n\in\mathbb{N}} \uparrow x_n \subseteq U$. Therefore, $\beta \in U$, and consequently, $x_n \in U$ for some $n \in \mathbb{N}$ or, equivalently, $\uparrow x_n \subseteq U$, proving that ΣP is ω-well-filtered.

6. Well-Filteredness of Scott Topology on Smyth Power Posets

In this section, we mainly discuss the following two questions:

Question 1. Is the Scott space $\Sigma\mathsf{K}(X)$ of a d-space X a d-space?

Question 2. Is the Scott space $\Sigma\mathsf{K}(X)$ of a well-filtered space X well-filtered?

First, Example 57 below shows that there is a second-countable Noetherian d-space X for which $\mathsf{K}(X)$ is not a dcpo, and hence, neither the Smyth power space $P_S(X)$ nor the Scott space $\Sigma\mathsf{K}(X)$ is a d-space, which gives a negative answer to Question 1.

In order to present the example, we need the following lemma.

Lemma 54. *(Schalk 1993, Lemma 7.26) For a locally compact T_0 space X, the Scott topology is coarser than the upper Vietoris topology on $\mathsf{K}(X)$, that is, $\sigma(\mathsf{K}(X)) \subseteq \mathscr{O}(P_S(X))$.*

Proof. It was proved by Schalk (1993, the proof of Lemma 7.26). We present a more direct proof here.

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Suppose that $\mathcal{U} \in \sigma(\mathsf{K}(X))$ and $K \in \mathcal{U}$. Let $\mathcal{K} = \{G \in \mathsf{K}(X) : K \subseteq \mathrm{int}G\}$. Now we show that $\mathcal{K}$ is filtered and $G = \bigcap \mathcal{K}$.

1° For each $U \in \mathcal{O}(X)$ with $K \subseteq U$, there is $G_U \in \mathcal{K}$ with $G_U \subseteq U$.

If $U \in \mathcal{O}(X)$ for which $K \subseteq U$, then for each $x \in K$, there is $K_x \in \mathsf{K}(X)$ such that $x \in \mathrm{int}\,K_x \subseteq K_x \subseteq U$ since X is locally compact. By the compactness of K, there is $\{x_1, x_2, ..., x_n\} \subseteq K$ such that $K \subseteq \bigcup_{i=1}^{n} \mathrm{int}K_{x_i}$. Let $G_U = \bigcup_{i=1}^{n} K_{x_i}$. Then, $K \subseteq \mathrm{int}G_U \subseteq G_U \subseteq U$, whence $G_U \in \mathcal{K}$ and $G_U \subseteq U$.

2° $\mathcal{K}$ is filtered.

Suppose that $G_1, G_2 \in \mathcal{K}$. Then, $K \subseteq \mathrm{int}G_1 \cap \mathrm{int}G_2$. Hence, by what was shown above, there is $G_3 \in \mathcal{K}$ with $G_3 \subseteq \mathrm{int}G_1 \cap \mathrm{int}G_2 \subseteq G_1 \cap G_2$, proving the filteredness of $\mathcal{K}$.

By 1° and 2°, $K \subseteq \bigcap \mathcal{K} \subseteq \{U \in \mathcal{O}(X) : K \subseteq U\} = K$, whence $K = \bigcap \mathcal{K} = \bigvee_{\mathsf{K}(X)} \mathcal{K}$ by Lemma 9. Since $K \in \mathcal{U} \in \sigma(\mathsf{K}(X))$, $G \in \mathcal{U}$ for some $G \in \mathcal{K}$. Hence, $K \in \Box_{\mathsf{K}(X)}\mathrm{int}\,G \subseteq \mathcal{U}$. Thus, $\mathcal{U} \in \mathcal{O}(P_S(X))$. □

By Corollary 25 and Lemma 54, we get the following corollary.

Corollary 55. *(Schalk 1993, Lemma 7.26) If X is a locally compact sober space (equivalently, a locally compact well-filtered space or a core-compact well-filtered space), then the upper Vietoris topology and the Scott topology on $\mathsf{K}(X)$ coincide.*

Considering Remark 14 and Lemma 54, we have the following question.

Question 56. For a core-compact T_0 space X, is the Scott topology coarser than the upper Vietoris topology on $\mathsf{K}(X)$?

Example 57. Let X be a countably infinite set (for example, $X = \mathbb{N}$) and X_{cof} the space equipped with the *co-finite topology* (the empty set and the complements of finite subsets of X are open). Then

(a) $\mathcal{C}(X_{cof}) = \{\emptyset, X\} \cup X^{(<\omega)}$, X_{cof} is T_1 and hence a d-space.

(b) $\mathsf{Irr}_c(X_{cof}) = \{\{x\} : x \in X\} \cup \{X\}$.

(c) $\mathsf{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$.

(d) X_{cof} is second-countable.
Clearly, $\mathcal{O}(X_{cof})$ is countable, and hence, X_{cof} is second-countable.

(e) X_{cof} is Noetherian and hence locally compact.
Since every subset of X is compact in X_{cof}, the space X_{cof} is a Noetherian space and hence a locally compact space.

(f) X_{cof} is a Rudin space.
By (e) and Proposition 38 (or by (d) and Corollary 108 below), X_{cof} is a Rudin space.

(g) $\mathsf{K}(X_{cof})$ is not a dcpo, and hence, X_{cof} is neither well-filtered nor sober.
$\mathcal{K} = \{X \setminus F : F \in X^{(<\omega)}\} \subseteq \mathsf{K}(X_{cof})$ is countable filtered and $\bigcap \mathcal{K}_X = X \setminus \bigcup X^{(<\omega)} = X \setminus X = \emptyset$, whence $\bigvee \mathcal{K}$ does not exist in $\mathsf{K}(X_{cof})$ by Lemma 9. Thus, $\mathsf{K}(X_{cof})$ is not a dcpo, whence by Remark 20 and Theorem 24, X_{cof} is neither well-filtered nor sober.

(h) The upper Vietoris topology and the Scott topology on $\mathsf{K}(X_{cof})$ agree.
By the local compactness of X_{cof} and Lemma 54, we have $\sigma(\mathsf{K}(X_{cof}) \subseteq \mathcal{O}(P_S(X_{cof}))$. Now we show that $\Box U \in \sigma(\mathsf{K}(X_{cof})$ for each $U \in \mathcal{O}(X_{cof}) \setminus \{\emptyset\}$. Clearly, $\Box U = \uparrow_{\mathsf{K}(X_{cof})}\Box U$. Suppose that $\mathcal{K}_D = \{K_d : d \in D\} \in \mathcal{D}(\mathsf{K}(X_{cof})$ and $\bigvee_{\mathsf{K}(X_{cof})} \mathcal{K}_D \in \Box U$. Then by Lemma 9 $\bigcap_{d\in D} K_d = \bigvee_{\mathsf{K}(X_{cof})} \mathcal{K}_D \subseteq U$ or, equivalently, $X \setminus U \subseteq \bigcup_{d\in D} (X \setminus K_d)$. Since $X \setminus U$ is finite and $\{X \setminus K_d : d \in D\}$ is directed, there is $d_0 \in D$ with $X \setminus U \subseteq X \setminus K_{d_0}$, whence $K_{d_0} \subseteq U$. Thus, $\Box U \in \sigma(\mathsf{K}(X_{cof})$. Therefore, $\mathcal{O}(P_S(X_{cof})) \subseteq \sigma(\mathsf{K}(X_{cof})$, and hence, $\sigma(\mathsf{K}(X_{cof}) = \mathcal{O}(P_S(X_{cof}))$.

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(i) $\Sigma\mathsf{K}(X_{cof})$ is not a d-space, and hence, it is neither a well-filtered space nor a sober space.
Since $\mathsf{K}(X_{cof})$ is not a dcpo, $\Sigma\mathsf{K}(X_{cof})$ is not a d-space. By Remark 20, $\Sigma\mathsf{K}(X_{cof})$ is neither a well-filtered space nor a sober space.

Now we investigate Question 2. First, as one of the main results of this paper, we have the following conclusion.

Theorem 58. *For a well-filtered space X, $\Sigma\mathsf{K}(X)$ is well-filtered.*

Proof. By Corollary 25, $\mathsf{K}(X)$ is a dcpo and $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$ (i.e., $\Box U \in \sigma(\mathsf{K}(X))$ for all $U \in O(X)$). Suppose that $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(\Sigma\mathsf{K}(X))$ is filtered, $\mathcal{U} \in \sigma(\mathsf{K}(X))$ and $\bigcap\limits_{d\in D} \mathcal{K}_d \subseteq \mathcal{U}$. If $\mathcal{K}_d \not\subseteq \mathcal{U}$ for each $d \in D$, that is, $\mathcal{K}_d \bigcap (\mathsf{K}(X) \setminus \mathcal{U}) \neq \emptyset$, then $\{\mathcal{K}_d : d \in D\} \in \mathsf{Irr}(P_S(\mathsf{K}(\Sigma\mathsf{K}(X))))$, and hence by Lemma 27, $\mathsf{K}(X) \setminus \mathcal{U}$ contains a minimal irreducible closed subset $\mathcal{A}$ that still meets all members $\mathcal{K}_d$. For each $d \in D$, let $K_d = \bigcup \uparrow_{\mathsf{K}(X)}(\mathcal{K}_d \cap \mathcal{A})$.

Claim 1: For each $d \in D$, $K_d \in \mathsf{K}(X)$ and $K_d \in \mathcal{A}$.

By $\mathcal{K}_d \in \mathsf{K}(\Sigma\mathsf{K}(X))$ and $\mathcal{A} \in \mathcal{C}(\Sigma\mathsf{K}(X))$, we have that $\uparrow_{\mathsf{K}(X)}(\mathcal{K}_d \cap \mathcal{A}) \in \mathsf{K}(\Sigma\mathsf{K}(X))$, and hence, $\uparrow_{\mathsf{K}(X)}(\mathcal{K}_d \cap \mathcal{A}) \in \mathsf{K}(P_s(X))$ by $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$. By Lemma 11, $K_d = \bigcup \uparrow_{\mathsf{K}(X)}(\mathcal{K}_d \cap \mathcal{A}) = \bigcup(\mathcal{K}_d \cap \mathcal{A}) \in \mathsf{K}(X)$. Since $\mathcal{A} = \downarrow_{\mathsf{K}(X)}\mathcal{A}$ and $\mathcal{K}_d \cap \mathcal{A} \neq \emptyset$, we have $K_d \in \mathcal{A}$.

Claim 2: $\{K_d : d \in D\} \subseteq \mathsf{K}(X)$ is filtered (by Claim 1 and the filteredness of $\{\mathcal{K}_d : d \in D\}$).

Claim 3: $K = \bigcap_{d\in D} K_d \in \mathsf{K}(X)$ and $K \in \mathcal{A}$.

By the well-filteredness of X, $K = \bigcap_{d\in D} K_d \in \mathsf{K}(X)$. By Claims 1, 2 and Lemma 9, $K = \bigvee_{\mathsf{K}(X)}\{K_d : d \in D\} \in \mathcal{A}$ since $\mathcal{A} \in \mathcal{C}(\Sigma\mathsf{K}(X))$.

Claim 4: For each $k \in K$, $\mathcal{A} \subseteq \Diamond_{\mathsf{K}(X)}\overline{\{k\}}$.

For each $d \in D$, we have $k \in K \subseteq K_d = \bigcup(\mathcal{K}_d \cap \mathcal{A})$, whence there is $G_d \in \mathcal{K}_d \cap \mathcal{A}$ such that $k \in G_d$, and consequently, $G_d \in \mathcal{K}_d \cap \mathcal{A} \cap \Diamond_{\mathsf{K}(K)}\overline{\{k\}}$. Therefore, $\mathcal{A} \cap \Diamond_{\mathsf{K}(K)}\overline{\{k\}} \in M(\{\mathcal{K}_d : d \in D\})$. By the minimality of $\mathcal{A}$ and $\Diamond_{\mathsf{K}(K)}\overline{\{k\}} \in \mathcal{C}(P_S(X)) \subseteq \mathcal{C}(\Sigma\mathsf{K}(X))$, we have $\mathcal{A} = \Diamond_{\mathsf{K}(K)}\overline{\{k\}} \bigcap \mathcal{A}$, that is, $\mathcal{A} \subseteq \Diamond_{\mathsf{K}(K)}\overline{\{k\}}$.

Claim 5: $\mathcal{A} = \downarrow_{\mathsf{K}(X)}K$.

By Claims 3 and 4, $\downarrow_{\mathsf{K}(X)}K \subseteq \mathcal{A} \subseteq \bigcap_{k\in K} \Diamond_{\mathsf{K}(K)}\overline{\{k\}}$. Clearly,

$$\begin{aligned} G \in \textstyle\bigcap_{k\in K} \Diamond_{\mathsf{K}(X)}\overline{\{k\}} &\Leftrightarrow \forall k \in K, G \in \Diamond_{\mathsf{K}(K)}\overline{\{k\}} \\ &\Leftrightarrow \forall k \in K, G \cap \overline{\{k\}} \neq \emptyset \\ &\Leftrightarrow \forall k \in K, k \in G \\ &\Leftrightarrow K \subseteq G. \end{aligned}$$

This implies that $\bigcap_{k\in K} \Diamond_{\mathsf{K}(K)}\overline{\{k\}} = \downarrow_{\mathsf{K}(X)}K$, and hence, $\mathcal{A} = \downarrow_{\mathsf{K}(X)}K$.

Claim 6: $K \in \bigcap_{d\in D} \mathcal{K}_d$.

For each $d \in D$, by $\mathcal{K}_d \bigcap \mathcal{A} \neq \emptyset$, $\mathcal{K}_d = \uparrow_{\mathsf{K}(X)}\mathcal{K}_d$ and $\mathcal{A} = \downarrow_{\mathsf{K}(X)}K$, we have $K \in \mathcal{K}_d$, whence $K \in \bigcap_{d\in D} \mathcal{K}_d \subseteq \mathcal{U}$, being a contradiction with $K \in \mathcal{A} \subseteq \mathsf{K}(X) \setminus \mathcal{U}$.

Therefore, there is $d_0 \in D$ such that $\mathcal{K}_{d_0} \subseteq \mathcal{U}$, proving that $\Sigma\mathsf{K}(X)$ is well-filtered. □

Definition 59. *A poset P is called a* well-filtered dcpo *if its Scott space ΣP is well-filtered. Let* $\mathbf{DCPO}_w$ *denote the full subcategory of* **DCPO** *of well-filtered dcpos.*

Proposition 60. *For any well-filtered space X, let $\Phi(X) = \mathsf{K}(X)$. Then $\Phi : \mathbf{Top}_w \longrightarrow \mathbf{DCPO}_w$ is a covariant functor, where for any $f : X \longrightarrow Y$ in $\mathbf{Top}_w$, $\Phi(f) : \Phi(X) \longrightarrow \Phi(Y)$ is defined by $\Phi(f)(K) = \uparrow f(K)$ for all $K \in \Phi(X)$.*

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Proof. Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{Top}_w$.

