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CONTRACTIVE SEMIGROUPS IN TOPOLOGICAL VECTOR SPACES, ON THE 100TH ANNIVERSARY OF STEFAN BANACH'S CONTRACTION PRINCIPLE

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Abstract

Celebrating 100 years of the Banach contraction principle, we prove some fixed point theorems having all ingredients of the principle, but dealing with common fixed points of a contractive semigroup of nonlinear mappings acting in a modulated topological vector space. This research follows the ideas of the author's recent papers ['On modulated topological vector spaces and applications', *Bull. Aust. Math. Soc.* **101** (2020), 325–332, and 'Normal structure in modulated topological vector spaces', *Comment. Math.* **60** (2020), 1–11]. Modulated topological vector spaces generalise, among others, Banach spaces and modular function spaces. The interest in modulars reflects the fact that the notions of 'norm like' but 'noneuclidean' (and not even necessarily convex) constructs to measure a level of proximity between complex objects are frequently used in science and technology. To prove our fixed point results in this setting, we introduce a new concept of Opial sets using analogies with the norm-weak and modular versions of the Opial property. As an example, the results of this work can be applied to spaces like L^p for $p > 0$, variable Lebesgue spaces $L^{p(\cdot)}$ where $1 \leq p(t) < +\infty$, Orlicz and Musielak–Orlicz spaces.

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1. Introduction

This year, 2022, the mathematical world is celebrating the 100th anniversary of Stefan Banach's fixed point theorem, typically referred to as the Banach contraction principle [1]. The standard formulation of the principle says that if $T : M \to M$ is a contraction, where M is a complete metric space, then T has a unique fixed point $z \in M$ and for every $x \in M$, $T^n(x) \to z$. Since 1922, thousands of variants and generalisations of the principle have been produced, providing new views on the main idea of the principle and enlarging its application domain. We can easily identify the four main ingredients of the Banach principle and all of them are usually present in such generalisations: (1) a set M equipped with a mathematical device to measure how distant from each other any two elements of the space are; (2) a contracting operation (or a family of such operations) acting on elements of M; (3) existence of a unique fixed point (or of a unique common fixed point) in M for this operation (or a family of operations); (4) a constructive method of approximating such a unique fixed point by the use of Picard's iterates. Any fixed point results that contain these four elements are definitely children or grand-children of Banach's principle.

In this paper, we present fixed point results that contain the four ingredients. As the set, we take a subset of a topological vector space equipped with a (not necessarily convex) modular as the device measuring the distance. We will prove the existence and uniqueness of a common fixed point for a contractive (in this modular sense) semigroup of nonlinear mappings acting in this set. And, yes, we will show that this fixed point is a limit of orbits being a semigroup equivalent of Picard's iterates. The paper uses the framework of modulated topological vector spaces (MTVSs) introduced in [10, 11]. Our fixed point theorems, besides being grand-children of Banach's theorem, generalise known results from the fixed point theory in Banach spaces (for example, Theorem 3.1 in the 2008 paper by Kirk and Xu [7]), and from the fixed point theory in modular function spaces (for example, Theorem 3.6 in Kozlowski's paper [9], see also [6, Theorem 7.1]). We need to emphasise that the results of the current paper (Theorems 3.5 and 3.9) do not assume convexity of a modular, nor do we assume any particular underlying measure theory structures as in [6, 9], which simplifies the exposition and widens the application domain. As a new technique, we introduce a concept of Opial sets, as a generalisation of the weak Opial property from Banach spaces. Interesting examples, including spaces as classical as L^p, demonstrate the advantages of the modular approach, since we know that many such spaces (like L^p spaces for $1 \leq p \neq 2$) do not satisfy the weak Opial condition.

2. Preliminaries

We use the framework of modulated topological vector spaces introduced by the author [10, 11]. We refer the reader to these papers for the detailed exposition of the theory and a list of examples. We need to note though that, in contrast with the cited papers, we do not assume here the convexity of the modular. This does not have any impact on the results of this theory that we need in the current work.

