1 Decoupling and Hierarchies of Scale

The world around us contains a cornucopia of length scales, ranging (at the time of writing) down to quarks and leptons at the smallest and up to the universe as a whole at the largest, with qualitatively new kinds of structures – nuclei, atoms, molecules, cells, organisms, mountains, asteroids, planets, stars, galaxies, voids, and so on – seemingly arising at every few decades of scales in between. So it is remarkable that all of this diversity seems to be described in all of its complexity by a few simple laws.

How can this be possible? Even given that the simple laws exist, why should it be possible to winkle out an understanding of what goes on at one scale without having to understand everything all at once? The answer seems to be a very deep property of nature called *decoupling*, which states that most (but not all) of the details of very small-distance phenomena tend to be largely irrelevant for the description of much larger systems. For example, not much needs to be known about the detailed properties of nuclei (apart from their mass and electrical charge, and perhaps a few of their multipole moments) in order to understand in detail the properties of electronic energy levels in atoms.

Decoupling is a very good thing, since it means that the onion of knowledge can be peeled one layer at a time: our initial ignorance of nuclei need not impede the unravelling of atomic physics, just as ignorance about atoms does not stop working out the laws describing the motion of larger things, like the behaviour of fluids or motion of the moon.

It so happens that this property of decoupling is also shared by the mathematics used to describe the laws of nature [1]. Since nowadays this description involves quantum field theories, it is gratifying that these theories as a group tend to predict that short distances generically decouple from long distances, in much the same way as happens in nature.

This book describes the way this happens in detail, with two main purposes in mind. One purpose is to display decoupling for its own sake since this is satisfying in its own right, and leads to deep insights into what precisely is being accomplished when writing down physical laws. But the second purpose is very practical; the simplicity offered by a timely exploitation of decoupling can often be the difference between being able to solve a problem or not. When exploring the consequences of a particular theory for short distance physics it is obviously useful to be able to identify efficiently those observables that are most sensitive to the theory's details and those from which they decouple. As a consequence the mathematical tools – *effective field theories* – for exploiting decoupling have become ubiquitous in some areas of theoretical physics, and are likely to become more common in many more.

The purpose of the rest of Chapter 1 is twofold. One goal is to sketch the broad outlines of decoupling, effective lagrangians and the physical reason why they work,

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all in one place. The second aim is to provide a toy model that can be used as a concrete example as the formalism built on decoupling is fleshed out in more detail in subsequent chapters.

1.1 An Illustrative Toy Model $^\diamond$

The first step is to set up a simple concrete model to illustrate the main ideas. To be of interest this model must possess two kinds of particles, one of which is much heavier than the other, and these particles must interact in a simple yet nontrivial way. Our focus is on the interactions of the two particles, with a view towards showing precisely how the heavy particle decouples from the interactions of the light particle at low energies.

To this end consider a complex scalar field, ϕ, with action[1]

$$S := -\int \mathrm{d}^4x \left[\partial_\mu \phi^* \partial^\mu \phi + V(\phi^* \phi)\right], \tag{1.1}$$

whose self-interactions are described by a simple quartic potential,

$$V(\phi^*\phi) = \frac{\lambda}{4}\left(\phi^*\phi - v^2\right)^2, \tag{1.2}$$

where λ and v^2 are positive real constants. The shape of this potential is shown in Fig. 1.1.

1.1.1 Semiclassical Spectrum

The simplest regime in which to explore the model's predictions is when $\lambda \ll 1$ and both v and $|\phi|$ are $O\left(\lambda^{-1/2}\right)$. This regime is simple because it is one for which the semiclassical approximation provides an accurate description. (The relevance of the semiclassical limit in this regime can be seen by writing $\phi := \varphi/\lambda^{1/2}$ and $v := \mu/\lambda^{1/2}$ with φ and μ held fixed as $\lambda \to 0$. In this case the action depends on λ only through an overall factor: $S[\phi, v, \lambda] = (1/\lambda)S[\varphi, \mu]$. This is significant because the action appears in observables only in the combination $S/\hbar$, and so the small-λ limit is equivalent to the small-$\hbar$ (classical) limit.)[2]

In the classical limit the ground state of this system is the field configuration that minimizes the classical energy,

$$E = \int \mathrm{d}^3x \left[\partial_t\phi^*\partial_t\phi + \nabla\phi^* \cdot \nabla\phi + V(\phi^*\phi)\right]. \tag{1.3}$$

Since this is the sum of positive terms it is minimized by setting each to zero; the classical ground state is any constant configuration (so $\partial_t\phi = \nabla\phi = 0$), with $|\phi| = v$ (so $V = 0$).

[1] Although this book presupposes some familiarity with quantum field theory, see Appendix C for a compressed summary of some of the relevant ideas and notation used throughout. Unless specifically stated otherwise, units are adopted for which $\hbar = c = 1$, so that time ~ length and energy ~ mass ~ 1/length, as described in more detail in Appendix A.

[2] The connection between small coupling and the semi-classical limit is explored more fully once power-counting techniques are discussed in §3.