Claim 1: $\Phi(f) : \Phi(X) \longrightarrow \Phi(Y)$ is well-defined.

By Theorem 58, $\Phi(X)$ and $\Phi(Y)$ are well-filtered dcpos. For any $K \in \Phi(X)$, since $f : X \longrightarrow Y$ is continuous, $\Phi(f)(K) = \uparrow f(K) \in \Phi(Y)$. Thus, $\Phi(f) : \Phi(X) \longrightarrow \Phi(Y)$ is well-defined.

Claim 2: $\Phi(f) : \Phi(X) \longrightarrow \Phi(Y)$ is Scott continuous.

Let $\{K_d : d \in D\} \in \mathcal{D}(\Phi(X))$ (note that $\Phi(X)$ is endowed with the Smyth order). Then by Lemma 9 and Proposition 21, we have that $\bigvee_{\Phi(X)}\{K_d : d \in D\} = \bigcap_{d\in D} K_d$ and

$$\begin{aligned}\Phi(f)(\textstyle\bigvee_{\Phi(X)}\{K_d : d \in D\}) &= \Phi(f)(\textstyle\bigcap_{d\in D} K_d)\\ &= \uparrow f(\textstyle\bigcap_{d\in D} K_d)\\ &= \uparrow \textstyle\bigcap_{d\in D} \uparrow f(K_d)\\ &= \textstyle\bigvee_{\Phi(Y)}\{\Phi(f)(K_d) : d \in D\}.\end{aligned}$$

It follows that $\Phi(f) : \Phi(X) \longrightarrow \Phi(Y)$ is Scott continuous by Lemma 12.

Claim 3: $\Phi(id_X) = id_{\Phi(X)}$

For each $K \in \Phi(X)$, $P_S(id_X)(K) = \uparrow id_X(K) = \uparrow K = K$.

Claim 4: For any morphism $g : Y \longrightarrow Z$ in $\mathbf{Top}_w$, $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$.

For any $K \in \Phi(X)$, $\Phi(g \circ f)(K) = \uparrow g \circ f(K) = \uparrow g(f(K)) = \uparrow g(\uparrow f(K)) = \Phi(g) \circ \Phi(f)(K)$. So $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$.

Thus, $\Phi : \mathbf{Top}_w \longrightarrow \mathbf{DCPO}_w$ is a covariant functor. □

Corollary 61. *For any well-filtered dcpo P, let $\Phi_S(P) = \mathsf{K}(\Sigma P)$. Then, $\Phi_S : \mathbf{DCPO}_w \longrightarrow \mathbf{DCPO}_w$ is a covariant functor, where for any $f : P \longrightarrow Q$ in $\mathbf{DCPO}_w$, $\Phi_S(f) : \Phi_S(P) \longrightarrow \Phi_S(Q)$ is defined by $\Phi_S(f)(K) = \uparrow f(K)$ for all $K \in \Phi_S(P)$.*

Example 68 shows that unlike Smyth power spaces (see Theorem 24), the converse of Theorem 58 does not hold.

From Theorems 41 and 58, we deduce the following result.

Corollary 62. *For a well-filtered space X, the following two conditions are equivalent:*

(1) $\Sigma\mathsf{K}(X)$ *is core-compact.*
(2) $\Sigma\mathsf{K}(X)$ *is locally compact.*

By Theorems 36 and 58, we have the following corollary.

Corollary 63. *For a well-filtered space X, the following three conditions are equivalent:*

(1) $\Sigma\mathsf{K}(X)$ *is sober.*
(2) $\Sigma\mathsf{K}(X)$ *is Rudin.*
(3) $\Sigma\mathsf{K}(X)$ *is well-filtered determined.*

Proposition 64. *Let X be a well-filtered space.*

(1) *If $\mathcal{K} \in \mathsf{K}(\Sigma\mathsf{K}(X))$, then $\bigcup \mathcal{K} \in \mathsf{K}(X)$.*
(2) *The mapping $\bigcup : \Sigma\mathsf{K}(\Sigma\mathsf{K}(X)) \longrightarrow \Sigma\mathsf{K}(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.*

Proof. (1): By Corollary 25, $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$. For $\mathcal{K} \in \mathsf{K}(\Sigma\mathsf{K}(X))$, we have $\mathcal{K} \in \mathsf{K}(P_S(X))$ since $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$. Then by Lemma 11, $\bigcup \mathcal{K} \in \mathsf{K}(X)$.

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(2): Suppose that $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(\Sigma\ \mathsf{K}(X))$ is directed (with the Smyth order) for which $\bigvee_{\mathsf{K}(\Sigma\mathsf{K}(X))}\{\mathcal{K}_d : d \in D\}$ exists. Then by Lemma 9, $\bigvee_{\mathsf{K}(\Sigma\mathsf{K}(X))}\{\mathcal{K}_d : d \in D\} = \bigcap_{d\in D}\mathcal{K}_d$. It follows that $\bigcup\bigvee_{\mathsf{K}(\Sigma\mathsf{K}(X))}\{\mathcal{K}_d : d \in D\} = \bigcup\bigcap_{d\in D}\mathcal{K}_d$ and $\bigvee_{d\in D}\bigcup\mathcal{K}_d = \bigcap_{d\in D}\bigcup\mathcal{K}_d = \bigcup_{\varphi\in\prod_{d\in D}\mathcal{K}_d}\bigcap_{d\in D}\varphi(d)$ by Lemma 9. For each $K \in \bigcap_{d\in D}\mathcal{K}_d$, define $\varphi_K \in \prod_{d\in D}\mathcal{K}_d$ by $\varphi_K(d) \equiv K$ for all $d \in D$. Then, $\bigcap_{d\in D}\varphi_K(d) = K$. Hence $\bigcup\bigcap_{d\in D}\mathcal{K}_d \subseteq \bigcup_{\varphi\in\prod_{d\in D}\mathcal{K}_d}\bigcap_{d\in D}\varphi(d)$.

Conversely, for each $\varphi \in \prod_{d\in D}\mathcal{K}_d$, $x \in \bigcap_{d\in D}\varphi(d)$, and $d' \in D$, we have that $\uparrow x \sqsupseteq \varphi(d') \in \mathcal{K}_{d'} = \uparrow_{\mathsf{K}(\Sigma\mathsf{K}(X))}\mathcal{K}'_d$, and consequently, $\uparrow x \in \bigcap_{d'\in D}\mathcal{K}'_d$ and hence, $\uparrow x \subseteq \bigcup\bigcap_{d\in D}\mathcal{K}_d$. It follows that $\bigcap_{d\in D}\varphi(d) \subseteq \bigcup\bigcap_{d\in D}\mathcal{K}_d$. Therefore, $\bigcup_{\varphi\in\prod_{d\in D}\mathcal{K}_d}\bigcap_{d\in D}\varphi(d) \subseteq \bigcup\bigcap_{d\in D}\mathcal{K}_d$.

Thus, $\bigcup\bigvee_{\mathsf{K}(\Sigma\mathsf{K}(X))}\{\mathcal{K}_d : d \in D\} = \bigvee_{d\in D}\bigcup\mathcal{K}_d$. By Lemma 12, $\bigcup : \Sigma\mathsf{K}(\Sigma\mathsf{K}(X)) \longrightarrow \Sigma\mathsf{K}(X)$ is continuous. □

Proposition 65. *Let X be a T_0 space. If the upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(X)$ (i.e., $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$), and $\Sigma\mathsf{K}(X)$ is well-filtered, then X is well-filtered.*

Proof. Suppose that $\{K_d : d \in D\} \subseteq \mathsf{K}(X)$ is filtered and $U \in \mathcal{O}(X)$ with $\bigcap_{d\in D}K_d \subseteq U$. Then, $\{\uparrow_{\mathsf{K}(X)}K_d : d \in D\} \subseteq \mathsf{K}(\Sigma\mathsf{K}(X))$ is filtered, $\Box U \in \mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$ and $\bigcap_{d\in D}\uparrow_{\mathsf{K}(X)}K_d \subseteq \Box U$. By the well-filteredness of $\Sigma\mathsf{K}(X)$, there is $d \in D$ such that $\uparrow_{\mathsf{K}(X)}K_d \subseteq \Box U$, and hence, $K_d \subseteq U$. Thus, X is well-filtered. □

Example 68 below shows that when X lacks the condition of $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$, Proposition 65 may not hold.

Corollary 66. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *The upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(X)$, and $\Sigma\mathsf{K}(X)$ is well-filtered.*
(3) *The upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(X)$, and $\Sigma\mathsf{K}(X)$ is a d-space.*
(4) *$\mathsf{K}(X)$ is a dcpo, and the upper Vietoris topology is coarser than the Scott topology on $\mathsf{K}(X)$.*

Proof. (1) ⇒ (2): By Corollary 25 and Theorem 58.

(2) ⇒ (3): By Remark 20.

(3) ⇒ (4): Trivial.

(4) ⇒ (1): By (4), $P_S(X)$ is a d-space, whence X is well-filtered by Theorem 24. □

7. Non-sobriety of Scott Topology on Smyth Power Poset of a Sober Space

In this section, we investigate the following question:

Question 3. Is the Scott space $\Sigma\mathsf{K}(X)$ of a sober space X sober?

First, the following example shows that there is a well-filtered space X for which the Scott space $\Sigma\mathsf{K}(X)$ is a first-countable sober c-space, but X is not sober although the Scott space $\Sigma\mathsf{K}(X)$ is sober by Corollary 94. Hence, by Corollaries 40 and 50, X is neither core-compact nor first-countable. So the sobriety of the Scott space $\Sigma\mathsf{K}(X)$ of a T_0 space X does not imply the sobriety of X in general.

Example 67. Let X be an uncountably infinite set and X_{coc} the space equipped with *the co-countable topology* (the empty set and the complements of countable subsets of X are open). Then

(a) $\mathcal{C}(X_{coc}) = \{\emptyset, X\}\bigcup X^{(\leq\omega)}$, X_{coc} is T_1 and hence a d-space, and the specialization order on X_{coc} is the discrete order.

(b) Neither X_{coc} nor $P_S(X_{coc})$ is first-countable.
For a point $x \in X$, suppose that there is a countable base $\{X \setminus C_n : n \in \mathbb{N}, C_n \in X^{(\leq\omega)}\}$ at x in X_{coc}. Let $C = \bigcup_{n\in\mathbb{N}} C_n$. Then, $C \in X^{(\leq\omega)}$. Select $t \in X \setminus (C \cup \{x\})$ and let $U = X \setminus \{t\}$. Then, $x \in U \in \mathscr{O}(X_{coc})$. But $X \setminus C_n \not\subseteq U$ for every $n \in \mathbb{N}$, a contradiction. Thus, X_{coc} is not first-countable. Since the first-countability is a hereditary property and X_{coc} is homeomorphic to the subspace $\mathscr{S}^u(X_{coc})$ of $P_S(X_{coc})$ (see Remark 4 or Proposition 102 below), $P_S(X_{coc})$ is not first-countable.

(c) $\mathsf{Irr}_c(X_{coc}) = \{\overline{\{x\}} : x \in X\} \cup \{X\} = \{\{x\} : x \in X\} \cup \{X\}$. Therefore, X_{coc} is not sober.

(d) $\mathsf{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$ and $\mathrm{int}\, K = \emptyset$ for all $K \in \mathsf{K}(X_{coc})$, and hence, X_{coc} is not locally compact.
Clearly, every finite subset is compact. Conversely, if $C \subseteq X$ is infinite, then C has an infinite countable subset $\{c_n : n \in \mathbb{N}\}$. Let $C_0 = \{c_n : n \in \mathbb{N}\}$ and $U_m = (X \setminus C_0) \cup \{c_m\}$ for each $m \in \mathbb{N}$. Then, $\{U_n : n \in \mathbb{N}\}$ is an open cover of C, but has no finite subcover. Hence, C is not compact. Thus, $\mathsf{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$. Clearly, $\mathrm{int}\, K = \emptyset$ for all $K \in \mathsf{K}(X_{coc})$. Hence, X_{coc} is not locally compact.

(e) X_{coc} is well-filtered and not core-compact.
Suppose that $\{F_d : d \in D\} \subseteq \mathsf{K}(X_{coc})$ is a filtered family and $U \in \mathscr{O}(X_{coc})$ with $\bigcap_{d\in D} F_d \subseteq U$. As $\{F_d : d \in D\}$ is filtered and all F_d are finite, $\{F_d : d \in D\}$ has a least element F_{d_0}, and hence, $F_{d_0} = \bigcap_{d\in D} F_d \subseteq U$, proving that X_{coc} is well-filtered. By (d) and Theorem 41, X_{coc} is not core-compact.

(f) $\mathsf{K}(X_{coc})$ is a Noetherian dcpo, and hence, $\Sigma\mathsf{K}(X_{coc}) = (\mathsf{K}(X_{coc}), \alpha(\mathsf{K}(X_{coc}))$ is first-countable.
Clearly, $\mathsf{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$ (with the Smyth order) is a Noetherian dcpo and $\sigma(\mathsf{K}(X_{coc})) = \alpha(\mathsf{K}(X_{coc}))$. For any $F \in \mathsf{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$, $\{\uparrow_{\mathsf{K}(X_{coc})} F\}$ is a base at F in $\Sigma\mathsf{K}(X_{coc})$. Hence $\Sigma\mathsf{K}(X_{coc})$ is first-countable.

(g) The upper Vietoris topology and the Scott topology on $\mathsf{K}(X_{coc})$ do not agree.
By (e) and Corollary 25, $\mathscr{O}(P_S(X_{coc})) \subseteq \sigma(\mathsf{K}(X_{coc}))$. For $F \in X^{(<\omega)} \setminus \{\emptyset\}$, $\uparrow_{\mathsf{K}(X_{coc})} F \in \alpha(\mathsf{K}(X_{coc})) = \sigma(\mathsf{K}(X_{coc}))$ but $\uparrow_{\mathsf{K}(X_{coc})} F \notin \mathscr{O}(P_S(X_{coc}))$ since there is no $G \in X^{(<\omega)}$ with $F \in \Box(X \setminus G) = (X \setminus G)^{(<\omega)} \setminus \{\emptyset\} \subseteq \uparrow_{\mathsf{K}(X_{coc})} F$. Thus, $\sigma(\mathsf{K}(X_{coc})) \not\subseteq \mathscr{O}(P_S(X_{coc}))$.

(h) The Scott space $\Sigma\mathsf{K}(X_{coc})$ is a sober c-space. So it is Rudin and well-filtered determined.
$\mathsf{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$ (with the Smyth order) is a Noetherian dcpo, and hence, it is an algebraic domain. By Theorem 15 and Proposition 18, $\Sigma\mathsf{K}(X)$ is a sober c-space. Hence by Theorem 36, $\Sigma\mathsf{K}(X)$ is Rudin and well-filtered determined.