First, let us recall the definition of a modular, a modular space and ρ-convergence together with associated notions (see papers [10, 11] and also [6, 8] for further details).

Definition 2.1. A functional $\rho : X \to [0, \infty]$ is called a modular if:

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(-x) = \rho(x)$;
(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular is called a convex modular if it is a convex function. The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0, \text{ as } \lambda \to 0\}$ is called a modular space.

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DEFINITION 2.2. Let ρ be a modular defined on a vector space X.

(a) We say that $\{x_n\}$, a sequence of elements of X_ρ, is ρ-convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$.
(b) A sequence $\{x_n\}$, where $x_n \in X_\rho$, is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(c) X_ρ is called ρ-complete if every ρ-Cauchy sequence is ρ-convergent to an $x \in X_\rho$.
(d) A set $B \subset X_\rho$ is called ρ-closed if for any sequence of $x_n \in B$, the convergence $x_n \xrightarrow{\rho} x$ implies that x belongs to B.
(e) A set $B \subset X_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B)$, which is defined by $\delta_\rho(B) = \sup\{\rho(x - y) : x \in B, y \in B\}$, is finite.
(f) A set $K \subset X_\rho$ is called ρ-compact if for any $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.
(g) A ρ-ball $B_\rho(x, r)$ is defined by $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \le r\}$.

We are now ready to introduce the main concept of a modulated topological vector space.

DEFINITION 2.3. Let ρ be a modular defined on a real vector space X and let τ be a linear, Hausdorff topology on X_ρ. The triplet (X_ρ, ρ, τ) is called a modulated topological vector space if the following two conditions are satisfied:

(i) ρ is sequentially τ-lower semi-continuous on X, that is, $\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$, provided $x_n \xrightarrow{\tau} x$;
(ii) if $x_n \xrightarrow{\rho} x$, then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\tau} x$, where $x,\ x_n \in X$.

PROPOSITION 2.4 [10, Proposition 2.5]. *Let (X_ρ, ρ, τ) be a ρ-complete modulated topological vector space. The following assertions follow immediately from the above definitions.*

(i) *Every τ-closed set is also ρ-closed.*
(ii) *Every ρ-compact set is also sequentially τ-compact.*
(iii) *Every ρ-ball $B_\rho(x, r)$ is τ-closed (and hence also ρ-closed).*

As shown in the cited papers, the typical examples of modulated topological vector space are: Banach spaces with ρ being the norm and τ the weak topology; modular function spaces with τ being the topology of the convergence in sub-measure, and in particular L^p for $p > 0$; variable Lebesgue spaces $L^{p(\cdot)}$ where $1 \le p(t) < +\infty$; Orlicz and Musielak–Orlicz spaces, where τ denotes the topology of convergence in a finite measure m.

3. Fixed point theorems for contractive semigroups

Let us start with the definition of a contractive semigroup in a modulated topological vector space.

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Definition 3.1. Let C be a nonempty subset of a modulated topological vector space (X_ρ, ρ, τ) and let $\mathcal{T} = \{T_t : t \geq 0\}$ be a one-parameter family of mappings from C into itself. Then $\mathcal{T}$ is called a contractive semigroup on C if:

(i) $T_0(x) = x$ for any $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for any $x \in C$ and any $s, t \geq 0$;
(iii) for each $t \geq 0$, T_t is a contraction with a constant L_t, $0 < L_t < 1$, that is, we have $\rho(T_t(x) - T_t(y)) \leq L_t \rho(x - y)$ for all $x, y \in C$;
(iv) $\limsup_{t\to+\infty} L_t < 1$.

An element $x_0 \in C$ is called a common fixed point for $\mathcal{T}$ if $T_t(x_0) = x_0$ for every $t \geq 0$. The set (possibly empty) of all common fixed points for $\mathcal{T}$ will be denoted by $F(\mathcal{T})$.

The next result follows immediately from this definition.