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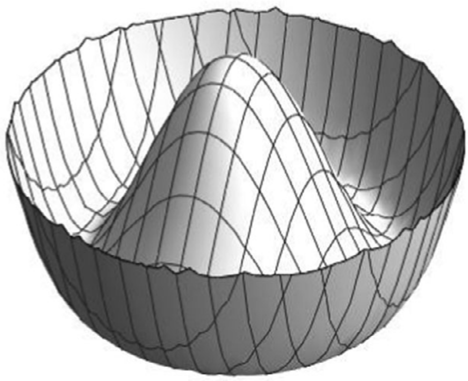

Fig. 1.1 The potential $V(\phi_R, \phi_I)$, showing its sombrero shape and the circular line of minima at $|\phi| = v$.

In the semi-classical regime, particle states are obtained by expanding the action about the classical vacuum, $\phi = v + \tilde{\phi}$,

$$S = -\int d^4x \left\{ \partial_\mu \tilde{\phi}^* \partial^\mu \tilde{\phi} + \frac{\lambda}{4} \left[v(\tilde{\phi} + \tilde{\phi}^*) + \tilde{\phi}^* \tilde{\phi} \right]^2 \right\}, \tag{1.4}$$

and keeping the leading (quadratic) order in the quantum fluctuation $\tilde{\phi}$. In terms of the field's real and imaginary parts, $\tilde{\phi} = \frac{1}{\sqrt{2}}(\tilde{\phi}_R + i\tilde{\phi}_I)$, the leading term in the expansion of S is

$$S_0 = -\frac{1}{2} \int d^4x \left[\partial_\mu \tilde{\phi}_R \, \partial^\mu \tilde{\phi}_R + \partial_\mu \tilde{\phi}_I \, \partial^\mu \tilde{\phi}_I + \lambda v^2 \, \tilde{\phi}_R^2 \right]. \tag{1.5}$$

The standard form (see §C.3.1) for the action of a free, real scalar field of mass m is proportional to $\partial_\mu \psi \, \partial^\mu \psi + m^2 \psi^2$, and so comparing with Eq. (1.5) shows $\tilde{\phi}_R$ represents a particle with mass $m_R^2 = \lambda v^2$ while $\tilde{\phi}_I$ represents a particle with mass $m_I^2 = 0$. These are the heavy and light particles whose masses provide a hierarchy of scales.

1.1.2 Scattering

For small λ the interactions amongst these particles are well-described in perturbation theory, by writing $S = S_0 + S_{\text{int}}$ and perturbing in the interactions

$$S_{\text{int}} = -\int d^4x \left[\frac{\lambda v}{2\sqrt{2}} \, \tilde{\phi}_R \left(\tilde{\phi}_R^2 + \tilde{\phi}_I^2 \right) + \frac{\lambda}{16} \left(\tilde{\phi}_R^2 + \tilde{\phi}_I^2 \right)^2 \right]. \tag{1.6}$$

Using this interaction, a straightforward calculation – for a summary of the steps involved see Appendix B – gives any desired scattering amplitude order-by-order in λ. Since small λ describes a semiclassical limit (because it appears systematically together with $\hbar$ in $S/\hbar$, as argued above), the leading contribution turns out to come from evaluating Feynman graphs with no loops[3] (*i.e.* tree graphs).

[3] A connected graph with no loops (or a 'tree' graph) is one which can be broken into two disconnected parts by cutting any internal line. Precisely how to count the number of loops and why this is related to powers of the small coupling λ is the topic of §3.

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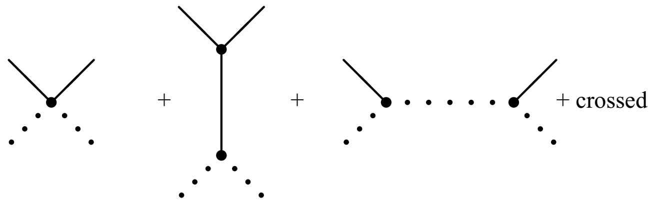

Fig. 1.2 The tree graphs that dominate $\tilde{\phi}_R\,\tilde{\phi}_I$ scattering. Solid (dotted) lines represent $\tilde{\phi}_R$ ($\tilde{\phi}_I$), and 'crossed' graphs are those with external lines interchanged relative to those displayed.

Consider the reaction $\tilde{\phi}_R(p) + \tilde{\phi}_I(q) \to \tilde{\phi}_R(p') + \tilde{\phi}_I(q')$, where $p^\mu = \{p^0, \mathbf{p}\}$ and $q^\mu = \{q^0, \mathbf{q}\}$ respectively denote the 4-momenta of the initial $\tilde{\phi}_R$ and $\tilde{\phi}_I$ particle, while p'^μ and q'^μ are 4-momenta of the final $\tilde{\phi}_R$ and $\tilde{\phi}_I$ states. The Feynman graphs of Fig. 1.2 give a scattering amplitude proportional to[4] $\mathcal{A}_{RI\to RI}\,\delta^4(p+q-p'-q')$, where the Dirac delta function, $\delta^4(p+q-p'-q')$, expresses energy–momentum conservation, and

$$\mathcal{A}_{RI\to RI} = 4\mathrm{i}\left(-\frac{\lambda}{8}\right) + \left(\frac{\mathrm{i}^2}{2}\right)\left(-\frac{\lambda v}{2\sqrt{2}}\right)^2 \left[\frac{24(-\mathrm{i})}{(p-p')^2+m_R^2} + \frac{8(-\mathrm{i})}{(p+q)^2} + \frac{8(-\mathrm{i})}{(p-q')^2}\right]$$
$$= -\frac{\mathrm{i}\lambda}{2} + \frac{\mathrm{i}(\lambda v)^2}{2m_R^2}\left[\frac{3}{1-2q\cdot q'/m_R^2} - \frac{1}{1-2p\cdot q/m_R^2} - \frac{1}{1+2p\cdot q'/m_R^2}\right]. \tag{1.7}$$