(i) X_{coc} is neither a Rudin space nor a WD space.
By (c), (e) and Theorem 36, X_{coc} is neither a Rudin space nor a WD space.

(j) The Smyth power space $P_S(X_{coc})$ is well-filtered but non-sober. Hence, it is neither a Rudin space nor a WD space.
By (c), (e), Theorems 19 and 24, $P_S(X_{coc})$ is well-filtered and non-sober. Hence, $P_S(X_{coc})$ is neither a Rudin space nor a WD space by Theorem 36.

(k) $P_S(X_{coc})$ is not core-compact.
By (j) and Corollary 40 (or Theorem 41), $P_S(X_{coc})$ is not core-compact.

The following example shows there is even a second-countable Noetherian T_0 space X such that the Scott space $\Sigma\mathsf{K}(X)$ is a second-countable sober space but X is not well-filtered (and hence not sober).

Example 68. Let $P = \mathbb{N} \cup \{\infty\}$ and define an order on P by $x \leq_P y$ iff $x = y$ or $x \in \mathbb{N}$ and $y = \infty$ (see Fig. 3).
Let $\tau = \{(\mathbb{N} \setminus F) \cup \{\infty\} : F \in \mathbb{N}^{(<\omega)}\} \cup \{\emptyset, P\} \cup \{\{\infty\}\}$. It is straightforward to verify that τ is a T_0 topology on P and the specialization order of (P, τ) agrees with the original order on P. Now we have

(a) $\mathscr{C}((P, \tau)) = \mathbb{N}^{(<\omega)} \cup \{\emptyset, P\} \cup \{\mathbb{N}\}$.

(b) $\mathsf{Irr}_c((P, \tau)) = \{\overline{\{n\}} = \{n\} : n \in \mathbb{N}\} \cup \{\overline{\{\infty\}} = P\} \cup \{\mathbb{N}\}$ and hence (P, τ) is not sober.

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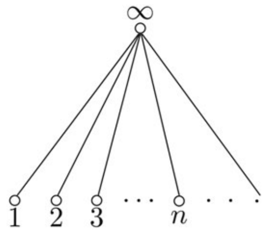

Figure 3. The poset P.

(c) $\mathsf{K}((P,\tau)) = \{A \cup \{\infty\} : A \subseteq \mathbb{N}\}$.

(d) (P,τ) is not well-filtered.
Let $\mathcal{K} = \{(\mathbb{N} \setminus F) \cup \{\infty\} : F \in \mathbb{N}^{(<\omega)}\}$. Then, $\mathcal{K} \subseteq \mathsf{K}((P,\tau))$ is a filtered family and $\bigcap \mathcal{K} = \{\infty\} \in \tau$. But there is no $F \in \mathbb{N}^{(<\omega)}$ with $(\mathbb{N} \setminus F) \cup \{\infty\} = \{\infty\}$. Thus, (P,τ) is not well-filtered. In fact, (P,τ) is not weak well-filtered in the sense of Lu et al. (2017).

(e) (P,τ) is Noetherian and second-countable, and hence, it is a Rudin space.
Since $|\tau| = \omega$, (P,τ) is second-countable. As every subset of P is compact in (P,τ), the space (P,τ) is a Noetherian space (and hence a locally compact space). Hence by Proposition 38, (P,τ) is a Rudin space.

(f) $\Sigma\mathsf{K}((P,\tau))$ is a second-countable sober space.
Clearly, $\mathsf{K}((P,\tau))$ is isomorphic with the algebraic lattice $2^{\mathbb{N}}$ (with the order of set inclusion) via the poset isomorphism $\varphi : \mathsf{K}((P,\tau)) \to 2^{\mathbb{N}}$ defined by $\varphi(A \cup \{\infty\}) = \mathbb{N} \setminus A$ for each $A \in 2^{\mathbb{N}}$ (note that the order on $\mathsf{K}((P,\tau))$ is the Smyth order). Hence $\Sigma\mathsf{K}((P,\tau)) \cong \Sigma 2^{\mathbb{N}}$. Clearly, $2^{\mathbb{N}}$ is an algebraic lattice, whence by Theorem 15 and Proposition 18, $\Sigma 2^{\mathbb{N}}$ is sober and hence $\Sigma\mathsf{K}((P,\tau))$ is sober. Clearly, $\Sigma 2^{\mathbb{N}}$ is second-countable since $\{\uparrow_{2^{\mathbb{N}}} F : F \in (2^{\mathbb{N}})^{(<\omega)}\}$ is a countable base of $\Sigma 2^{\mathbb{N}}$. So $\Sigma\mathsf{K}((P,\tau))$ is second-countable.

(g) $P_S((P,\tau))$ is second-countable.
Clearly, $\{\Box U : U \in \tau\}$ is a countable base of $P_S((P,\tau))$ (note that $|\tau| = \omega$). Hence $P_S((P,\tau))$ is second-countable.

(h) $\sigma(\mathsf{K}((P,\tau)) \subseteq \mathcal{O}(P_S((P,\tau)))$ but $\mathcal{O}(P_S((P,\tau)) \nsubseteq \sigma(\mathsf{K}((P,\tau)))$.
Since (P,τ) is locally compact, $\sigma(\mathsf{K}((P,\tau)) \subseteq \mathcal{O}(P_S((P,\tau)))$ by Lemma 54. Clearly, $\Box\{\infty\} = \{\{\infty\}\} \in \mathcal{O}(P_S((P,\tau))$. Now we show that $\Box\{\infty\} \notin \sigma(\mathsf{K}((P,\tau)))$. By Lemma 9, $\bigvee\{F \cup \{\infty\} : F \in (\mathbb{N})^{(<\omega)} \setminus \{\emptyset\}\} = \bigcap\{F \cup \{\infty\} : F \in (\mathbb{N})^{(<\omega)} \setminus \{\emptyset\}\} = \{\infty\} \in \Box\{\infty\}$, but there is no $F \in (\mathbb{N})^{(<\omega)} \setminus \{\emptyset\}$ with $F \cup \{\infty\} \in \Box\{\infty\} = \{\{\infty\}\}$. Thus, $\Box\{\infty\} \notin \sigma(\mathsf{K}((P,\tau)))$.

In the following, we will construct a sober space X for which the Scott space $\Sigma\mathsf{K}(X)$ is non-sober (see Theorem 91 below).

Let $\mathcal{L} = \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, where $\mathbb{N}$ is the set of natural numbers with the usual order. Define an order $\le$ on $\mathcal{L}$ as follows:

$(i_1, j_1, k_1) \le (i_2, j_2, k_2)$ if and only if:
(1) $i_1 = i_2, j_1 = j_2, k_1 \le k_2 \le \infty$; or
(2) $i_2 = i_1 + 1, k_1 \le j_2, k_2 = \infty$.

$\mathcal{L}$ is a known dcpo constructed by Jia in Jia (2018, Example 2.6.1). It can be easily depicted as in Fig. 4 taken from Jia (2018).

For each $(n,m) \in \mathbb{N} \times \mathbb{N}$, let
$\mathcal{L}_{(n,m)} = \{(n,,l) : l \in \mathbb{N} \cup \{\infty\}$ (the (n,m)*th line* of $\mathcal{L}$),
$\mathcal{L}_n = \bigcup_{j\in\mathbb{N}} \mathcal{L}_{(n,j)}$ (the *nth row* of $\mathcal{L}$) and $\mathcal{L}_0 = \emptyset$,
$\mathcal{L}_n^{\infty} = \{(n,j,\infty) : j \in \mathbb{N}\}$ (the set of all maximal elements of $\mathcal{L}_n$),
$\mathcal{L}_n^{<\infty} = \mathcal{L}_n \setminus \mathcal{L}_n^{\infty}$ (the set of all finite height elements of $\mathcal{L}_n$),
$\mathcal{L}^{\infty} = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n^{\infty} = \{(i,j,\infty) : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ (the set of all maximal elements of $\mathcal{L}$),

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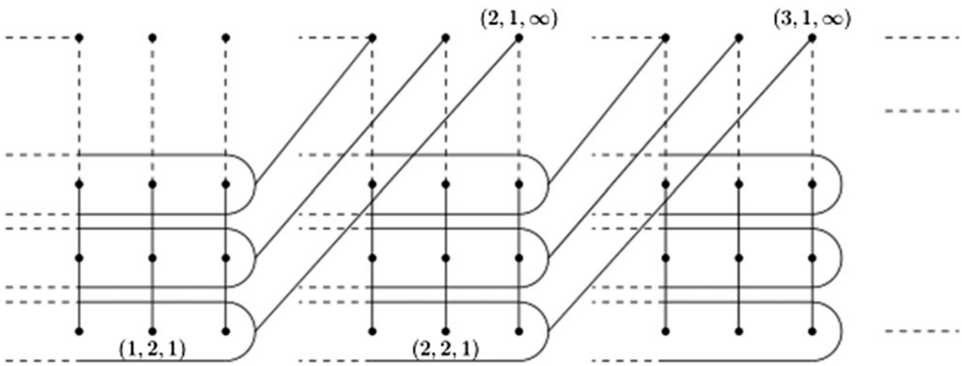

Figure 4. A non-sober well-filtered dcpo $\mathscr{L}$.

$$\mathscr{L}^{<\infty} = \mathscr{L} \setminus \mathscr{L}^{\infty} \text{ (the set of all elements of finite height)},$$
$$\mathscr{L}_{(\leq n)} = \bigcup_{i=1}^{n} \mathscr{L}_i = \{(i,j,l) : i \leq n, j \in \mathbb{N}, l \in \mathbb{N} \cup \{\infty\}\},$$
$$\mathscr{L}_{(\geq n)} = \bigcup_{i \geq n} \mathscr{L}_i = \{(i,j,l) : i \geq n, j \in \mathbb{N}, l \in \mathbb{N} \cup \{\infty\}\},$$
$$\mathscr{L}^{\infty}_{(<n+1)} = \mathscr{L}^{\infty}_{(\leq n)} = \bigcup_{i=1}^{n} \mathscr{L}^{\infty}_i = \{(i,j,\infty) : i \leq n, j \in \mathbb{N}\},$$
$$\mathscr{L}^{\infty}_{(>n)} = \mathscr{L}^{\infty}_{(\geq n+1)} = \bigcup_{i \geq n} \mathscr{L}^{\infty}_i = \{(i,j,\infty) : i > n, j \in \mathbb{N}\}, \text{ and}$$
$$\mathscr{L}^{\infty}_{(n,\geq m)} = \{(n,j,\infty) : j \geq m\}.$$

Definition 69. *For $x = (i,j,k) \in \mathscr{L}$, k is called the* height *of x and is denoted by $h(x)$ (i.e., $h(x) = k$). If $k \in \mathbb{N}$ (resp., $k = \infty$), then we say that x is a* point of finite height *(resp.,* point of infinite height*). For a nonempty subset A of $\mathscr{L}$, if there is $n \in \mathbb{N}$ with $\{h(a) : a \in A\} \subseteq \downarrow n$, then A is said a* subset of finite height*; otherwise ,A is said to be a* subset of infinite height*. The* height *of A is defined by*

$$h(A) = \begin{cases} \max\{h(a) : a \in A\}, & \text{if } A \text{ is finite height} \\ \infty, & \text{otherwise.} \end{cases}$$

For simplicity, let $h(\emptyset) = 0$.

Lemma 70. *Let $i \in \mathbb{N}$ and U be a nonempty Scott open set of $\mathscr{L}$. If $U \cap \mathscr{L}_i \neq \emptyset$, then there is $j' \in \mathbb{N}$ such that $\mathscr{L}^{\infty}_{(i+1,\geq j')} = \{(i+1, j, \infty) : j \geq j'\} \subseteq U$.*

Proof. Since $U = \uparrow U$ and $U \cap \mathscr{L}_i \neq \emptyset$, we can assume that $(i, j(i), \infty) \in U$ for some $j(i) \in \mathbb{N}$. As $\bigvee_{l \in \mathbb{N}} (i, j(i), l) = (i, j(i), \infty) \in U \in \sigma(\mathscr{L})$, there is $l(i) \in \mathbb{N}$ such that $(i, j(i), l(i)) \in U$. Let $j' = l(i)$. Then, $\mathscr{L}^{\infty}_{(i+1,\geq j')} \subseteq \uparrow (i, j(i), l(i)) \subseteq U$. □

Lemma 71. *Suppose that D is an infinite directed subset of $\mathscr{L}$. Then, there is a unique $(i_D, j_D, \infty) \in \mathscr{L}^{\infty}$ such that (i_D, j_D, ∞) is a largest element of D or the following three conditions are satisfied:*

(i) $(i_D, j_D, \infty) \notin D$,
(ii) $D \subseteq \{(i_D, j_D, l) : l \in \mathbb{N}\}$, *and*
(iii) $(i_D, j_D, \infty) = \bigvee_{\mathscr{L}} D$.

Hence, $\mathscr{L}$ is a dcpo.

Proof. If there is $(i_0, j_0, \infty) \in D \cap \mathscr{L}^\infty$, then for each $d = (i_d, j_d, l_d) \in D$, there is $d^* = (i_{d^*}, j_{d^*}, l_{d^*}) \in D$ such that $(i_0, j_0, \infty) \leq d^* = (i_{d^*}, j_{d^*}, l_{d^*})$ and $d = (i_d, j_d, l_d) \leq d^* = (i_{d^*}, j_{d^*}, l_{d^*})$, whence $l_{d^*} = \infty, i_{d^*} = i_0, j_{d^*} = j_0$ (i.e., $d^* = (i_0, j_0, \infty)$) and $d \leq d^* = (i_0, j_0, \infty)$. Hence, $(i_D, j_D, \infty) = (i_0, j_0, \infty)$ is the (unique) largest element of D.

Now suppose that $D \cap \mathscr{L}^\infty = \emptyset$, that is, $D \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Select a $d_1 = (i_{d_1}, j_{d_1}, l_{d_1}) \in D$. Then for each $d = (i_d, j_d, l_d) \in D$, by the directedness of D, there is $d' = (i_{d'}, j_{d'}, l_{d'}) \in D$ such that $d_1 = (i_{d_1}, j_{d_1}, l_{d_1}) \leq d' = (i_{d'}, j_{d'}, l_{d'})$ and $d = (i_d, j_d, l_d) \leq d' = (i_{d'}, j_{d'}, l_{d'})$. Hence, $i_{d'} = i_d = i_{d_0}, j_{d'} = j_d = j_{d_0}$ and $l_{d_0} \leq l_{d'}, l_d \leq l_{d'}$. Let $i_D = i_{d_1}$ and $j_D = j_{d_1}$. Then, $D \subseteq \{(i_D, j_D, l) : l \in \mathbb{N}\}$. Clearly, (i_D, j_D, ∞) is the unique element of $\mathscr{L}^\infty$ satisfying Conditions (i)–(iii).