Proposition 3.2. *If $\mathcal{T}$ is a contractive semigroup, then $F(\mathcal{T})$ consists at most of one element.*

The following 'convergence of orbits' lemma provides a constructive method of approximating a common fixed point knowing its existence.

Lemma 3.3. *Let $z \in F(\mathcal{T})$, where $\mathcal{T} = \{T_t : t \geq 0\}$ is a contractive semigroup on $C \subset X_\rho$. Then $T_t(u) \xrightarrow{\rho} z$ for every $u \in C$.*

Proof. Let us fix $u \in C$. Note that for any $s, t \geq 0$,

$$\rho(T_{t+s}(u) - z) = \rho(T_{t+s}(u) - T_t(z)) \leq L_t \rho(T_s(u) - z).$$

Hence,

$$\limsup_{s\to\infty} \rho(T_s(u) - z) = \limsup_{s\to\infty} \rho(T_{t+s}(u) - z) \leq L_t \limsup_{s\to\infty} \rho(T_s(u) - z)$$

for any $t \geq 0$, which implies that $\limsup_{s\to\infty} \rho(T_s(u) - z) = 0$ as $\limsup_{t\to+\infty} L_t < 1$. The same reasoning can be applied to lim inf giving the required convergence. □

Let us recall the concept of ρ-types.

Definition 3.4. Given a nonempty subset C of a modulated topological vector space (X_ρ, ρ, τ), and a sequence $\{x_n\}$ of elements from C (alternatively, a net $\{x_t\}_{t\in[0,+\infty)}$), a function $\Phi : C \to [0, +\infty]$ defined for any $z \in X$ by

$$\Phi(z) = \limsup_{n\to\infty} \rho(x_n - z),$$

alternatively by

$$\Phi(z) = \limsup_{t\to\infty} \rho(x_t - z),$$

is called a ρ-type on C.

We are now ready to prove our first fixed point result.

THEOREM 3.5. *Let C be a nonempty ρ-bounded subset of a ρ-complete modulated topological vector space (X_ρ, ρ, τ) and let $\mathcal{T} = \{T_t : t \geq 0\}$ be a contractive semigroup on C. Assume that there exists an $x \in C$ such that the ρ-type Φ defined for $u \in C$ by $\Phi(u) = \limsup_{t\to\infty} \rho(T_t(x) - u)$ attains its minimum in C. Then there exists a unique common fixed point $z \in F(\mathcal{T})$. Moreover, $\rho(T_t(u) - z) \to 0$ for every $u \in C$.*

PROOF. In view of Proposition 3.2 and Lemma 3.3, we need only to prove the existence of a common fixed point. By our assumption, there exists $z \in C$ such that $\Phi(z) = \inf\{\Phi(y) : y \in C\}$. We shall prove now that $\Phi(z) = 0$. To this end, we note first that for any $s, t \geq 0$,

$$\rho(T_{t+s}(x) - T_t(z)) \leq L_t \rho(T_s(x) - z),$$

and that by letting $s \to +\infty$, we get $\Phi(T_t(z)) \leq L_t \Phi(z)$, which, by letting t tend to infinity, implies that

$$\Phi(z) \leq \limsup_{t\to+\infty} \Phi(T_t(z)) \leq \limsup_{t\to+\infty} L_t \Phi(z).$$

Since $\limsup_{t\to+\infty} L_t < 1$, we conclude that $\Phi(z) = 0$, and hence, by the definition of Φ, for any $s \geq 0$,

$$\begin{aligned} 0 \leq \limsup_{t\to+\infty} \rho(T_t(x) - T_s(z)) &= \limsup_{t\to+\infty} \rho(T_{t+s}(x) - T_s(z)) \\ &\leq \limsup_{t\to+\infty} \rho(T_t(x) - z) = \Phi(z) = 0. \end{aligned}$$

Hence, $T_t(x) \xrightarrow{\rho} z$ and $T_t(x) \xrightarrow{\rho} T_s(z)$ for any $s \geq 0$, which, by the uniqueness of the ρ-limit, implies that $T_s(z) = z$ for any $s \geq 0$, that is, $z \in F(\mathcal{T})$, as claimed. □