Here the factors like 4, 24 and 8 in front of various terms count the combinatorics of how many ways each particular graph can contribute to the amplitude. The second line uses energy–momentum conservation, $(p-p')^\mu = (q'-q)^\mu$, as well as the kinematic conditions $p^2 = -(p^0)^2 + \mathbf{p}^2 = -m_R^2$ and $(q')^2 = q^2 = -(q^0)^2 + \mathbf{q}^2 = 0$, as appropriate for relativistic particles whose energy and momenta are related by $E = p^0 = \sqrt{\mathbf{p}^2 + m^2}$.

Notice that the terms involving the square bracket arise at the same order in λ as the first term, despite nominally involving two powers of $S_{\rm int}$ rather than one (provided that the square bracket itself is order unity). To see this, keep in mind $m_R^2 = \lambda v^2$ so that $(\lambda v/m_R)^2 = \lambda$.

For future purposes it is useful also to have the corresponding result for the reaction $\tilde{\phi}_I(p) + \tilde{\phi}_I(q) \to \tilde{\phi}_I(p') + \tilde{\phi}_I(q')$. A similar calculation, using instead the Feynman graphs of Fig. 1.3, gives the scattering amplitude

$$\mathcal{A}_{II\to II} = 24\mathrm{i}\left(-\frac{\lambda}{16}\right) + 8\left(\frac{\mathrm{i}^2}{2}\right)\left(-\frac{\lambda v}{2\sqrt{2}}\right)^2$$
$$\times\left[\frac{-\mathrm{i}}{(p+q)^2+m_R^2} + \frac{-\mathrm{i}}{(p-p')^2+m_R^2} + \frac{-\mathrm{i}}{(p-q')^2+m_R^2}\right]$$
$$= -\frac{3\mathrm{i}\lambda}{2} + \frac{\mathrm{i}(\lambda v)^2}{2m_R^2}\left[\frac{1}{1+2p\cdot q/m_R^2} + \frac{1}{1-2q\cdot q'/m_R^2} + \frac{1}{1-2p\cdot q'/m_R^2}\right]. \tag{1.8}$$

[4] See Exercise 1.1 and Appendix B for the proportionality factors.

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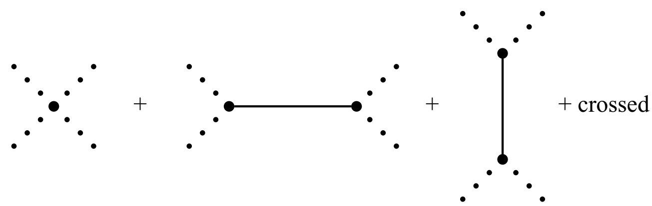

Fig. 1.3 The tree graphs that dominate the $\tilde{\phi}_I\,\tilde{\phi}_I$ scattering amplitude. Solid (dotted) lines represent $\tilde{\phi}_R$ and $\tilde{\phi}_I$ particles.

1.1.3 The Low-Energy Limit

For the present purposes it is the low-energy regime that is of most interest: when the centre-of-mass kinetic energy and momentum transfers during scattering are very small compared with the mass of the heavy particle. This limit is obtained from the above expressions by taking $|p\cdot q|$, $|p\cdot q'|$ and $|q\cdot q'|$ all to be small compared with m_R^2.

Taylor expanding the above expressions shows that both $\mathcal{A}_{RI\to RI}$ and $\mathcal{A}_{II\to II}$ are suppressed in this limit by powers of $(q$ or $q')/m_R$, in addition to the generic small perturbative factor λ:

$$\mathcal{A}_{RI\to RI} \simeq 2\mathrm{i}\lambda \left(\frac{q\cdot q'}{m_R^2}\right) + \mathcal{O}\left(m_R^{-4}\right), \tag{1.9}$$

while

$$\mathcal{A}_{II\to II} \simeq 2\mathrm{i}\lambda \left[\frac{(p\cdot q)^2 + (p\cdot q')^2 + (q\cdot q')^2}{m_R^4}\right] + \mathcal{O}\left(m_R^{-6}\right). \tag{1.10}$$

Both of these expressions use 4-momentum conservation, and kinematic conditions like $q^2 = 0$ *etc.* to simplify the result, and both expressions end up being suppressed by powers of q/m_R and/or q'/m_R once this is done.

The basic simplicity of physics at low energies arises because physical quantities typically simplify when Taylor expanded in powers of any small energy ratios (like scattering energy/m_R in the example above). It is this simplicity that ultimately underlies the phenomenon of decoupling: in the toy model the low-energy implications of the very energetic $\tilde{\phi}_R$ states ultimately can be organized into a sequence in powers of m_R^{-2}, with only the first few terms relevant at very low energies.