For any $D \in \mathscr{D}(\mathscr{L})$, if D has a largest element s, then $s = \vee D$; if D has no largest element, then D is infinite directed subset of $\mathscr{L}$, whence there is a unique $(i, j, \infty) \in \mathscr{L}^\infty$ such that Conditions (i)–(iii) are satisfied. Then, $(i, j, \infty) = \bigvee_{\mathscr{L}} D$. Hence, $\mathscr{L}$ is a dcpo. □

Corollary 72. *Suppose that A is a nonempty subset of $\mathscr{L}$ with $A = \downarrow \max(A)$ and D is an infinite directed subset of A having no largest element. Let $(i_D, j_D, \infty) \in \mathscr{L}^\infty$ be the unique maximal element of $\mathscr{L}$ satisfying Conditions (i)–(iii) in Lemma 71. Then, $(i_D, j_D, \infty) \in \max(A)$ or $|\max(A) \cap \mathscr{L}^\infty_{i_D+1}| = \omega$.*

Proof. Since D has no largest element, by Lemma 71, there is a unique $(i_D, j_D, \infty) \in \mathscr{L}^\infty$ such that the following three conditions are satisfied:

(i) $(i_D, j_D, \infty) \notin D$,
(ii) $D \subseteq \{(i_D, j_D, l) : l \in \mathbb{N}\}$, and
(iii) $(i_D, j_D, \infty) = \bigvee_{\mathscr{L}} D$.

Suppose that $(i_D, j_D, \infty) \notin \max(A)$. We will show that $\max(A) \cap \mathscr{L}^\infty_{i_D+1}$ is infinite. For each $d = (i_D, j_D, l_d) \in D$, by $A = \downarrow \max(A)$, there is $(i(d), j(d), l(d)) \in \max(A)$ with $d = (i_D, j_D, l_d) \leq (i(d), j(d), l(d))$. If $i(d) = i_D$, then $j(d) = j_D$ and $l_d \leq l(d)$. Since $D \subseteq A \cap \{(i_D, j_D, l) : l \in \mathbb{N}\}$ is infinite, $\{l_{d'} : d' = (i_D, j_D, l_{d'}) \in D\} \subseteq \mathbb{N}$ is infinite and $(i_D, j_D, l(d)) = (i(d), j(d), l(d)) \in \max(A)$, we have that $l(d) = \infty$, which is in contradiction with $(i_D, j_D, \infty) \notin \max(A)$. Therefore, $i(d) = i_D + 1$ and hence $l(d) = \infty$ and $l_d \leq j(d)$ by $(i_D, j_D, l_d) \leq (i(d), j(d), l(d)) = (i_D + 1, j(d), l(d))$. Since $\{l_d : d = (i_D, j_D, l_d) \in D\} \subseteq \mathbb{N}$ is infinite, $\{(i(d), j(d), l(d)) = (i_D + 1, j(d), \infty) : d \in D\} \subseteq \max(A) \cap \mathscr{L}^\infty$ is infinite (note that $l_d \leq j(d)$ for each $d = (i_D, j_D, l_d) \in D$). Thus, $\max(A) \cap \mathscr{L}^\infty_{i_D+1}$ is infinite. □

Corollary 73. *Suppose that A is a nonempty subset of $\mathscr{L}$ with $A = \downarrow \max(A)$ and D is an infinite directed subset of A for which $\vee D$ does not exist in A. Let $(i_D, j_D, \infty) \in \mathscr{L}^\infty$ be the unique maximal element of $\mathscr{L}$ satisfying Conditions (i)–(iii) in Lemma 71. Then, $(i_D, j_D, \infty) \notin \max(A)$ and $|\max(A) \cap \mathscr{L}^\infty_{i_D+1}| = \omega$.*

Remark 74. Suppose that A is a nonempty subset of $\mathscr{L}$ with $A = \downarrow \max(A)$. Since $\max(A) = (\max(A) \cap \mathscr{L}^\infty) \cup (\max(A) \cap \mathscr{L}^{<\infty})$, we have $A = \downarrow \max(A) = \downarrow (\max(A) \cap \mathscr{L}^\infty) \cup \downarrow (\max(A) \cap \mathscr{L}^{<\infty})$. Clearly, $\max(A) \cap \mathscr{L}^{<\infty} \cap \downarrow (\max(A) \cap \mathscr{L}^\infty) = \emptyset$ and $\max(A) \cap \mathscr{L}^\infty \cap \downarrow (\max(A) \cap \mathscr{L}^{<\infty}) = \emptyset$.

Lemma 75. *Let $A \subseteq \mathscr{L}$ be a nonempty set and $A \neq \mathscr{L}$. Then, A is Scott closed if and only if $A = \downarrow \max(A)$ and one of the following three conditions is satisfied:*

(1) *$A \subseteq \mathscr{L}^{<\infty}$ (or equivalently, $\max(A) \subseteq \mathscr{L}^{<\infty}$).*
(2) *$A \cap \mathscr{L}^\infty \neq \emptyset$ and $|A \cap \mathscr{L}^\infty_n| < \omega$ for each $n \in \mathbb{N}$.*
(3) *$i(A) = \max\{i \in \mathbb{N} : |A \cap L^\infty_i| = \omega\}$ exists and $\mathscr{L}_n \subseteq A$ for each $n \leq i(A) - 1$.*

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Proof. Suppose that A is Scott closed. Then, $A = \downarrow\max(A)$ by Lemma 17. Now we show that A satisfies one of Conditions (1)–(3). If neither Condition (1) nor Condition (2) holds, then there is some $i_0 \in \mathbb{N}$ such that $A \cap L_{i_0}^{\infty}$ is infinite. We first show that A satisfies the following property Q:

(Q) For $n \in \mathbb{N}$, if $A \cap L_n^{\infty}$ is infinite, then $\mathscr{L}_i \subseteq A$ for each $i \leq n-1$.

If $n = 1$, then $\mathscr{L}_{n-1} = \mathscr{L}_0 = \emptyset \subseteq A$. Now we assume $2 \leq n$. For each $(j, l) \in \mathbb{N} \times \mathbb{N}$, since $A \cap L_n^{\infty}$ is infinite (i.e., $\{j' \in \mathbb{N} : (n, j', \infty) \in A\}$ is infinite), there is $j' \in \mathbb{N}$ such that $(n, j', \infty) \in A$ and $l \leq j'$, whence $(n-1, j, l) \leq (n, j', \infty)$. Thus, $\{(n-1, j, l) : (j, l) \in \mathbb{N} \times \mathbb{N}\} \subseteq \downarrow(A \cap L_n^{\infty}) \subseteq \downarrow A = A$. For each $j \in \mathbb{N}$, since $(n-1, j, \infty) = \bigvee_{l \in \mathbb{N}} (n-1, j, l)$ and A is Scott closed, we have $(n-1, j, \infty) \in A$. Hence, $\mathscr{L}_{n-1} \subseteq A$. In particular, $\mathscr{L}_{n-1}^{\infty} \subseteq A$. Then by induction, we get that $\mathscr{L}_i \subseteq A$ for any $1 \leq i \leq n-1$.

By property Q, if $\{i \in \mathbb{N} : |A \cap L_i^{\infty}| = \omega\}$ is infinite, then for each $n \in \mathbb{N}$, $\mathscr{L}_{n-1} \subseteq A$. Hence, $\mathscr{L} = \bigcup_{n \in \mathbb{N}} \mathscr{L}_n \subseteq A$, which contradicts $A \neq \mathscr{L}$. Hence $\{i \in \mathbb{N} : |A \cap L_i^{\infty}| = \omega\}$ is a nonempty finite subset of $\mathbb{N}$, and consequently, $i(A) = \max\{i \in \mathbb{N} : |A \cap L_i^{\infty}| = \omega\}$ exists. By property Q, we have $\mathscr{L}_i \subseteq A$ for each $i \leq i(A) - 1$. This completes the proof of Condition (3).

Conversely, assume that $A = \downarrow\max(A)$ and one of Conditions (1)–(3) is satisfied. We will show that A is Scott closed.

Case 1. Condition (1) or Condition (2) holds.

Suppose that D is a directed subset of A. If D has a largest element s, then $\vee D = s \in A$. Now suppose that D has no largest element. Then, D is infinite, whence by Lemma 71, there is a unique $(i_D, j_D, \infty) \in \mathscr{L}^{\infty}$ such that the following three conditions are satisfied:

(i) $(i_D, j_D, \infty) \notin D$,
(ii) $D \subseteq \{(i_D, j_D, l) : l \in \mathbb{N}\}$, and
(iii) $(i_D, j_D, \infty) = \bigvee_{\mathscr{L}} D$.

If $(i_D, j_D, \infty) \notin A$, then by Corollary 72, $\max(A) \cap \mathscr{L}_{i_D+1}^{\infty}$ is infinite, which is a contradiction with Condition (1) or Condition (2). So $\bigvee_{\mathscr{L}} D = (i_D, j_D, \infty) \in A$. Thus, A is Scott closed.

Case 2. $i(A) = \max\{i \in \mathbb{N} : |A \cap L_i^{\infty}| = \omega\}$ exists and $\mathscr{L}_n \subseteq A$ for each $n \leq i(A) - 1$.

Suppose that D is a directed subset of A. When D has a largest element, we clearly have $\bigvee_{\mathscr{L}} D \in A$. Now we assume that D has no largest element. Then, D is infinite, whence by Lemma 71, there is a unique $(i_D, j, \infty) \in \mathscr{L}^{\infty}$ satisfying Conditions (i)–(iii) in Lemma 71. If $(i_D, j_D, \infty) \in A$ or $i_D \leq i(A) - 1$, then $\bigvee_{\mathscr{L}} D = (i_D, j_D, \infty) \in A$ or $\bigvee_{\mathscr{L}} D = (i_D, j_D, \infty) \in \mathscr{L}_{i_D} \subseteq A$. If $i_D \geq i(A)$ and $(i_D, j_D, \infty) \notin A$, then by Corollary 72, $\max(A) \cap \mathscr{L}_{i_D+1}^{\infty}$ is infinite. It follows that $i_D + 1 \leq i(A)$, which is a contradiction with $i_D \geq i(A)$. So $A \in \mathscr{C}(\Sigma\mathscr{L})$. □

Remark 76. The condition $A = \downarrow\max(A)$ is necessary. For example, for any $(i, j) \in \mathbb{N} \times \mathbb{N}$ the set $A_{(i,j)} = \{(i, j, l) : l \in \mathbb{N}\}$ is a lower set and $\max(A) = \emptyset$, whence $A \neq \downarrow\max(A)$. Clearly, $A_{(i,j)} \subseteq \mathscr{L}^{<\infty}$ and $\vee\{(i, j, l) : l \in \mathbb{N}\} = (i, j, \infty) \notin A_{(i,j)}$. So $A_{(i,j)}$ is not Scott closed.

Proposition 77. *Let $A \subseteq \mathscr{L}$ be a nonempty lower set. Then, the following four conditions are equivalent:*

(1) *A is Scott closed.*
(2) *For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, if $(i, j, \infty) \notin A$, then $A \cap \{(i, j, l) : l \in \mathbb{N}\}$ is finite.*
(3) *For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, $\mathscr{L}_{(i,j)} \subseteq A$ or $(i, j, \infty) \notin A$ and $A \cap \{(i, j, l) : l \in \mathbb{N}\}$ is finite.*
(4) *For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, $\mathscr{L}_{(i,j)} \subseteq A$ or $A \cap \mathscr{L}_{(i,j)}$ is finite.*

Proof. (1) ⇒ (2): Trivial. (2) ⇔ (3) ⇔ (4): Trivial (since A is a lower set).

(2) ⇒ (1): Suppose that D is a directed subset of A. When D has a largest element, then $\bigvee_{\mathscr{L}} D \in A$. Now assume that D has no largest element. Then, D is infinite, whence by Lemma 71, there is

a unique $(i_D, j_D, \infty) \in \mathscr{L}^\infty$ satisfying Conditions (i)-(iii) in Lemma 71. Since $D \subseteq A \cap \{(i_D, j_D, l) : l \in \mathbb{N}\}$ is infinite, by (2), $\bigvee_{\mathscr{L}} D = (i_D, j_D, \infty) \in A$. Thus, A is Scott closed. □

Dually, as a direct corollary of Proposition 77, we have the following.

Corollary 78. *Let $U \subseteq \mathscr{L}$ be an upper set. Then, the following four conditions are equivalent:*

(1) *U is Scott open.*
(2) *For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, if $(i, j, \infty) \in U$, then $U \cap \mathscr{L}_{(i,j)}$ is infinite.*
(3) *For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, $U \cap \mathscr{L}_{(i,j)} = \emptyset$ or $(i, j, \infty) \in U$ and $U \cap \mathscr{L}_{(i,j)}$ is infinite.*
(4) *For each $(i, j) \in \mathbb{N} \times \mathbb{N}$, $U \cap \mathscr{L}_{(i,j)} = \emptyset$ or $U \cap \mathscr{L}_{(i,j)}$ is infinite.*

By Lemma 75 and Proposition 77, we get the following.

Corollary 79. *Let $A \subseteq \mathscr{L}$ be a nonempty lower set and $A \neq \mathscr{L}$. Then, A is Scott closed if and only if one of the following three conditions is satisfied:*

(1) *$A \subseteq \mathscr{L}^{<\infty}$ (or equivalently, $\max(A) \subseteq \mathscr{L}^{<\infty}$) and $A \cap \{(i, j, l) : l \in \mathbb{N}\}$ is finite for each $(i, j) \in \mathbb{N} \times \mathbb{N}$.*
(2) *$A \cap \mathscr{L}^\infty \neq \emptyset$, $|A \cap \mathscr{L}_i^\infty| < \omega$ for each $i \in \mathbb{N}$ and $A \cap \{(i, j, l) : l \in \mathbb{N}\}$ is finite for each $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $(i, j, \infty) \notin A$.*
(3) *$i(A) = \max\{i \in \mathbb{N} : |A \cap L_i^\infty| = \omega\}$ exists, $\mathscr{L}_n \subseteq A$ for each $n \leq i(A) - 1$, and $A \cap \{(i, j, l) : l \in \mathbb{N}\}$ is finite for each $(i, j) \in \mathbb{N}$ with $(i, j, \infty) \notin A$.*

Lemma 80. $\mathsf{Irr}_c(\Sigma\mathscr{L}) = \{\overline{\{x\}} = \downarrow x : x \in \mathscr{L}\} \cup \{\mathscr{L}\}$.