To be able to effectively use Theorem 3.5, we need to have practical ways of assessing for which sets C the ρ-types attain their minimum. In the Banach space case, as is well known, this will be true if C is a nonempty, convex, bounded and weakly compact set (see, for example, [3, 12]). However, this result depends on some specific Banach space characteristics, like a triangle property of norms, and on the fact that a closed convex subset of a weakly compact set is weakly compact itself. Since in the setting of modulated topological vector spaces these properties are generally not available, we are going to introduce a powerful technique of τ-Opial sets, which will allow us to prove our second fixed point result, Theorem 3.9, and to provide a list of examples and applications. We shall begin by recalling a standard result, written below in the language of modulated topological vector spaces. We write l.s.c. to mean lower semi-continuous.

LEMMA 3.6. *Let C be a nonempty, sequentially τ-compact subset of a ρ-complete modulated topological vector space (X_ρ, ρ, τ). Let $\Psi : C \to [0, +\infty)$. If Ψ is sequentially τ-l.s.c., then Ψ attains its minimum in C.*

PROOF. Denote $\Psi_0 = \inf\{\Psi(y) : y \in C\}$. Let $\Psi_0 = \lim_{n\to\infty} \Psi(y_n)$ for some sequence $\{y_n\}$ of elements of C. By the sequential τ-compactness of C, we can choose a

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subsequence $\{y_{n_k}\}$ such that $y_{n_k} \xrightarrow{\tau} y$ for a $y \in C$. Using the assumption that Ψ is sequentially τ-l.s.c.,

$$\Psi_0 \leq \Psi(y) \leq \liminf_{k\to\infty} \Psi(y_{n_k}) = \Psi_0. \qquad \square$$

DEFINITION 3.7. Let C be a nonempty subset of a modulated topological vector space (X_ρ, ρ, τ). We say that C is a τ-Opial set (or, in short, a (τ-O) set) if for every $y \in C$ and every sequence $\{x_n\}$ of elements of C with $x_n \xrightarrow{\tau} x$ for an $x \in C$,

$$\liminf_{n\to\infty} \rho(x_n - y) = \liminf_{n\to\infty} \rho(x_n - x) + \rho(x - y).$$

THEOREM 3.8. *Let C be a nonempty, sequentially τ-compact, ρ-bounded subset of a ρ-complete modulated topological vector space (X_ρ, ρ, τ). If C is a τ-Opial set, then every ρ-type Φ on C is sequentially τ-l.s.c. and attains its minimum in C. Moreover, if $\{y_n\}$ is a sequence of elements of C such that $y_n \xrightarrow{\tau} y$, then*

$$\Phi(y) + \liminf_{n\to\infty} \rho(y_n - y) \leq \liminf_{n\to\infty} \Phi(y_n). \tag{3.1}$$

PROOF. Observe first that $\Phi(y) < +\infty$ for every $y \in C$ because C is ρ-bounded. It is obvious that to show that Φ is sequentially τ-l.s.c. (and hence that it attains its minimum, by Lemma 3.6), it is enough to prove inequality (3.1). To this end, let Φ be defined by a sequence $\{x_n\}$ of elements of C, that is, $\Phi(z) = \limsup_{n\to\infty} \rho(x_n - z)$ for any $z \in C$. Let us fix $y \in C$. Let $\{x_{p(n)}\}$ be a subsequence of $\{x_n\}$ such that $x_{p(n)} \xrightarrow{\tau} x$ for some $x \in C$ (recall that C is sequentially τ-compact) and that $\Phi(y) = \lim_{n\to\infty} \rho(x_{p(n)} - y)$. Let us fix temporarily $m \in \mathbb{N}$ and observe that

$$\Phi(y_m) = \limsup_{n\to\infty} \rho(x_n - y_m) \geq \limsup_{n\to\infty} \rho(x_{p(n)} - y_m) \geq \liminf_{n\to\infty} \rho(x_{p(n)} - y_m). \tag{3.2}$$