1.2 The Simplicity of the Low-Energy Limit ◇

Now imagine that your task is to build an experiment to test the above theory by measuring the cross section for scattering $\tilde{\phi}_I$ particles from various targets, using only accelerators whose energies, E, do not reach anywhere near as high as the mass m_R. Since the experiment is more difficult if the scattering is rare, the suppression of the order-λ cross sections by powers of q/m_R and/or q'/m_R at low energies presents a potential problem. But maybe this suppression is an accident of the leading, $\mathcal{O}(\lambda)$,

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prediction? If the $O(\lambda^2)$ result is not similarly suppressed, then it might happen that $\mathcal{A} \simeq \lambda^2$ is measurable even if $\mathcal{A} \simeq \lambda(E/m_R)^2$ is not.

It turns out that the suppression of $\tilde{\phi}_I$ scattering at low energies persists order-by-order in the λ expansion, so any hope of evading it by working to higher orders would be in vain. But the hard way to see this is to directly compute the $O(\lambda^n)$ amplitude as a complete function of energy, and then take the low-energy limit. It would be much more efficient if it were possible to zero in directly on the low-energy part of the result *before* investing great effort into calculating the complete answer. Any simplicity that might emerge in the low-energy limit then would be much easier to see.

Indeed, a formalism exists precisely for efficiently identifying the nature of physical quantities in the low-energy limit – *effective field theories* – and it is this formalism that is the topic of this book. This formalism exists and is so useful because one is often in the situation of being faced with a comparatively simple low-energy limit of some, often poorly understood, more complicated system.

The main idea behind this formalism is to take advantage of the low-energy approximation as early as possible in a calculation, and the best way to do so is directly, once and for all, in the action (or Hamiltonian or Lagrangian), rather than doing it separately for each independent observable. But how can the low-energy expansion be performed directly in the action?

1.2.1 Low-Energy Effective Actions

To make this concrete for the toy model discussed above, a starting point is the recognition that the low-energy limit, Eq. (1.10), of $\mathcal{A}_{II\to II}$ has precisely the form that would be expected (at leading order of perturbation theory) if the $\tilde{\phi}_I$ particles scattered only through an *effective* interaction of the form $S_{\rm eff} = S_{\rm eff\,0} + S_{\rm eff\,int}$, with

$$S_{\rm eff\,0} = -\frac{1}{2}\int d^4x\; \partial_\mu\tilde{\phi}_I\,\partial^\mu\tilde{\phi}_I, \tag{1.11}$$

and

$$S_{\rm eff\,int} = \frac{\lambda}{4m_R^4}\int d^4x\; (\partial_\mu\tilde{\phi}_I\,\partial^\mu\tilde{\phi}_I)(\partial_\nu\tilde{\phi}_I\,\partial^\nu\tilde{\phi}_I), \tag{1.12}$$

up to terms of order λ^2 and/or m_R^{-6}.

What is less obvious at this point, but nonetheless true (and argued in detail in the chapters that follow), is that this same effective interaction, Eqs. (1.11) and (1.12), also correctly captures the leading low-energy limit of other scattering processes, such as for $\tilde{\phi}_I\tilde{\phi}_I \to \tilde{\phi}_I\tilde{\phi}_I\tilde{\phi}_I\tilde{\phi}_I$ and reactions involving still more $\tilde{\phi}_I$ particles. That is, *all* amplitudes obtained from the full action, Eqs. (1.5) and (1.6), precisely agree with those obtained from the effective action, Eqs. (1.11) and (1.12), provided that the predictions of both theories are expanded only to leading order in λ and m_R^{-2} [2].

Given that a low-energy action like $S_{\rm eff}$ exists, it is clear that it is much easier to study the system's low-energy limit by first computing $S_{\rm eff}$ and then using $S_{\rm eff}$ to work out any observable of interest, than it is to calculate all observables using $S_0 + S_{\rm int}$ of Eqs. (1.5) and (1.6), and only then expanding them to find their low-energy form.

As an example of this relative simplicity, because each factor of $\tilde{\phi}_I$ appears differentiated in Eq. (1.12), it is obvious that the amplitudes for more complicated

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scattering processes computed with it are also suppressed by high powers of the low-energy scattering scale. For instance, the amplitude for $\tilde{\phi}_I\tilde{\phi}_I \to N\tilde{\phi}_I$ (into N final particles) computed using tree graphs built using just the quartic interaction $S_{\rm eff\,int}$ would be expected to give an amplitude proportional to at least

$$\mathcal{A}_{II\to I\cdots I} \propto \lambda^{N/2}\left(\frac{\text{scattering energy}}{m_R}\right)^{N+2} \tag{1.13}$$

in the low-energy limit. Needless to say, this type of low-energy suppression is much harder to see when using the full action, Eqs. (1.5) and (1.6).

It may seem remarkable that an interaction like $S_{\rm eff}$ exists that completely captures the leading low-energy limit of the full theory in this way. But what is even more remarkable is that a similar effective action also exists that reproduces the predictions of the full theory to *any* fixed higher order in λ and m_R^{-2}. This more general effective action replaces Eq. (1.12) by

$$S_{\rm eff\,int} = \int d^4x\; \mathfrak{L}_{\rm eff\,int}, \tag{1.14}$$

where

$$\mathfrak{L}_{\rm eff\,int} = a\,(\partial_\mu\tilde{\phi}_I\partial^\mu\tilde{\phi}_I)(\partial_\nu\tilde{\phi}_I\partial^\nu\tilde{\phi}_I) + b\,(\partial_\mu\tilde{\phi}_I\partial^\mu\tilde{\phi}_I)(\partial_\nu\tilde{\phi}_I\partial^\nu\tilde{\phi}_I)(\partial_\rho\tilde{\phi}_I\partial^\rho\tilde{\phi}_I) + \cdots, \tag{1.15}$$

where the ellipses represent terms involving additional powers of $\partial_\mu\hat{\phi}_I$ and/or its derivatives, though only a finite number of such terms is required in order to reproduce the full theory to a fixed order in λ and m_R^{-2}.