Proof. Clearly, $\{\overline{\{x\}} = \downarrow x : x \in \mathscr{L}\} \subseteq \mathsf{Irr}_c(\Sigma\mathscr{L})$. It was shown in Jia (2018, Example 2.6.1) that $\mathscr{L} \in \mathsf{Irr}_c(\Sigma\mathscr{L})$. This can be easily proved by Lemma 75 . Suppose that $B, C \in \mathscr{C}(\Sigma\mathscr{L})$ and $\mathscr{L} = B \cup C$. Let $\mathbb{N}_B = \{n \in \mathbb{N} : |B \cap L_n^\infty| = \omega\}$ and $\mathbb{N}_C = \{n \in \mathbb{N} : |C \cap L_n^\infty| = \omega\}$. As $\mathscr{L} = B \cup C$, we have $\mathbb{N} = \mathbb{N}_B \cup \mathbb{N}_C$, and hence, at least one of $\mathbb{N}_B$ and $\mathbb{N}_C$ is infinite. Without loss of generality, assume $\mathbb{N}_B$ is infinite. Then by property Q in the proof of Lemma 75, we have $B = \mathscr{L}$. Thus, $\mathscr{L} \in \mathsf{Irr}_c(\Sigma\mathscr{L})$.

Conversely, suppose that $A \in \mathsf{Irr}_c(\Sigma\mathscr{L})$ and $A \neq \mathscr{L}$.

Case 1. $\max(A) \cap \mathscr{L}^{<\infty} \neq \emptyset$ (i.e., A has a maximal point of finite height).

Select an $(i, j, l) \in \max(A) \cap \mathscr{L}^{<\infty}$. Then, $l \in \mathbb{N}$. Let $B = \downarrow (\max(A) \setminus \{(i, j, l)\})$ (B may be the empty set). Then, $A = B \cup \downarrow (i, j, l)$. Clearly, $\downarrow (i, j, l)$ is Scott closed. By Lemma 75 or Proposition 77, it is easy to verify that B is Scott closed. Indeed, we can give a direct proof of the Scott closedness of B. Suppose that D is a directed subset of B. When D has a largest element, we clearly have $\bigvee_{\mathscr{L}} D \in D \subseteq B$. Now we assume that D has no largest element. Then, D is infinite, whence by Lemma 71, there is a unique $(i_D, j_,\infty) \in \mathscr{L}^\infty$ satisfying Conditions (i)-(iii) in Lemma 71. Since A is Scott closed, $(i_D, j_D, \infty) = \bigvee_{\mathscr{L}} D \in A$. As $l \in \mathbb{N}$, $(i_D, j_D, \infty) \notin \downarrow (i, j, l)$, whence $(i_D, j_D, \infty) \in \max(A) \setminus \{(i, j, l)\} \subseteq B$. Thus, B is Scott closed. By the irreducibility of A, we have $A = B$ or $A = \downarrow (i, j, l)$. If $A = B$, then there is $(i^*, j^*, l^*) \in \max(A) \setminus \{(i, j, l)\}$ with $(i, j, l) \leq (i^*, j^*, l^*)$. Hence, $(i, j, l) = (i^*, j^*, l^*)$ by $(i, j, l) \in \max(A)$, a contradiction. Therefore, $A = \downarrow (i, j, l) = \mathrm{cl}_{\sigma(\mathscr{L})}\{(i, j, l)\}$.

Case 2. $\max(A) \subseteq \mathscr{L}^\infty$ (i.e., every maximal point of A is a point of infinite height).

Since $A \neq \emptyset$, $A = \downarrow \max(A)$ and $\max(A) \subseteq \mathscr{L}^\infty$, we have that $\max(A) \cap \mathscr{L}^\infty \neq \emptyset$. If $|A \cap L_n^\infty| < \omega$ for all $n \in \mathbb{N}$, then select any point $(i, j, \infty) \in \max(A)$ and let $n(A) = i$. We clearly have that $|A \cap L_{n(A)+1}^\infty| < \omega$. If $|A \cap L_m^\infty| = \omega$ for some $m \in \mathbb{N}$, then by property Q in the proof of Lemma 75, the set $\{n \in \mathbb{N} : |A \cap L_n^\infty| = \omega\}$ is a nonempty finite set, whence $n(A) = \max\{n \in \mathbb{N} : |A \cap L_n^\infty| = \omega\}$ exists. Then, $A \cap L_{n(A)+1}^\infty$ is finite. Select an $(n(A), j, \infty) \in A \cap L_{n(A)}^\infty$. Let $B = \downarrow (\max(A) \setminus \{(n(A), j, \infty)\})$. Then, $A = B \cup \downarrow (n(A), j, \infty)$. Clearly, $\downarrow (n(A), j, \infty)$ is Scott closed.

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Now we show that B is Scott closed. Suppose that D is a directed subset of B. If D has a largest element s, then $\bigvee_{\mathscr{L}} D = s \in D \subseteq B$; if D has no largest element, then D is infinite, whence by Lemma 71, there is a unique $(i_D, j_D, \infty) \in \mathscr{L}^{\infty}$ such that the following three conditions are satisfied:

(i) $(i_D, j_D, \infty) \notin D$,
(ii) $D \subseteq \{(i_D, j_D, l) : l \in \mathbb{N}\}$, and
(iii) $(i_D, j_D, \infty) = \bigvee_{\mathscr{L}} D$.

Since A is Scott closed, $(i_D, j_D, \infty) = \bigvee_{\mathscr{L}} D \in A$, whence $(i_D, j_D, \infty) \in \max(A)$. Now we show that $(i_D, j_D, \infty) \neq (n(A), j, \infty)$. Suppose, on the contrary, that $(i_D, j_D, \infty) = (n(A), j, \infty)$. For each $d = (i_D, j_D, l_d) \in D$, there is an $(i_d, j_d, \infty) \in \max(A) \setminus \{(n(A), j, \infty)\}$ such that $d = (i_D, j_D, l_d) \leq (i_d, j_d, \infty)$. Since $(i_d, j_d, \infty) \neq (n(A), j, \infty) = (i_D, j_D, \infty)$, we have that $i_d = i_D + 1 = n(A) + 1$ and $l_d \leq j_d$. As $D \subseteq \{(i_D, j_D, l) : l \in \mathbb{N}\}$ is infinite, the set $\{(n(A) + 1, l_d, \infty) = (i_D + 1, l_d, \infty) : d \in D\}$ is infinite, and consequently, $A \cap L^{\infty}_{n(A)+1}$ is infinite, a contradiction. So $(i_D, j_D, \infty) \neq (n(A), j, \infty)$ and hence $\bigvee_{\mathscr{L}} D = (i_D, j_D, \infty) \in \max(A) \setminus \{(n(A), j, \infty)\} \subseteq B$. Thus, B is Scott closed.

By the irreducibility of A, we have $A = B$ or $A = \downarrow (n(A), j, \infty)$. It follows from $(n(A), j, \infty) \notin B$ that $A = \downarrow (n(A), j, \infty) = \mathrm{cl}_{\sigma(\mathscr{L})}\{(n(A), j, \infty)\}$.

All these together deduce that $\mathsf{Irr}_c(\Sigma\mathscr{L}) = \{\overline{\{x\}} = \downarrow x : x \in \mathscr{L}\} \cup \{\mathscr{L}\}$. □

Lemma 81. *(Jia 2018, Example 2.6.1) For a nonempty saturated subset $K \subseteq \mathscr{L}$, K is compact in $\Sigma\mathscr{L}$ if and only if the following three conditions are satisfied:*

(1) $\min(K) \cap \mathscr{L}^{<\infty}$ *is finite,*
(2) $\{n \in \mathbb{N} : \mathscr{L}^{\infty}_n \cap K \neq \emptyset\}$ *is finite, and*
(3) *there is a unique $(i_K, j_K) \in \mathbb{N} \times \mathbb{N}$ such that $K \cap (\mathscr{L}^{\infty}_{(<i_K)} \cup \mathscr{L}^{\infty}_{(i_K, \geq j_K)}) = \{(i_K, j_K, \infty)\}$ (or equivalently, $K \cap (\mathscr{L}_{(<i_K)} \cup \mathscr{L}^{\infty}_{(i_K, \geq j_K)}) = \{(i_K, j_K, \infty)\}$).*

Corollary 82. *For any filtered family $\{K_d : d \in D\} \subseteq \mathsf{K}(\Sigma\mathscr{L})$, $\bigcap_{d\in D} K_d \neq \emptyset$.*

Proof. We can assume that D is directed and $K_{d_2} \subseteq K_{d_1}$ iff $d_1 \leq d_2$ (indeed, D can be defined an order by $d_1 \leq d_2$ iff $K_{d_2} \subseteq K_{d_1}$). For each $d \in D$, by Lemma 81, there exists a unique $(i_d, j_d) \in \mathbb{N} \times \mathbb{N}$ such that $(\mathscr{L}^{\infty}_{(<i_d)} \cup \mathscr{L}^{\infty}_{(i_d, \geq j_d)}) \cap K_d = \{(i_d, j_d, \infty)\}$ and the set $\mathbb{N}_d = \{n \in \mathbb{N} : \mathscr{L}^{\infty}_n \cap K_d \neq \emptyset\}$ is finite. Select a $d_0 \in D$. Then for each $d \in D$ with $d_0 \leq d$ (whence $K_d \subseteq K_{d_0}$), we have that $i_d \in \mathbb{N}_d \subseteq \mathbb{N}_{d_0}$ and $i_{d_0} \leq i_d$ (otherwise, $i_{d_0} > i_d$ would imply that $(i_d, j_d, \infty) \in (\mathscr{L}^{\infty}_{(<i_d)} \cup \mathscr{L}^{\infty}_{(i_d, \geq j_d)}) \cap K_d \subseteq L^{\infty}_{(<i_{d_0})} \cap K_{d_0}$, which contradicts $L^{\infty}_{(<i_{d_0})} \cap K_{d_0} = \emptyset$). Let $D_{d_0} = \{d \in D : d_0 \leq d\}$ and $D_i = \{d \in D_{d_0} : i_d = i\}$ for each $i \in \mathbb{N}_{d_0}$. Since $\mathbb{N}_{d_0}$ is finite, D_{d_0} is directed and $D_{d_0} = \bigcup_{i\in\mathbb{N}_{d_0}} D_i$, there is $i_0 \in \mathbb{N}_{d_0}$ such that D_{i_0} is a cofinal subset of D_{d_0} and hence a cofinal subset of D, more precisely, for each $d \in D$, there is $d^* \in D$ such that $d^* \in \uparrow d_0 \cap \uparrow d$ and $i_{d^*} = i_0$.

Clearly, D_{i_0} is also directed. Select a $d_1 \in D_{i_0}$. Then, $D_{d_1} = \{d \in D_{i_0} : d_1 \leq d\}$ is a directed and cofinal subset of D_{i_0} and hence a directed and cofinal subset of D. For each $d \in D_{d_1}$ (note that $K_d \subseteq K_{d_1}$), we have that $i_{d_1} = i_d = i_0$, $(i_d = i_0, j_d, \infty) \in K_d \subseteq K_{d_1}$ and $\mathscr{L}^{\infty}_{(i_0, \geq j_{d_1})} \cap K_{d_1} = \{(i_0, j_{d_1}, \infty)\}$. It follows that $j_d \leq j_{d_1}$. For each $1 \leq j \leq j_{d_1}$, let $\widetilde{D}_j = \{d \in D_{d_1} : j_d = j\}$. Since $\{1, 2, ..., j_{d_1}\}$ is finite, D_{d_1} is directed and $D_{d_1} = \bigcup_{i\in\mathbb{N}_{d_0}} \widetilde{D}_j$, there is $1 \leq j_0 \leq j_{d_1}$ such that $\widetilde{D}_{j_0}$ is a cofinal subset of D_{d_1} and hence a cofinal subset of D; more precisely, for each $d \in D$, there is $d' \in D$ such that $d' \in \uparrow d_0 \cap \uparrow d_1 \cap \uparrow d$, $i_{d'} = i_0$ and $j_{d'} = j_0$. It follows that $(i_0, j_0, \infty) \in \bigcap_{d\in D} K_d$. □

Proposition 83. *(Jia 2018, Example 2.6.1) $\Sigma\mathscr{L}$ is well-filtered but non-sober.*

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Proof. By Lemma 80, $\mathscr{L}$ is an irreducible closed subset of $\Sigma\mathscr{L}$ but has no largest element, so $\Sigma\mathscr{L}$ is non-sober.

The well-filteredness of $\Sigma\mathscr{L}$ was proved in Jia (2018) (see Jia 2018, the proof of Claim 2.6.4). Using Topological Rudin Lemma, Lemma 80 and Corollary 82, we can give a short proof of the well-filteredness of $\Sigma\mathscr{L}$. Suppose that $\{K_d : d \in D\} \subseteq \mathsf{K}(\Sigma\mathscr{L})$ is a filtered family and $U \in \sigma(\mathscr{L})$ with $\bigcap_{d\in D} K_d \subseteq U$. Assume, on the contrary, that $K_d \not\subseteq U$ for each $d \in D$ (whence $U \neq \mathscr{L}$). Then by Lemma 27, $\mathscr{L} \setminus U$ contains a minimal irreducible closed subset A that still meets all members K_d. By Corollary 82, $\bigcap_{d\in D} K_d \neq \emptyset$, whence $U \neq \emptyset$ and $A \neq \mathscr{L}$. It follows from Lemma 80 that $A = \overline{\{x\}}$ for some $x \in \mathscr{L}$. Then, $x \in \bigcap_{d\in D} K_d \subseteq U$, which contradicts $x \in A \subseteq \mathscr{L} \setminus U$. Thus $\Sigma\mathscr{L}$ is well-filtered. $\square$

Definition 84. *Let X be a T_0 space for which X is irreducible (i.e., $X \in \mathsf{Irr}_c(X)$). Choose a point $\top$ such that $\top \notin X$. Then ,$(\mathscr{C}(X) \setminus \{X\}) \cup \{X \cup \{\top\}\}$ (as the set of all closed sets) is a topology on $X \cup \{\top\}$. The resulting space is denoted by $X_\top$. Define a mapping $\zeta_X : X \to X_\top$ by $\zeta_X(x) = x$ for each $x \in X$. Clearly, η_X is a topological embedding.*

As X is T_0, $X_\top$ is also T_0 and $\overline{\{\top\}} = X \cup \{\top\}$ in $X_\top$. Hence, $\top$ is a largest element of $X_\top$ and for $x, y \in X$, $x \leq_X y$ iff $x \leq y$ in $X_\top$. It is worthy noting that the set $\{\top\}$ is not open in $X_\top$.

Remark 85. If X is not irreducible, then there exist $A, B \in \mathscr{C}(X) \setminus \{X\}$ such that $X = A \cup B$, whence $(\mathscr{C}(X) \setminus \{X\}) \cup \{X \cup \{\top\}\}$ is not a topology on $X \cup \{\top\}$.

Lemma 86. *Let X be a T_0 space for which X is irreducible. Then, $\mathsf{K}(X_\top) = \{G \cup \{\top\} : G \in \mathsf{K}(X)\} \cup \{\{\top\}\}$.*

Proof. Clearly, $\mathscr{O}(X_\top) = \{U \cup \{\top\} : U \in \mathscr{O}(X) \setminus \{\emptyset\}\} \cup \{\emptyset\}$.