Since C is a (τ-O) set, it follows that

$$\liminf_{n\to\infty} \rho(x_{p(n)} - y_m) = \liminf_{n\to\infty} \rho(x_{p(n)} - x) + \rho(x - y_m),$$

and hence by (3.2),

$$\Phi(y_m) \geq \liminf_{n\to\infty} \rho(x_{p(n)} - x) + \rho(x - y_m).$$

By taking $m \to \infty$, we get

$$\liminf_{m\to\infty} \Phi(y_m) \geq \liminf_{n\to\infty} \rho(x_{p(n)} - x) + \liminf_{m\to\infty} \rho(x - y_m). \tag{3.3}$$

Using (τ-O) again, this time with $\liminf_{m\to\infty} \rho(x - y_m)$, and substituting into (3.3),

$$\liminf_{n\to\infty} \Phi(y_n) \geq \liminf_{n\to\infty} \rho(x_{p(n)} - x) + \liminf_{n\to\infty} \rho(y_n - y) + \rho(y - x). \tag{3.4}$$

However,

$$\Phi(y) = \limsup_{n\to\infty} \rho(x_n - y) = \lim_{n\to\infty} \rho(x_{p(n)} - y) = \liminf_{n\to\infty} \rho(x_{p(n)} - y), \tag{3.5}$$

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and using (τ-O) on the right-hand side of (3.5),

$$\Phi(y) = \liminf_{n\to\infty} \rho(x_{p(n)} - x) + \rho(x - y),$$

which, combined with (3.4), gives us finally

$$\Phi(y) + \liminf_{n\to\infty} \rho(y_n - y) \le \liminf_{n\to\infty} \Phi(y_n),$$

as claimed. □

Combining Theorems 3.5 and 3.8, we immediately obtain the following fixed point result.

THEOREM 3.9. *Let C be a ρ-bounded and sequentially τ-compact subset of a ρ-complete modulated topological vector space (X_ρ, ρ, τ). Let $\mathcal{T} = \{T_t : t \ge 0\}$ be a contractive semigroup on C. If C is a τ-Opial set, then there exists a unique common fixed point $z \in F(\mathcal{T})$. Moreover, $\rho(T_t(u) - z) \to 0$ for every $u \in C$.*

While the notion of an Opial set as defined in Definition 3.7, alluding to the celebrated weak Opial property [13], is actually a novel concept, it has been known since the 1996 work by Khamsi [4] (see also [6, Theorem 4.7]) that in every Δ_2 modular function space defined by a convex, orthogonally additive modular ρ, every ρ-bounded set is an Opial set. Subject to some technicalities, the same also holds without Δ_2. This fact gives a wide range of function spaces where bounded sets are Opial sets, including L^p for $p \ge 1$, variable Lebesgue spaces $L^{p(\cdot)}$ where $1 \le p(t) < +\infty$, Orlicz and Musielak–Orlicz spaces. Let us recall the fact observed already in [13] that the weak Opial property does not hold in L^p spaces for $1 \le p \ne 2$, and hence there are bounded sets which are not Opial sets with $\rho = \|\cdot\|_p$, but, as seen above, they are Opial sets with $\rho(x) = \int_{[0,1]} |x(t)|^p \, dm(t)$. It is also important to notice the nonconvex modular case for $0 < p < 1$ where all sets are Opial sets, due to the fact that $\lim_{n\to+\infty}\{\|x_n\|_p^p - \|x_n - x\|_p^p\} = \|x\|_p^p$, whenever $x_n \to x$ almost everywhere, as shown already in [2].

In view of these remarks, Theorem 3.9 is a generalisation of the results proven in the context of regular convex modular function spaces (see, for example, [5, Theorem 4.2], [9, Theorem 3.6] and [6, Theorem 7.1]) that can be applied to spaces like L^p for $p > 0$, variable Lebesgue spaces $L^{p(\cdot)}$ where $1 \le p(t) < +\infty$, Orlicz and Musielak–Orlicz spaces.

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