In principle, the coefficients a and b in Eq. (1.15) are given as a series in λ once the appropriate power of m_R is extracted on dimensional grounds,

$$a = \frac{1}{m_R^4}\left[\frac{\lambda}{4} + a_2\lambda^2 + O(\lambda^3)\right] \quad\text{and}\quad b = \frac{1}{m_R^8}\left[b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + O(\lambda^4)\right], \tag{1.16}$$

which displays explicitly the order-λ value for a found above that reproduces low-energy scattering in the full theory. Explicit calculations in later sections also show $b_1 = 0$. More generally, to the extent that the leading (classical, or tree-level) part of the action should be proportional to $1/\lambda$ once m_R is eliminated for v using $m_R^2 = \lambda v^2$ (as is argued above, and in more detail in Eq. (2.24) and §3), it must also be true that b_2 vanishes.

1.2.2 Why It Works

Why is it possible to find an effective action capturing the low-energy limit of a theory, along the lines described above? The basic idea goes as follows.

It is not in itself surprising that there is some sort of Hamiltonian describing the time evolution of low-energy states. After all, in the full theory time evolution is given by a unitary operation

$$|\psi_f(t)\rangle = U(t,t')\,|\psi_i(t')\rangle, \tag{1.17}$$

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where $U(t,t') = \exp[-\mathrm{i}H(t-t')]$ with a Hamiltonian[5] $H = H(\hat{\phi}_R, \hat{\phi}_l)$ depending on both the heavy and light fields. But if the initial state has an energy $E_i < m_R$ it cannot contain any $\hat{\phi}_R$ particles, and energy conservation then precludes $\hat{\phi}_R$ particles from ever being produced by subsequent time evolution.

This means that time evolution remains a linear and unitary transformation even when it is restricted to low-energy states. That is, suppose we define

$$U_{\text{eff}}(t,t') := P_\Lambda\, U(t,t')\, P_\Lambda := \exp\left[-\mathrm{i}H_{\text{eff}}(t-t')\right], \tag{1.18}$$

with $P_\Lambda^2 = P_\Lambda$ being the projection operator onto states with low energy $E < \Lambda \ll m_R$. P_Λ commutes with H and so also with time evolution. Because $H_{\text{eff}} = P_\Lambda H P_\Lambda$ if H is hermitian then so must be H_{eff} and so if $U(t,t')$ is unitary then so must be $U_{\text{eff}}(t,t')$ when acting on low-energy states.

Furthermore, because the action of H_{eff} is well-defined for states having energy $E < \Lambda$, it can be written as a linear combination of products of creation and annihilation operators for the $\hat{\phi}_l$ field only (since these form a basis for operators that transform among only low-energy states).[6] As a consequence, it must be possible to write $H_{\text{eff}} = H_{\text{eff}}[\hat{\phi}_l]$, without making any reference to the heavy field $\hat{\phi}_R$ at all.

But there is no guarantee that the expression for $H_{\text{eff}}[\hat{\phi}_l]$ obtained in this way is anywhere as simple as is $H[\hat{\phi}_R, \hat{\phi}_l]$. So the real puzzle is why the effective interaction found above is so simple. In particular, why is it local,

$$H_{\text{eff}}[\hat{\phi}_l] = \int \mathrm{d}^3x\; \mathcal{H}_{\text{eff}}(x), \tag{1.19}$$

with $\mathcal{H}_{\text{eff}}(x)$ a simple polynomial in $\hat{\phi}_l(x)$ and its derivatives, all evaluated at the same spacetime point?

Ultimately, the simplicity of this local form can be traced to the uncertainty principle. Interactions, like Eq. (1.12), in H_{eff} not already present in H describe the influence on low-energy $\hat{\phi}_l$ particles of virtual processes involving heavy $\hat{\phi}_R$ particles. These virtual processes are not ruled out by energy conservation even though the production of real $\hat{\phi}_R$ particles is forbidden. One way to understand why they are possible is because the uncertainty principle effectively allows energy conservation to be violated,[7] $E_f = E_i + \Delta E$, but only over time intervals that are sufficiently short, $\Delta t \lesssim \hbar/\Delta E$. The effects of virtual $\hat{\phi}_R$ particles are necessarily localized in time over intervals that are of order $1/m_R$, which are unobservably short for observers restricted to energies $E \ll m_R$. Consequently, they are described at these energies by operators all evaluated at effectively the same time.