First, if $K \in \mathsf{K}(X_\top) \setminus \{\{\top\}\}$, then $G = K \setminus \{\top\}$ is a nonempty saturated subset of X. Now we verify that G is a compact subset of X. Suppose that $\{U_i : i \in I\} \subseteq \mathscr{O}(X) \setminus \{\emptyset\}$ is an open cover of G. Then, $\{U_i \cup \{\top\} : i \in I\} \subseteq \mathscr{O}(X_\top)$ is an open cover of $K = G \cup \{\top\}$. By the compactness of K, there is $I_0 \in I^{(<\omega)}$ such that $K \subseteq \bigcup_{i\in I_0} U_i \cup \{\top\}$, whence $G = K \setminus \{\top\} \subseteq \bigcup_{i\in I_0} U_i$, proving that $G \in \mathsf{K}(X)$.

Conversely, assume that $G \in \mathsf{K}(X)$ and $\{W_j : j \in J\} \subseteq \mathscr{O}(X_\top) \setminus \{\emptyset\}$ is an open cover of $K = G \cup \{\top\}$. Then, K is saturated and for each $j \in J$, there is $V_j \in \mathscr{O}(X)$ such that $W_j = V_j \cup \{\top\}$. Hence, $\{V_j : j \in J\} \subseteq \mathscr{O}(X)$ is an open cover of G. By the compactness of G, there is $J_0 \in J^{(<\omega)}$ such that $G \subseteq \bigcup_{j\in J_0} V_j$, whence $K = G \cup \{\top\} \subseteq \bigcup_{j\in J_0} W_j$. So $K \in \mathsf{K}(X_\top)$.

Thus, $\mathsf{K}(X_\top) = \{G \cup \{\top\} : G \in \mathsf{K}(X)\} \cup \{\{\top\}\}$. $\square$

Lemma 87. *Suppose that X is a non-sober T_0 space for which $\mathsf{Irr}_c(X) = \{\overline{\{x\}} : x \in X\} \cup \{X\}$. Then, $\langle X_\top, \zeta_X\rangle$ is a sobrification of X.*

Proof. Since X is non-sober and $\mathsf{Irr}_c(X) = \{\overline{\{x\}} : x \in X\} \bigcup \{X\}$, $X \neq \overline{\{x\}}$ for every $x \in X$. It is well-known that the space X^s with the canonical mapping $\eta_X : X \longrightarrow X^s$, $\eta_X(x) = x$, is a sobrification of X (see, e.g., Gierz et al. 2003, Exercise V-4.9). For $C \in \mathscr{C}(X)$, we have

$$\Box_{\mathsf{Irr}_c(X)} C = \{A \in \mathsf{Irr}_c(X) : A \subseteq C\} = \begin{cases} \{\overline{\{c\}} : c \in C\}, & C \neq X, \\ \{\overline{\{x\}} : x \in X\} \cup \{X\}, & C = X. \end{cases}$$

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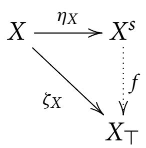
Define a mapping $f : X^s \to X_\top$ by

$$f(A) = \begin{cases} x & A = \overline{\{x\}}, x \in X, \\ \top & A = X. \end{cases}$$

For each $C \in (\mathscr{C}(X) \setminus \{X\}) \bigcup \{X_\top\}$ and $B \in \mathscr{C}(X)$, we have $f^{-1}(C) = \Box_{\mathsf{Irr}_c(X)} C$ and

$$f(\Box_{\mathsf{Irr}_c(X)} B) = \begin{cases} B & B \neq X, \\ X \cup \{\top\} & B = X. \end{cases}$$

Thus, f is a homeomorphism.



So $\langle X_\top, \zeta_X = f \circ \eta_X \rangle$ is a sobrification of X. □

The following corollary is straightforward from Lemma 80, Proposition 83, and Lemma 87.

Corollary 88. *$\langle (\Sigma\mathscr{L})_\top, \zeta_{\mathscr{L}} \rangle$ is a sobrification of $\Sigma\mathscr{L}$, where $\zeta_{\mathscr{L}} : \Sigma\mathscr{L} \to (\Sigma\mathscr{L})_\top$ is defined by $\zeta_{\mathscr{L}}(x) = x$ for each $x \in \mathscr{L}$.*

Note that although the set $\{\top\}$ is open in $\Sigma\mathscr{L}_\top$ (or equivalently, $\top$ is a compact element of the dcpo $\mathscr{L} \cup \{\top\}$), it is not open in $(\Sigma\mathscr{L})_\top$.

By Lemmas 80 and 86, we get the following.

Corollary 89. $\mathsf{K}((\Sigma\mathscr{L})_\top) = \{G \cup \{\top\} : G \in \mathsf{K}(\Sigma\mathscr{L}))\} \cup \{\{\top\}\}$.

Lemma 90. *$\{\top\}$ is a compact element in the dcpo $\mathsf{K}((\Sigma\mathscr{L})_\top)$. Hence, $\{\{\top\}\}$ is open in the Scott space $\Sigma\mathsf{K}((\Sigma\mathscr{L})_\top)$.*

Proof. By Proposition 83, $\Sigma\mathscr{L}$ is well-filtered, whence $\mathsf{K}(\Sigma\mathscr{L})$ (with the Smyth order) is a dcpo. So by Corollary 89, $\mathsf{K}((\Sigma\mathscr{L})_\top)$ is a dcpo. Now we show that $\{\top\} \ll \{\top\}$ in $\mathsf{K}((\Sigma\mathscr{L})_\top)$. Suppose that $\{K_d : d \in D\}$ is a directed subset of $\mathsf{K}((\Sigma\mathscr{L})_\top)$ and $\{\top\} \sqsubseteq \bigvee_{d \in D} K_d$. Then by Lemma 9, $\bigvee_{d \in D} K_d = \bigcap_{d \in D} K_d$, and hence, $\bigcap_{d \in D} K_d \subseteq \{\top\}$. It follows that $\bigcap_{d \in D} (K_d \setminus \{\top\}) = \emptyset$. By Corollaries 82 and 89, there is $d \in D$ such that $K_d \setminus \{\top\} = \emptyset$, that is, $K_d = \{\top\}$. Thus, $\{\top\}$ is a compact element in $\mathsf{K}((\Sigma\mathscr{L})_\top)$. Hence, $\{\{\top\}\} \in \sigma(\mathsf{K}((\Sigma\mathscr{L})_\top))$. □

Theorem 91. *The Scott space $\Sigma\mathsf{K}((\Sigma\mathscr{L})_\top)$ of the sober space $(\Sigma\mathscr{L})_\top$ is non-sober.*

Proof. For simplicity, let $\mathscr{A} = \{G \cup \{\top\} : G \in \mathsf{K}(\Sigma\mathscr{L})\}$.

Claim 1: $\mathscr{A}$ is a closed subset of $\Sigma\mathsf{K}((\Sigma\mathscr{L})_\top)$.

By Corollary 89 and Lemma 90, $\mathscr{A}$ is Scott closed.

Claim 2: $\mathscr{A}$ is irreducible.

Suppose that $\mathscr{U}, \mathscr{V} \in \sigma(\mathsf{K}((\Sigma\mathscr{L})_\top))$ and $\mathscr{A} \bigcap \mathscr{U} \neq \emptyset \neq \mathscr{A} \bigcap \mathscr{V}$. Then by Corollary 89, there are some $G_1, G_2 \in \mathsf{K}(\Sigma\ \mathscr{L}))$ such that $G_1 \cup \{\top\} \in \mathscr{A} \bigcap \mathscr{U}$ and $G_2 \cup \{\top\} \in \mathscr{A} \bigcap \mathscr{V}$, whence $(i_1, j_1, \infty) \in G_1$ and $(i_2, j_2, \infty) \in G_2$ for some $(i_1, j_1), (i_2, j_2) \in \mathbb{N} \times \mathbb{N}$. Hence by Corollary 89,$\uparrow(i_1, j_1, \infty) \cup \{\top\} \in \mathscr{A} \bigcap \mathscr{U}$ and $\uparrow(i_2, j_2, \infty) \cup \{\top\} \in \mathscr{A} \bigcap \mathscr{V}$ (note that $\mathscr{U}, \mathscr{V}$ are upper

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sets and $G_1 \cup \{\top\} \sqsubseteq \uparrow(i_1, j_1, \infty) \cup \{\top\}$, $G_2 \cup \{\top\} \sqsubseteq \uparrow(i_2, j_2, \infty) \cup \{\top\}$). Without loss of generality, we assume $i_1 \leq i_2$. Since $\bigvee_{l\in\mathbb{N}} (\uparrow(i_1, j_1, l) \cup \{\top\}) = \bigcap_{l\in\mathbb{N}} (\uparrow(i_1, j_1, l) \cup \{\top\}) = \uparrow(i_1, j_1, \infty) \cup \{\top\} \in \mathscr{U} \in \sigma(\mathsf{K}((\Sigma\mathscr{L})_\top))$, we have some $l_1 \in \mathbb{N}$ such that $\uparrow(i_1, j_1, l_1) \in \mathscr{U}$. So by Corollary 89, $\uparrow(i_1 + 1, l_1, \infty) \cup \{\top\} \in \mathscr{U}$ since $(i_1, j_1, l_1) \leq (i_1 + 1, l_1, \infty)$ and $\mathscr{U}$ is an upper set. Then by induction, we have $\uparrow(i_2, j', \infty) \cup \{\top\} \in \mathscr{U}$ for some $j' \in \mathbb{N}$. Again, since $\bigvee_{l\in\mathbb{N}} (\uparrow(i_2, j', l) \cup \{\top\}) = \bigcap_{l\in\mathbb{N}} (\uparrow(i_2, j', l) \cup \{\top\}) = \uparrow(i_2, j', l) \cup \{\top\} \in \mathscr{U} \in \sigma(\mathsf{K}((\Sigma\mathscr{L})_\top))$ and $\bigvee_{l\in\mathbb{N}} (\uparrow(i_2, j_2, l) \cup \{\top\}) = \bigcap_{l\in\mathbb{N}} (\uparrow(i_2, j_2, l) \cup \{\top\}) = \uparrow(i_2, j_2, \infty) \cup \{\top\} \in \mathscr{V} \in \sigma(\mathsf{K}((\Sigma\mathscr{L})_\top))$, we have some $k_1, k_2 \in \mathbb{N}$ such that $\uparrow(i_2, j', k_1) \in \mathscr{U}$ and $\uparrow(i_2, j_2, k_2) \in \mathscr{V}$. Take $m = \max\{k_1, k_2\}$. Then, $\uparrow(i_2, m, \infty) \cup \{\top\} \in \mathscr{A} \bigcap \mathscr{U} \bigcap \mathscr{V}$. Thus, $\mathscr{A} \in \mathsf{Irr}_c(\Sigma\mathsf{K}((\Sigma\mathscr{L})_\top))$.

Claim 3: $\mathscr{A}$ has no largest element.

Clearly, $\{\uparrow(i, j, \infty) \cup \{\top\} : i, j \in \mathbb{N}\}$ is the set of all maximal elements of $\mathscr{A}$, and hence, $\mathscr{A}$ has no largest element.

By Claims 1–3, the Scott space $\Sigma\mathsf{K}((\Sigma\mathscr{L})_\top)$ is non-sober. □

Question 92. For a dcpo (especially, a complete lattice) P with the sober Scott topology, is the Scott space $\Sigma\mathsf{K}(\Sigma P)$ sober?

We know that every T_2 space is sober, and hence, the Scott space $\Sigma\mathsf{K}(X)$ is well-filtered by Theorem 58. In the next section, we will show that for a locally compact (especially, compact) T_2 space X, the Soctt space $\Sigma\mathsf{K}(X)$ is sober (see Corollary 99 below).

By Theorem 19, Corollary 25, Theorem 91, and Corollary 99 below, we naturally pose the following question.

Question 93. For a T_2 space X, is the Scott space $\Sigma\mathsf{K}(X)$ sober?

8. Local Compactness, First-Countability, and Sobriety of Scott Topology on Smyth Power Posets

In this section, we study the question under what conditions the Scott space $\Sigma\mathsf{K}(X)$ of a sober space X is sober. This question is related to the investigation of conditions under which the upper Vietoris topology coincides with the Scott topology on $\mathsf{K}(X)$, and further, it is closely related to the local compactness and first-countability of X.

First, by Corollaries 40, 50 and Theorem 58, we get the following.

Corollary 94. *If X is a well-filtered space for which the Scott space $\Sigma\mathsf{K}(X)$ is first-countable or core-compact (especially, locally compact), then $\Sigma\mathsf{K}(X)$ is sober.*

For the local compactness of Smyth power spaces, we have the following.

Lemma 95. *(Lyu et al. 2022, Theorem 3.1) For a T_0 space X, the following conditions are equivalent:*

(1) *X is locally compact.*
(2) *$P_S(X)$ is core-compact.*
(3) *$P_S(X)$ is locally compact.*
(4) *$P_S(X)$ is locally hypercompact.*
(5) *$P_S(X)$ is a c-space.*

The following corollary follows directly from Proposition 37 and Lemma 95.

Corollary 96. *For a locally compact T_0 space X, the Smyth power space $P_S(X)$ is a* DC *space.*

Concerning the Scott space $\Sigma\mathsf{K}(X)$ of a locally compact T_0 space X, we have the following question.

Question 97. For a locally compact T_0 space X, is the Scott space $\Sigma\mathsf{K}(X)$ a Rudin space or a WD space?

Proposition 98. *Let X be a locally compact sober space. Then,*

(1) *the Scott space $\Sigma\mathsf{K}(X)$ and the Symth power space of X coincide, that is, $\Sigma\mathsf{K}(X) = P_S(X)$.*
(2) *$\mathsf{K}(X)$ is a continuous domain.*
(3) *$\Sigma\mathsf{K}(X)$ is a sober c-space.*

Proof. By Corollary 25 and Lemma 54, $\Sigma\mathsf{K}(X) = P_S(X)$. By Gierz et al. (2003, Proposition I-1.24.2), $\mathsf{K}(X)$ is a continuous semilattice, and hence by Theorem 15 and Proposition 18, $\Sigma\mathsf{K}(X)$ is a sober c-space. □

Corollary 99. *If X is a locally compact T_2 (especially, a compact T_2) space, then*

(1) *$\Sigma\mathsf{K}(X) = P_S(X)$.*
(2) *$\mathsf{K}(X)$ is a continuous domain.*
(3) *$\Sigma\mathsf{K}(X)$ is a sober c-space.*

By Theorem 15, Propositions 18 and 98, we have the following corollary.