In relativistic theories, large momenta necessarily involve large energies and since the uncertainty principle relates large momenta to short spatial distances, a similar argument can be made that the effect of large virtual momentum transfers on the

[5] The convention here is to use $\tilde{\phi}$ to denote the fluctuation when this is a non-operator field (appearing within a path integral, say) and instead use $\hat{\phi}$ for the quantum operator fluctuation field.
[6] See the discussion around Eq. (C.9) of Appendix C for details.
[7] More precisely, energy need not be conserved at each vertex when organized in old-fashioned Rayleigh–Schrödinger perturbation theory from undergraduate quantum mechanics classes. Once reorganized into manifestly relativistic Feynman–Schwinger–Dyson perturbation theory energy actually *is* preserved at each vertex, but internal particles are not on-shell: $E \neq \sqrt{\mathbf{p}^2 + m^2}$. Either way the locality consequences are the same.

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low-energy theory can also be captured by effective interactions localized at a single spatial point. Together with the localization in time just described, this shows that the effects of very massive particles are local in both space and time, as found in the toy model above.

Locality arises explicitly in relativistic calculations when expanding the propagators of massive particles in inverse powers of m_R, after which they become local in spacetime since

$$G(x,y) := \langle 0|T\hat{\phi}_R(x)\hat{\phi}_R(y)|0\rangle = -\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}}{p^2+m_R^2} \tag{1.20}$$

$$\simeq -\frac{\mathrm{i}}{m_R^2}\sum_{k=0}^{\infty}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(-\frac{p^2}{m_R^2}\right)^k e^{ip(x-y)} = -\frac{\mathrm{i}}{m_R^2}\sum_{k=0}^{\infty}\left(\frac{\Box}{m_R^2}\right)^k \delta^4(x-y),$$

where the 'T' denotes time ordering, $p(x-y) := p\cdot(x-y) = p_\mu(x-y)^\mu$ and $\Box = \partial_\mu\partial^\mu = -\partial_t^2 + \nabla^2$ is the covariant d'Alembertian operator.

The upshot is this: to any fixed order in $1/m_R$ the full theory usually can be described by a local effective lagrangian.[8] The next sections develop tools for its efficient calculation and use.

1.2.3 Symmetries: Linear vs Nonlinear Realization

Before turning to the nitty gritty of how the effective action is calculated and used, it is worth first pausing to extract one more useful lesson from the toy model considered above. The lesson is about symmetries and their low-energy realization, and starts by asking why it is that the self-interactions among the light $\hat{\phi}_I$ particles – such as the amplitudes of Eqs. (1.9) and (1.10) – are so strongly suppressed at low energies by powers of $1/m_R^2$.

That is, although it is natural to expect some generic suppression of low-energy interactions by powers of $1/m_R^2$, as argued above, why does nothing at all arise at zeroeth order in $1/m_R$ despite the appearance of terms like $\lambda\hat{\phi}_I^4$ in the full toy-model potential? And why are there so very many powers of $1/m_R$ in the case of $2\hat{\phi}_I \to N\hat{\phi}_I$ scattering in the toy model? (Specifically, why is the amplitude for two $\hat{\phi}_I$ particles scattering to N $\hat{\phi}_I$ particles suppressed by $(1/m_R)^{N+2}$?)

This suppression has a very general origin, and can be traced to a symmetry of the underlying theory [3–5]. The symmetry in question is invariance under the $U(1)$ phase rotation, $\phi \to e^{\mathrm{i}\omega}\phi$, of Eqs. (1.1) and (1.2). In terms of the real and imaginary parts this acts as

$$\begin{pmatrix}\phi_R\\ \phi_I\end{pmatrix} \to \begin{pmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{pmatrix}\begin{pmatrix}\phi_R\\ \phi_I\end{pmatrix}. \tag{1.21}$$

A symmetry such as this that acts linearly on the fields is said to be *linearly realized*. As summarized in Appendix C.4, if the symmetry is also linearly realized on particle states then these states come in multiplets of the symmetry, all elements of which share the same couplings and masses. However (as is also argued in

[8] For nonrelativistic systems locality sometimes breaks down in space (*e.g.* when large momenta coexist with low energy). It can also happen that the very existence of a Hamiltonian (without expanding the number of degrees of freedom) breaks down for open systems – the topic of §16.

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Appendix C.4) linear transformations of the fields – such as (1.21) – are insufficient to infer that the symmetry also acts linearly for particle states, $|\mathbf{p}\rangle = \mathfrak{a}^*_{\mathbf{p}}|0\rangle$, unless the ground-state, $|0\rangle$, is also invariant. If a symmetry of the action does not leave the ground state invariant it is said to be *spontaneously broken*.

For instance, in the toy model the ground state satisfies $\langle 0|\phi(x)|0\rangle = v$, and so the ground state is only invariant under $\phi \to e^{i\omega}\phi$ when $v = 0$. Indeed, for the toy model if $v = 0$ both particle masses are indeed equal: $m_R = m_I = 0$, as are all of their self-couplings. By contrast, when $v \neq 0$ the masses of the two types of particles differ, as does the strength of their cubic self-couplings. Although $\phi \to e^{i\omega}\phi$ always transforms linearly, the symmetry acts inhomogeneously on the deviation $\hat{\phi} = \phi - v = \frac{1}{\sqrt{2}}(\hat{\phi}_R + i\hat{\phi}_I)$ that creates and destroys the particle states. It is because the deviation does not transform linearly (and homogeneously) that the arguments in Appendix C.4 no longer imply that particle states need have the same couplings and masses when $v \neq 0$.

To see why this symmetry should suppress low-energy $\hat{\phi}_I$ interactions, consider how it acts within the low-energy theory. Even though ϕ transforms linearly in the full theory, because the low-energy theory involves only the single real field $\hat{\phi}_I$, the symmetry cannot act on it in a linear and homogeneous way. To see what the action of the symmetry becomes purely within the low-energy theory, it is useful to change variables to a more convenient set of fields than $\hat{\phi}_R$ and $\hat{\phi}_I$.

To this end, define the two real fields χ and ξ by[9]

$$\phi = \left(v + \frac{\chi}{\sqrt{2}}\right) e^{i\xi/\sqrt{2}v}. \tag{1.22}$$

These have the advantage that the action of the $U(1)$ symmetry, $\phi \to e^{i\omega}\phi$ takes a particularly simple form,

$$\xi \to \xi + \sqrt{2}\, v\, \omega, \tag{1.23}$$

with χ unchanged, so ξ carries the complete burden of symmetry transformation.

In terms of these fields the action, Eq. (1.1), becomes

$$S = -\int d^4x \left[\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\left(1 + \frac{\chi}{\sqrt{2}\,v}\right)^2 \partial_\mu\xi\partial^\mu\xi + V(\chi)\right], \tag{1.24}$$

with

$$V(\chi) = \frac{\lambda}{4}\left(\sqrt{2}\, v\, \chi + \frac{\chi^2}{2}\right)^2. \tag{1.25}$$

Expanding this action in powers of χ and ξ gives the perturbative action $S = S_0 + S_{\rm int}$, with unperturbed contribution

$$S_0 = -\frac{1}{2}\int d^4x \left[\partial_\mu\chi\partial^\mu\chi + \partial_\mu\xi\partial^\mu\xi + \lambda v^2\chi^2\right]. \tag{1.26}$$

This shows that χ is an alternative field representation for the heavy particle, with $m_\chi^2 = m_R^2 = \lambda v^2$. ξ similarly represents the massless field.

It also shows the symmetry is purely realized on the massless state, as an inhomogeneous shift (1.23) rather than a linear, homogeneous transformation.

[9] Numerical factors are chosen here to ensure fields are canonically normalized.

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Such a transformation – often called a *nonlinear realization* of the symmetry (both to distinguish it from the linear realization discussed above, and because the transformations turn out in general to be nonlinear when applied to non-abelian symmetries) – is a characteristic symmetry realization in the low-energy limit of a system which spontaneously breaks a symmetry.

The interactions in this representation are given by

$$S_{\text{int}} = -\int d^4x \left[\left(\frac{\chi}{\sqrt{2}\,v} + \frac{\chi^2}{4\,v^2}\right)\partial_\mu\xi\partial^\mu\xi + \frac{\lambda v}{2\sqrt{2}}\chi^3 + \frac{\lambda}{16}\chi^4\right]. \tag{1.27}$$

For the present purposes, what is important about these expressions is that ξ always appears differentiated. This is a direct consequence of the symmetry transformation, Eq. (1.23), which requires invariance under constant shifts: $\xi \to \xi + \text{constant}$. Since this symmetry forbids a ξ mass term, which would be $\propto m_I^2\,\xi^2$, it ensures ξ remains exactly massless to all orders in the small expansion parameters. ξ is what is called a *Goldstone boson* for the spontaneously broken $U(1)$ symmetry: it is the massless scalar that is guaranteed to exist for spontaneously broken (global) symmetries. Because ξ appears always differentiated it is immediately obvious that an amplitude describing N_i ξ particles scattering into N_f ξ particles must be proportional to at least $N_i + N_f$ powers of their energy, explaining the low-energy suppression of light-particle scattering amplitudes in this toy model.

For instance, explicitly re-evaluating the Feynman graphs of Fig. 1.3, using the interactions of Eq. (1.27) instead of (1.6), gives the case $N_i = N_f = 2$ as

$$\begin{aligned}
&\mathcal{A}_{\xi\xi\to\xi\xi} \\
&= 0 + 8\left(\frac{i^2}{2}\right)\left(-\frac{1}{\sqrt{2}\,v}\right)^2\left[\frac{-i(p\cdot q)(p'\cdot q')}{(p+q)^2+m_R^2} + \frac{-i(p\cdot p')(q\cdot q')}{(p-p')^2+m_R^2} + \frac{-i(p\cdot q')(q\cdot p')}{(p-q')^2+m_R^2}\right] \\
&= \frac{2i\lambda}{m_R^4}\left[\frac{(p\cdot q)^2}{1+2p\cdot q/m_R^2} + \frac{(q\cdot q')^2}{1-2q\cdot q'/m_R^2} + \frac{(p\cdot q')^2}{1-2p\cdot q'/m_R^2}\right],
\end{aligned} \tag{1.28}$$

in precise agreement with Eq. (1.8) – as may be seen explicitly using the identity $(1+x)^{-1} = 1 - x + x^2/(1+x)$ – but with the leading low-energy limit much more explicit.

This representation of the toy model teaches several things. First, it shows that scattering amplitudes (and, more generally, arbitrary physical observables) do not depend on which choice of field variables are used to describe a calculation [8–10]. Some kinds of calculations (like loops and renormalization) are more convenient using the variables $\hat{\phi}_R$ and $\hat{\phi}_I$, while others (like extracting consequences of symmetries) are easier using χ and ξ.