Corollary 100. *Let P be a quasicontinuous domain. Then*

(1) *the upper Vietoris topology agrees with the Scott topology on $\mathsf{K}(\Sigma P)$.*
(2) *$\mathsf{K}(\Sigma P)$ is a continuous semilattice.*
(3) *the Scott space $\Sigma\mathsf{K}(\Sigma P)$ is a sober c-space.*

Now we discuss the first-countability of the Scott topology on Smyth power posets. First, for the Smyth power spaces and sobrifications of T_0 spaces, we have the following conclusion. ZZZ

Proposition 101. *(Brecht et al. 2019; Xu et al. 2021a,c) For a T_0 space, the following conditions are equivalent:*

(1) *X is second-countable.*
(2) *$P_S(X)$ is second-countable.*
(3) *X^s is second-countable.*

Since first-countability is a hereditary property, by Remarks 3 and 4, we get the following result.

Proposition 102. *Let X be a T_0 space. If X^s is first-countable or $P_S(X)$ is first-countable, then X is first-countable.*

Example 67 shows that unlike the Smyth power space, the first-countability of the Scott space $\Sigma\mathsf{K}(X)$ of a T_0 space X does not imply the first-countability of X in general.

The converse of Proposition 102 does not hold in general, as shown in Example 53 and the following example. It also shows that even for a compact Hausdorff first-countable space X, the Scott space $\Sigma\mathsf{K}(X)$ and the Smyth power space of X may not be first-countable.

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There is even a T_0 space X for which the Scott space $\Sigma\mathsf{K}(X)$ is second-countable but X is not fist-countable (see Example 120 below). So for the Scott topology on Smyth power posets, the analogous results to Propositions 101 and 102 do not hold.

Example 103. Consider in the plane $\mathbb{R}^2$ two concentric circles $C_i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = i\}$, where $i = 1, 2$, and their union $X = C_1 \cup C_2$; the projection of C_1 onto C_2 from the point $(0, 0)$ is denoted by p. On the set X, we generate a topology by defining a neighborhood system $\{B(z) : z \in X\}$ as follows: $B(z) = \{z\}$ for $z \in C_2$ and $B(z) = \{U_j(z) : j \in \mathbb{N}\}$ for $z \in C_1$, where $U_j = V_j \bigcup p(V_j \setminus \{z\})$ and V_j is the arc of C_1 with center at z and of length $1/j$. The space X is called the *Alexandroff double circle* (cf. Engelking 1989). The following conclusions about X are known (see, for example, Engelking 1989, Example 3.1.26).

(a) X is Hausdorff and first-countable.
(b) X is compact and locally compact.
(c) X is not separable, and hence not second-countable.
(d) C_1 is a compact subspace of X.
(e) C_2 is a discrete subspace of X.

There is no countable base at C_1 in $P_S(X)$. Thus, $P_S(X)$ is not first-countable. For details, see Xu et al. (2021*c*, Example 4.4). By Corollary 99, $\Sigma\mathsf{K}(X) = P_S(X)$, whence the Scott space $\Sigma\mathsf{K}(X)$ is not first-countable.

Proposition 104. *(Xu et al. 2021c, Proposition 4.5) Let X be a first-countable T_0 space. If* $\min(K)$ *is countable for any $K \in \mathsf{K}(X)$, then $P_S(X)$ is first-countable.*

For a metric space (X, d), $x \in X$ and a positive number r, let $B(x, r) = \{y \in Y : d(x, y) < r\}$ be the *r-ball* about x. For a set $A \subseteq X$ and a positive number r, by the *r-ball* about A we mean the set $B(A, r) = \bigcup_{a \in A} B(a, r)$.

The following result is well-known (cf. Engelking 1989).

Lemma 105. *Let (X, d) be a metric space and K a compact set of X. Then for any open set U containing K, there is an $r > 0$ such that $K \subseteq B(K, r) \subseteq U$.*

Proposition 106. *For a metric space (X, d), $P_S((X, d))$ is first-countable.*

Proof. For $K \in \mathsf{K}((X, d))$, let $\mathscr{B}_K = \{B(K, 1/n) : n \in \mathbb{N}\}$. Then by Lemma 105, $\mathscr{B}_K = \{B(K, 1/n) : n \in \mathbb{N}\}$ is a countable base at K in $P_S((X, d))$. Thus, $P_S((X, d))$ is first-countable. □

For a countable T_0 space X, it is easy to see that X is second-countable iff X is first-countable. Indeed, let $X = \{x_n : n \in \mathbb{N}\}$. If X is first-countable, then for each $n \in \mathbb{N}$, there is a countable base $\mathscr{B}_n$ at x_n. Let $\mathscr{B} = \bigcup_{n \in \mathbb{N}} \mathscr{B}_n$. Then, $\mathscr{B}$ is a countable base of X. Thus, X is second-countable. Therefore, by Propositions 101 and 102, we have the following.

Corollary 107. *For a countable T_0 space X, the following conditions are equivalent:*

(1) *X is first-countable.*
(2) *X is second-countable.*
(3) *X^s is first-countable.*
(4) *X^s is second-countable.*
(5) *$P_S(X)$ is first-countable.*
(6) *$P_S(X)$ is second-countable.*

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It is worth noting that the Scott topology on a countable complete lattice may not be first-countable, see Xu et al. (2020*a*, Example 4.8).

By Proposition 48, Theorem 49, Proposition 101, and Corollary 107, we deduce the following two results.

Corollary 108. *(Xu et al. 2021a, Corollaries 5.7 and 5.8) Every second-countable (especially, countable first-countable) T_0 space is an ω-Rudin space.*

Corollary 109. *Every second-countable (especially, countable first-countable) ω-well-filtered space is sober.*

For a T_0 space X with a first-countable Smyth power space, we have a similar result to Lemma 54.

Lemma 110. *Let X be a T_0 space for which the Smyth power space $P_S(X)$ is first-countable. Then, the Scott topology is coarser than the upper Vietoris topology on $\mathsf{K}(X)$.*

Proof. See the proof of Xu et al. (2021*c*, Theorem 5.7). □

The following conclusion is straightforward from Theorem 24, Corollaries 25, 50, and Lemma 110.

Corollary 111. *(Xu et al. 2021c, Theorem 5.7) Let X be a well-filtered space for which the Smyth power space $P_S(X)$ is first-countable. Then*

(1) *the upper Vietoris topology agrees with the Scott topology on $\mathsf{K}(X)$.*
(2) *the Scott space $\Sigma\mathsf{K}(X)$ is a first-countable sober space.*

By Proposition 104 and Corollary 111, we obtain the following.

Corollary 112. *(Xu et al. 2021c, Corollary 5.10) Let X be a first-countable well-filtered space X in which all compact subsets are countable (especially, $|X| \leq \omega$). Then*

(1) *the upper Vietoris topology agrees with the Scott topology on $\mathsf{K}(X)$.*
(2) *the Scott space $\Sigma\mathsf{K}(X)$ is a first-countable sober space.*

Let X_{coc} be the space in Example 67. Then, X_{coc} is well-filtered and not first-countable, and the Scott space $\Sigma\mathsf{K}(X)$ is a first-countable sober c-space, but $\sigma(\mathsf{K}(X_{coc}) \not\subseteq \mathcal{O}(P_S(X_{coc}))$.

By Example 67, Lemma 110, and Corollary 111, we naturally pose the following four questions.

Question 113. For a first-countable T_0 space X, is the Scott topology coarser than the upper Vietoris topology on $\mathsf{K}(X)$?

Question 114. For a first-countable well-filtered (or equivalently, a first-countable sober) space X, does the upper Vietoris topology and the Scott topology on $\mathsf{K}(X)$ coincide?

Question 115. For a first-countable T_2 space X, does the upper Vietoris topology and the Scott topology on $\mathsf{K}(X)$ coincide?

Question 116. For a first-countable well-filtered (or equivalently, a first-countable sober) space X, is the Scott space $\Sigma\mathsf{K}(X)$ sober?

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Since every metric space is T_2 (and hence sober), by Propositions 98, 106, and Corollary 111, we get the following conclusion.

Corollary 117. *Let (X, d) be a metric space. Then,*

(1) *the upper Vietoris topology agrees with the Scott topology on $\mathsf{K}((X, d))$.*
(2) *the Scott space $\Sigma\mathsf{K}((X, d))$ is a first-countable sober space.*

If, in addition, (X, d) is locally compact (especially, compact), then

(3) *$\mathsf{K}((X, d))$ is a continuous semilattice.*
(4) *the Scott space $\Sigma\mathsf{K}((X, d))$ is a c-space.*

The following two conclusions follow directly from Proposition 101, Lemma 110, and Corollary 111.

Corollary 118. *Let X be a second-countable T_0 space. Then, the Scott topology is coarser than the upper Vietoris topology on $\mathsf{K}(X)$.*

Corollary 119. *Let X be a second-countable well-filtered space (or equivalently, a second-countable sober space). Then,*

(1) *the Scott topology agrees with the upper Vietoris topology on $\mathsf{K}(X)$.*
(2) *the Scott space $\Sigma\mathsf{K}(X)$ is a second-countable sober space.*

The following example shows that there is a countable Hausdorff space X for which the Scott space $\Sigma\mathsf{K}(X)$ is second-countable but X is not first-countable (and hence $P_S(X)$ is not first-countable).

Example 120. Let p be a point in $\beta(\mathbb{N}) \setminus \mathbb{N}$, where $\beta(\mathbb{N})$ is the Stone-Cěch compactification of the discrete space of natural numbers, and consider on $X = \mathbb{N} \cup \{p\}$ the induced topology (cf. Gierz et al. 2003, Example II-1.25). Then

(a) $|X| = \omega$ and X is a non-discrete Hausdorff space and hence a sober space.
(b) $\mathsf{K}(X) = X^{(<\omega)} \setminus \{\emptyset\}$ and $\mathrm{int}K = \emptyset$ for each $K \in \mathsf{K}(X)$ with $p \in K$. So X is not locally compact.
(c) $\mathsf{K}(X)$ is a Noetherian poset and $|\mathsf{K}(X)| = \omega$. Hence, the Scott space $\Sigma\mathsf{K}(X)$ is a second-countable sober c-space.
Clearly, $\mathsf{K}(X) = X^{(<\omega)} \setminus \{\emptyset\}$ (with the Smyth order) is Noetherian (and hence algebraic) and $|\mathsf{K}(X)| = \omega$ since $|X| = \omega$. Therefore, $\sigma(\mathsf{K}(X)) = \alpha(\mathsf{K}(X))$ and $\{\uparrow_{\mathsf{K}(X)} F : F \in X^{(<\omega)} \setminus \{\emptyset\}\}$ is a countable base of $\Sigma\mathsf{K}(X)$. By Theorem 15 and Proposition18, $\Sigma\mathsf{K}(X)$ is a sober c-space.
(d) the upper Vietoris topology and the Scott topology on $\mathsf{K}(X)$ does not coincide, or more precisely, $\sigma(\mathsf{K}(X)) \not\subseteq \mathcal{O}(P_S(X))$.
By Corollary 25, $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$. Clearly, for any $F \in X^{(<\omega)} \setminus \{\emptyset\}$, $\uparrow_{\mathsf{K}(X)} F \in \sigma(\mathsf{K}(X))$ but $\uparrow_{\mathsf{K}(X)} G \notin \mathcal{O}(P_S(X))$ for any $G \in \sigma(\mathsf{K}(X))$ with $p \in G$, proving that $\sigma(\mathsf{K}(X)) \not\subseteq \mathcal{O}(P_S(X))$. We can also get this result by (b), (c) and Xu et al. (2021*c*, Theorem 3.10).
(e) Neither X nor $P_S(X)$ is first-countable.
By (d), Proposition 104 and Lemma 110, neither $P_S(X)$ nor X is first-countable (cf. Engelking 1989, Corollary 3.6.17).

The above example also shows that if the Smyth power space $P_S(X)$ is replaced with the Scott space $\Sigma\mathsf{K}(X)$ in the conditions of Lemma 110 and Corollary 111, the analogous results to Lemma 110 and Corollary 111 do not hold.

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By Proposition 101, Lemma 110, Corollaries 111 and 119, we raise the following question.

Question 121. For a second-countable T_0 space X, is the Scott space $\Sigma\mathsf{K}(X)$ second-countable?

9. Rudin Property and Well-Filtered Determinedness of Smyth Power Spaces and Scott Topology on Smyth Power Posets

Firstly, we discuss the Rudin property and well-filtered determinedness of Smyth power spaces. The following result was proved in Xu et al. (2020*b*).

Proposition 122. *(Xu et al. 2020b, Theorem 7.21) Let X be a T_0 space. If $P_S(X)$ is well-filtered determined, then X is well-filtered determined.*

By Theorems 24 and 36, we have the following.

Proposition 123. *Let X be a well-filtered space. Then, the following conditions are equivalent:*

(1) *X is a Rudin space.*
(2) *X is a* WD *space.*
(3) *$P_S(X)$ is a Rudin space.*
(4) *$P_S(X)$ is a* WD *space.*

It is still not known whether the converse of Proposition 122 holds (i.e., whether the Smyth power space $P_S(X)$ of a well-filtered determined T_0 space X is well-filtered determined) (see Xu et al. 2020*b*, Question 8.6).

Theorem 124. *Let X be a T_0 space. If $P_S(X)$ is a Rudin space, then X is a Rudin space.*

Proof. Let $A \in \mathsf{Irr}_c(X)$. Then by Lemma 5, $\overline{\xi_X(A)} = \Diamond A \in \mathsf{Irr}_c(P_S(X))$, where $\xi_X : X \longrightarrow P_S(X)$ is the canonical embedding (see Remark 4). Since $P_S(X)$ is a Rudin space, there is a filtered family $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(P_S(X))$ such that $\Diamond A \in m(\{\mathcal{K}_d : d \in D\})$. For each $d \in D$, let $K_d = \bigcup \mathcal{K}_d$. Then by Lemma 11, $\{K_d : d \in D\} \subseteq \mathsf{K}(X)$ is filtered. Clearly, $A \in M(\{K_d : d \in D\})$. For any proper closed subset B of A, we have that $\Diamond B \in \mathcal{C}(P_S(X))$ and $\Diamond B$ is a proper closed subset of $\Diamond A$ (for any $a \in A \setminus B$, $\uparrow a \in \Diamond A \setminus \Diamond B$). By the minimality of $\Diamond A$, there is a $d \in D$ such that $\Diamond B \cap \mathcal{K}_d = \emptyset$, and consequently, $B \cap K_d = \emptyset$. Thus $B \notin M(\{K_d : d \in D\})$, and hence, $A \in m(\{K_d : d \in D\})$. □

Question 125. Is the Smyth power space $P_S(X)$ of a Rudin space X still a Rudin space?

Now we discuss the Rudin property and well-filtered determinedness of the Scott topology on Smyth power posets.

First, even for a sober space X (whence it is both a Rudin space and a WD space by Theorem 36), the Scott space $\Sigma\mathsf{K}(X)$ may not be a WD space (and hence not a Rudin space). Indeed, let $(\Sigma\mathcal{L})_\top$ be as in Theorem 91. Then, $(\Sigma\mathcal{L})_\top$ is a sober space. By Theorems 58 and 91, the Scott space $\Sigma\mathsf{K}((\Sigma\mathcal{L})_\top)$ is well-filtered but non-sober. Hence by Theorem 36, $\Sigma\mathsf{K}((\Sigma\mathcal{L})_\top)$ is neither a Rudin space nor a WD space.

Conversely, Example 67 shows that there is a well-filtered space X such that

(a) the Scott space $\Sigma\mathsf{K}(X)$ is a first-countable sober c-space, and hence, $\Sigma\mathsf{K}(X)$ is both Rudin and WD.
(b) X is neither a Rudin space nor a WD space.
(c) the Smyth power space $P_S(X)$ is neither a Rudin space nor a WD space.

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Then for a T_0 space X, we investigate some sufficient conditions under which the well-filtered determinedness (resp. the Rudin property) of Scott space $\Sigma\mathsf{K}(X)$ implies that of X.

Definition 126. *A T_0 space X is said to have* property S *if for each $A \in \mathsf{Irr}_c(X)$, $\{\uparrow a : a \in A\} \in \mathsf{Irr}(\Sigma\mathsf{K}(X))$ or $\Diamond A \in \mathsf{Irr}_c(\Sigma\mathsf{K}(X))$. A poset P is said to have property S if ΣP has property S.*

Remark 127. Let X be a T_0 space and $A \in \mathsf{Irr}_c(X)$.

(1) Since $\xi_X : X \longrightarrow P_S(\mathsf{K}(X)), x \mapsto \uparrow x$, is continuous, by Lemmas 2 and 5, we have that $\{\uparrow a : a \in A\} \in \mathsf{Irr}(P_S(X))$ and $\mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow a : a \in A\} = \Diamond A \in \mathsf{Irr}_c(P_S(X))$
(2) If $\xi_X^\sigma : X \longrightarrow \Sigma\mathsf{K}(X), x \mapsto \uparrow x$, is continuous, then by Lemma 2, X has property S.
(3) If $\sigma(\mathsf{K}(X)) \subseteq \mathcal{O}(P_S(X))$, then $\xi_X^\sigma : X \longrightarrow \Sigma\mathsf{K}(X)$ is continuous by Remark 4, and hence, X has property S.
(4) For a poset P, by Lemmas 12 and 9, the mapping $\xi_P^\sigma : \Sigma P \longrightarrow \Sigma\mathsf{K}(\Sigma P), x \mapsto \uparrow x$, is continuous. Therefore, P has property S.

Proposition 128. *Suppose that a T_0 space X has property S and $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$. If $\Sigma\mathsf{K}(X)$ is a Rudin space, then X is a Rudin space.*

Proof. Let $A \in \mathsf{Irr}_c(X)$. Then by the property S of X, $\{\uparrow a : a \in A\} \in \mathsf{Irr}(\Sigma\mathsf{K}(X))$ or $\Diamond A \in \mathsf{Irr}_c(\Sigma\mathsf{K}(X))$.

Case 1: $\{\uparrow a : a \in A\} \in \mathsf{Irr}(\Sigma\mathsf{K}(X))$.

Since $\Sigma\mathsf{K}(X)$ is a Rudin space, there is a filtered family $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(\Sigma\mathsf{K}(X))$ such that $\mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow a : a \in A\} \in m(\{\mathcal{K}_d : d \in D\})$. As $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$, we have that $\mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow a : a \in A\} \subseteq \Diamond A \in \mathcal{C}(P_S(X)) \subseteq \mathcal{C}(\Sigma\,\mathsf{K}(X))$ and $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(P_S(X))$. Therefore, $\Diamond A \in M(\{\mathcal{K}_d : d \in D\})$. For each $d \in D$, let $K_d = \bigcup \mathcal{K}_d$. Then by Lemma 11, $\{K_d : d \in D\} \subseteq \mathsf{K}(X)$ is filtered. Since $\Diamond A \in M(\{\mathcal{K}_d : d \in D\})$, $A \in M(\{K_d : d \in D\})$. Now we show that $A \in m(\{K_d : d \in D\})$. Suppose that B is a proper closed subset B of A. Then, there is $a \in A \cap (X \setminus B)$, and hence $\uparrow a \in \Box(X \setminus B) \in \mathcal{O}(P_S(X) \subseteq \sigma(\mathsf{K}(X))$. Clearly, $\{\uparrow b : b \in B\} \cap \Box(X \setminus B) = \emptyset$, and consequently, $\uparrow a \notin \mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow b : b \in B\}$. Therefore, $\mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow b : b \in B\}$ is a proper subset of $\mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow a : a \in A\}$. By $\mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow a : a \in A\} \in m(\{\mathcal{K}_d : d \in D\})$, there is $d_0 \in D$ such that $\mathrm{cl}_{\sigma(\mathsf{K}(X))}\{\uparrow b : b \in B\} \cap \mathcal{K}_{d_0} = \emptyset$, and hence $\{\uparrow b : b \in B\} \cap \mathcal{K}_{d_0} = \emptyset$. Since $\mathcal{K}_{d_0} = \uparrow_{\mathsf{K}(X)} \mathcal{K}_{d_0}$, we have that $B \cap K_{d_0} = B \cap (\bigcup \mathcal{K}_{d_0}) = \emptyset$. Thus, $A \in m(\{K_d : d \in D\})$.

Case 2: $\Diamond A \in \mathsf{Irr}_c(\Sigma\mathsf{K}(X))$.

Since $\Sigma\mathsf{K}(X)$ is a Rudin space, there is a filtered family $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(\Sigma\mathsf{K}(X))$ such that $\Diamond A \in m(\{\mathcal{K}_d : d \in D\})$. As carried out in the proof of Case 1, A is a Rudin set of X.

Thus, X is a Rudin space. □

Corollary 129. *Suppose that X is a well-filtered space with property S. If $\Sigma\mathsf{K}(X)$ is a* WD *space (especially, a Rudin space), then both $\Sigma\mathsf{K}(X)$ and X are sober.*

Proof. By Theorem 58, $\Sigma\mathsf{K}(X)$ is well-filtered. As $\Sigma\mathsf{K}(X)$ is WD (if $\Sigma\mathsf{K}(X)$ is Rudin, then by Proposition 34 it is WD), by Theorem 36, $\Sigma\mathsf{K}(X)$ is sober. Hence, by Theorem 36 and Corollary 25, $\Sigma\mathsf{K}(X)$ is Rudin and $\mathcal{O}(P_S(X)) \subseteq \sigma(\mathsf{K}(X))$, and consequently, X is Rudin by Proposition 128. It follows from Theorem 36 that X is sober. □

By Remark 127 and Corollary 129, we have the following corollary.

Corollary 130. *Let X be a well-filtered space. If $\xi_X^\sigma : X \longrightarrow \Sigma\mathsf{K}(X)$ is continuous and $\Sigma\mathsf{K}(X)$ is a* WD *space (especially, a Rudin space), then both $\Sigma\mathsf{K}(X)$ and X are sober.*

As an immediate corollary of Corollary 130, we get one the main results of Xu et al. (2021*b*).

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Corollary 131. *(Xu et al. 2021b, Theorem 2) Suppose that X is a well-filtered space and $\xi_X^\sigma : X \longrightarrow \Sigma\mathsf{K}(X)$ is continuous. If $\Sigma\mathsf{K}(X)$ is sober, then X is sober. Therefore, if X is non-sober, then the Scott space $\Sigma\mathsf{K}(X)$ is non-sober.*

Example 67 shows that when X lacks the property S or the continuity of $\xi_X^\sigma : X \longrightarrow \Sigma\mathsf{K}(X)$, Proposition 128, Corollaries 129, 130, and 131 may not hold.

By Remark 127, Proposition 128, Corollaries 129 and 131, we deduce the following three corollaries.

Corollary 132. *Let P be a poset. If $\mathcal{O}(P_S(\Sigma P)) \subseteq \sigma(\mathsf{K}(\Sigma P))$ and $\mathsf{K}(\Sigma P)$ is a Rudin poset, then P is a Rudin dcpo.*

Corollary 133. *Let P be a well-filtered dcpo. If $\mathsf{K}(\Sigma P)$ is a WD dcpo (especially, a Rudin dcpo), then both $\mathsf{K}(\Sigma P)$ and P are sober dcpos.*

Corollary 134. *Let P be a well-filtered dcpo. If $\mathsf{K}(\Sigma P)$ is a sober dcpo, then P is a sober dcpo. Therefore, if P is not a sober dcpo, then $\mathsf{K}(\Sigma P)$ is not a sober dcpo.*

Example 135. Let L be the Isbell's lattice constructed in Isbell (1982). Then,

(a) L is not a sober dcpo (see Isbell 1982).
(b) L is a well-filtered dcpo by Proposition 22
(c) L is neither a Rudin dcpo nor a WD dcpo by (a), (b) and Corollary 44.
(d) $\mathsf{K}(\Sigma L)$ is a well-filtered dcpo by Theorem 58.
(e) $\mathsf{K}(\Sigma L)$ is a spatial frame (see Xu et al. 2021b, Lemma 1).
(f) $\mathsf{K}(\Sigma L)$ is not a sober dcpo by (a) and Corollary 134.
(g) $\mathsf{K}(\Sigma L)$ is neither a Rudin dcpo nor a WD dcpo by (a), (b) and Corollary 133.

Proposition 136. *Suppose that X is a T_0 space for which $\sigma(\mathsf{K}(X)) \subseteq \mathcal{O}(P_S(X))$. If $\Sigma\mathsf{K}(X)$ is well-filtered determined, then X is well-filtered determined.*

Proof. By Remark 127, X has property S. Let $A \in \mathsf{Irr}_c(X)$, Y a well-filtered space and $f : X \longrightarrow Y$ a continuous mapping. Then by $\sigma(\mathsf{K}(X)) \subseteq \mathcal{O}(P_S(X))$, Lemma 6 and Theorem 58, $\Sigma\mathsf{K}(Y)$ is well-filtered and $P_S^\sigma(f) : P_S(X) \longrightarrow \Sigma\mathsf{K}(Y)$ is continuous, where $P_S^\sigma(f)(K) = \uparrow f(K)$ for all $K \in \mathsf{K}(X)$. By assumption, $\{\uparrow a : a \in A\} \in \mathsf{Irr}(\Sigma\mathsf{K}(X))$ or $\Diamond A \in \mathsf{Irr}_c(\Sigma\mathsf{K}(X))$, and hence by the well-filtered determinedness of $\Sigma\mathsf{K}(Y)$ and the continuity of $P_S^\sigma(f)$, there exists a unique $Q \in \mathsf{K}(Y)$ such that $\overline{\{\uparrow f(a) : a \in A\}} = \overline{P_S^\sigma(f)(\{\uparrow a : a \in A\})} = \overline{\{Q\}} = \downarrow_{\mathsf{K}(Y)} Q$ in $\Sigma\mathsf{K}(Y)$ or $\overline{P_S^\sigma(f)(\Diamond A)} = \overline{\{Q\}} = \downarrow_{\mathsf{K}(Y)} Q$ in $\Sigma\mathsf{K}(Y)$. For the case of $\overline{P_S^\sigma(f)(\Diamond A)} = \overline{\{Q\}}$ in $\Sigma\mathsf{K}(Y)$, since $\overline{P_S^\sigma(f)(\Diamond A)} = \overline{P_S^\sigma(f)(\mathrm{cl}_{\mathcal{O}(P_S(X))}\xi_X(A))} = \overline{P_S^\sigma(f)(\xi_X(A))} = \overline{\{\uparrow f(a) : a \in A\}}$ in $\Sigma\mathsf{K}(Y)$, we also have $\overline{\{\uparrow f(a) : a \in A\}} = \overline{\{Q\}}$ in $\Sigma\mathsf{K}(Y)$. Since Y is well-filtered, by Corollary 25, $\mathcal{O}(P_S(Y)) \subseteq \sigma(\mathsf{K}(Y))$. Hence by Remark 1, $\mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow f(a) : a \in A\} = \mathrm{cl}_{\mathcal{O}(P_S(X))}\{Q\}$.

Claim 1: Q is supercompact.

Let $\{V_j : j \in J\} \subseteq \mathcal{O}(Y)$ with $Q \subseteq \bigcup_{j\in J} V_j$, i.e., $Q \in \Box \bigcup_{j\in J} V_j$. Since $\mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow f(a) : a \in A\} = cl_{\mathcal{O}(P_S(X))}\{Q\}$, we have $\{\uparrow f(a) : a \in A\} \cap \Box \bigcup_{j\in J} V_j \neq \emptyset$. Then, there exists $a_0 \in A$ and $j_0 \in J$ such that $\uparrow f(a_0) \subseteq V_{j_0}$, and consequently, $\{\uparrow a : a \in A\} \cap \Box U_{j_0} \neq \emptyset$. By $\mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow f(a) : a \in A\} = \mathrm{cl}_{\mathcal{O}(P_S(X))}\{Q\}$ again, we have $Q \in \Box U_{j_0}$, that is, $Q \subseteq U_{j_0}$.

Hence, by Heckmann and Keimel (2013, Fact 2.2), there exists $y_Q \in Y$ such that $Q = \uparrow y_Q$.

Claim 2: $\overline{f(A)} = \overline{\{y_Q\}}$ in Y.

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For each $y \in f(A)$, by $\mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow f(a) : a \in A\} = cl_{\mathcal{O}(P_S(X))}\{\uparrow y_Q\}$, we have that $\uparrow y \in \mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow y_Q\} = \downarrow_{\mathsf{K}(Y)} \uparrow y_Q$, whence $\uparrow y_Q \subseteq \uparrow y$, i.e., $y \in \overline{\{y_Q\}}$. This implies that $f(A) \subseteq \overline{\{y_Q\}}$. In addition, since $\uparrow y_Q \in \mathrm{cl}_{\mathcal{O}(P_S(X))}\{\uparrow f(a) : a \in A\} \subseteq \Diamond\overline{f(A)}$, $\uparrow y_Q \cap \overline{f(A)} \neq \emptyset$. It follows that $y_Q \in \overline{f(A)}$. Therefore, $\overline{f(A)} = \overline{\{y_Q\}}$.

By Claim 2, $A \in \mathsf{WD}(X)$, proving that X is well-filtered determined. □

Corollary 137. *For a poset P, if $\sigma(\mathsf{K}(\Sigma P)) \subseteq \mathcal{O}(P_S(\Sigma P))$ and $\mathsf{K}(\Sigma P))$ is a* WD *poset, then P is a* WD *poset.*

From Corollaries 44 and 137, we deduce the following result.

Corollary 138. *If P is a well-filtered dcpo, $\sigma(\mathsf{K}(\Sigma P)) \subseteq \mathcal{O}(P_S(\Sigma P))$ and $\mathsf{K}(\Sigma P)$ is a* WD *dcpo (especially, a Rudin dcpo), then both $\mathsf{K}(\Sigma P)$ and P are sober dcpos.*

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