Second, this example shows that it is worthwhile to use the freedom to perform field redefinitions to choose those fields that make life as simple as possible. In particular, it is often very useful to make symmetries of the high-energy theory as explicit as possible in the low-energy theory as well.

Third, this example shows that once restricted to the low-energy theory it need not be true that a symmetry remains linearly realized by the fields [11–13], even if this were true for the full underlying theory including the heavy particles. The necessity of realizing symmetries nonlinearly arises once the scales defining the low-energy theory (*e.g.* $E \ll m_R$) are smaller than the mass difference (*e.g.* m_R) between particles that are related by the symmetry in the full theory, since in this case some of the states required to fill out a linear multiplet are removed as part of the high-energy theory.

1.3 Summary

This first chapter defines a toy model, in which a complex scalar field, ϕ, self-interacts *via* a potential $V = \frac{\lambda}{4}(\phi^*\phi - v^2)^2$ that preserves a $U(1)$ symmetry: $\phi \to e^{i\omega}\phi$. Predictions for particle masses and scattering amplitudes are made as a function of the model's two parameters, λ and v, in the semiclassical regime $\lambda \ll 1$. This model is used throughout the remaining chapters of Part I as a vehicle for illustrating how the formalism of effective field theories works in a concrete particular case.

The semiclassical spectrum of the model has two phases. If $v = 0$ the $U(1)$ symmetry is preserved by the semiclassical ground state and there are two particles whose couplings and masses are the same because of the symmetry. When $v \neq 0$ the symmetry is spontaneously broken, and one particle is massless while the other gets a nonzero mass $m = \sqrt{\lambda}\, v$.

The model's symmetry-breaking phase has a low-energy regime, $E \ll m$, that provides a useful illustration of low-energy methods. In particular, the massive particle decouples at low energies in the precise sense that its virtual effects only play a limited role for the low-energy interactions of the massless particles. In particular, explicit calculation shows the scattering of massless particles at low energies in the full theory to be well-described to leading order in λ and E/m in terms of a simple local 'effective' interaction with lagrangian density $\mathfrak{L}_{\rm eff} = a_{\rm eff}(\partial_\mu \xi\, \partial^\mu \xi)^2$, with effective coupling: $a_{\rm eff} = \lambda/(4m^4)$. The $U(1)$ symmetry of the full theory appears in the low-energy theory as a shift symmetry $\xi \to \xi + \text{constant}$.

Exercises

Exercise 1.1 Use the Feynman rules coming from the action $S = S_0 + S_{\rm int}$ given in Eqs. (1.5) and (1.6) to evaluate the graphs of Fig. 1.2. Show from your result that the corresponding S-matrix element is given by

$$\langle \hat{\phi}_R(p'), \hat{\phi}_I(q')|\mathcal{S}\,|\,\hat{\phi}_R(p), \hat{\phi}_I(q)\rangle = -i(2\pi)^4 \mathcal{A}_{RI\to RI}\, \delta^4(p+q-p'-q'),$$

with $\mathcal{A}_{RI\to RI}$ given by Eq. (1.7). Taylor expand your result for small q, q' to verify the low-energy limit given in Eq. (1.9). [Besides showing the low-energy decoupling of Goldstone particles, getting right the cancellation that provides this suppression in these variables is a good test of – and a way to develop faith in – your understanding of Feynman rules.]

Exercise 1.2 Using the Feynman rules coming from the action $S = S_0 + S_{\rm int}$ given in Eqs. (1.5) and (1.6) evaluate the graphs of Fig. 1.3 to show

$$\langle \hat{\phi}_I(p'), \hat{\phi}_I(q')|\mathcal{S}\,|\,\hat{\phi}_I(p), \hat{\phi}_I(q)\rangle = -i(2\pi)^4 \mathcal{A}_{II\to II}\, \delta^4(p+q-p'-q'),$$

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with $\mathcal{A}_{II\to II}$ given by Eq. (1.8). Taylor expand your result for small q, q' to verify the low-energy limit given in Eq. (1.10).

Exercise 1.3 Using the toy model's leading effective interaction $S = S_{\text{eff}\,0} + S_{\text{eff}\,\text{int}}$, with Feynman rules drawn from (1.11) (1.12), draw the graphs that produce the dominant contributions – *i.e.* carry the fewest factors of λ and (external energy)/m_R – to the scattering process $\hat{\phi}_I + \hat{\phi}_I \to 4\hat{\phi}_I$. Show that these agree with the estimate (1.13) in their prediction for the leading power of λ and of external energy.

Exercise 1.4 Using the Feynman rules coming from the action $S = S_0 + S_{\text{int}}$ given in Eqs. (1.26) and (1.27) evaluate the graphs of Fig. 1.3 to show

$$\langle \xi(p'), \xi(q')|\mathcal{S}\,|\,\xi(p), \xi(q)\rangle = -\mathrm{i}(2\pi)^4 \mathcal{A}_{\xi\xi\to\xi\xi}\, \delta^4(p + q - p' - q'),$$

with $\mathcal{A}_{\xi\xi\to\xi\xi}$ given by Eq. (1.28). [Comparing this result to the result in Exercise 1.2 provides an illustration of Borcher's theorem [8–10], which states that scattering amplitudes remain unchanged by a broad class of local field redefinitions.]

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