RING THEORETIC PROPERTIES OF MATRIX RINGS

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Morita Theory. K. Morita has shown that, given two rings R and S, there is an isomorphism between the category of left R-modules and the category of left S-modules if and only if there exists an R-S bimodule U such that

(1) U is a progenerator in the category of left R-modules, and

(2) $S \cong (\mathrm{End}_R U)^{opp}$ as rings.[2]

If $S = R_{(n)}$, the ring of $n \times n$ matrices with entries in R, then R^n satisfies the two properties above when viewed as the R-$R_{(n)}$ bimodule of $1 \times n$ matrices over R. In this case the inverse isomorphisms may be defined directly. They will be used to show systematically that $R_{(n)}$ has certain ring theoretic properties if and only if R has the same property.[3]

The Isomorphisms. Given two categories $\mathcal{C}$ and $\mathcal{C}'$, two (covariant) functors $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{C}' \to \mathcal{C}$ will be called inverse isomorphisms if GF and FG are naturally equivalent to the identity functors on $\mathcal{C}$ and $\mathcal{C}'$ respectively.

Let R be a ring with unit. e_{ij} will denote the element of

[1] I would like to thank Professor I. Connell, the director of my research, for his assistance.

[2] See Morita [6] and Bass [2].

[3] These isomorphisms were defined in my Master's thesis before I was aware of Morita theory.

$R_{(n)}$ whose only non-zero entry is 1 in the ij^{th} position. $\mathcal{M}$ will denote the category of left R-modules and $\mathcal{N}$ the category of left $R_{(n)}$-modules. We define functors $S: \mathcal{N} \to \mathcal{M}$ and $T: \mathcal{M} \to \mathcal{N}$ as follows. For $M \in |\mathcal{N}|$ let $S(M) = e_{11}M$. If $r \in R$, let $\hat{r}$ denote the scalar matrix rI, where I is the identity matrix. The action of R on $S(M)$ is defined by $r(e_{11}m) = \hat{r}e_{11}m = e_{11}\hat{r}e_{11}m \in S(M)$. If $\varphi: M \to N$ is a mapping in $\mathcal{N}$, let $S(\varphi)(e_{11}m) = \varphi(e_{11}m) = e_{11}\varphi(m) \in S(N)$. For $M \in |\mathcal{M}|$ let $T(M)$ be a direct sum of n copies of M. The action of $R_{(n)}$ on $T(M)$ is defined by $(\sum_{i,j} r_{ij}e_{ij})(m_1, \ldots, m_n) = (\sum_j r_{1j}m_j, \ldots, \sum_j r_{nj}m_j)$. If $\varphi: M \to N$ is a mapping in $\mathcal{M}$, let $T(\varphi)(m_1, \ldots, m_n) = (\varphi(m_1), \ldots, \varphi(m_n))$. It is easily checked that S and T are functors between $\mathcal{M}$ and $\mathcal{N}$.

PROPOSITION 1. S and T are inverse isomorphisms.

Proof. Let $I_{\mathcal{M}}$ denote the identity functor on $\mathcal{M}$ and $I_{\mathcal{N}}$ the identity functor on $\mathcal{N}$. We must exhibit natural isomorphisms $\mu: I_{\mathcal{M}} \to ST$ and $\nu: I_{\mathcal{N}} \to TS$. Let $M \in |\mathcal{N}|$. Define $\nu(M)(m) = (e_{11}m, \ldots, e_{1n}m) = (e_{11}e_{11}m, \ldots, e_{11}e_{1n}m) \in TS(M)$. If $\nu(M)(m) = 0$, then $e_{1j}m = 0$ for every j and hence $m = \sum_j e_{j1}e_{1j}m = 0$. A typical element of $TS(M)$ is of the form $(e_{11}m_1, \ldots, e_{11}m_n) = \nu(M)(\sum_j e_{j1}m_j)$, where each $m_j \in M$. $\nu(M)$ clearly preserves sums. Let $\sum_{i,j} r_{ij}e_{ij} \in R_{(n)}$.

$$\begin{aligned}(\sum_{i,j} r_{ij}e_{ij})\nu(M)(m) &= (\sum_{i,j} r_{ij}e_{ij})(e_{11}m, \ldots, e_{1n}m) \\ &= (\sum_j r_{1j}e_{1j}m, \ldots, \sum_j r_{nj}e_{1j}m) \\ &= (e_{11}\sum_{i,j} r_{ij}e_{ij}m, \ldots, e_{1n}\sum_{i,j} r_{ij}e_{ij}m) \\ &= \nu(M)(\sum_{i,j} r_{ij}e_{ij}m).\end{aligned}$$

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Therefore $\nu(M)$ is an isomorphism. If $\varphi: M \to N$ is an $R_{(n)}$-homomorphism then

$$\begin{aligned}\nu(N)(\varphi(m)) &= (e_{11}\varphi(m), \ldots, e_{1n}\varphi(m)) \\ &= (\varphi(e_{11}m), \ldots, \varphi(e_{1n}m)) \\ &= (S(\varphi)(e_{11}m), \ldots, S(\varphi)(e_{1n}m)) \\ &= TS(\varphi)(e_{11}m, \ldots, e_{1n}m) \\ &= TS(\varphi)(\nu(M)(m)) .\end{aligned}$$

Thus ν is natural.

Now let $M \in |\mathfrak{M}|$. A typical element of $ST(M)$ is of the form $(m, 0, \ldots, 0)$ with $m \in M$. Let $\mu(M)(m) = (m, 0, \ldots, 0)$. μ is clearly an isomorphism. If $\varphi: M \to N$ is an R-homomorphism then $\mu(N)(\varphi(m)) = (\varphi(m), 0, \ldots, 0) = T(\varphi)(m, 0, \ldots, 0) = ST(\varphi)(m, 0, \ldots, 0) = ST(\varphi)(\mu(M)(m))$. Thus μ is natural.

PROPOSITION 2. S and T are exact functors.

Proof. Consider an exact sequence $M \xrightarrow{\varphi} P \xrightarrow{\psi} Q$ in $\mathfrak{M}$. $T(\psi)T(\varphi) = T(\psi\varphi) = T(0) = 0$. If $p = (p_1, \ldots, p_n) \in T(P)$ and $T(\psi)(p) = (\psi(p_1), \ldots, \psi(p_n)) = 0$ then $\psi(p_j) = 0$ for all j, that is, $p_j \in \operatorname{Ker}(\psi) = \operatorname{Im}(\varphi)$ for all j. Therefore there exist $m_j \in M$, $j = 1, \ldots, n$ such that $\varphi(m_j) = p_j$. $p = (p_1, \ldots, p_n) = (\varphi(m_1), \ldots, \varphi(m_n)) = T(\varphi)(m_1, \ldots, m_n) \in \operatorname{Im}(T(\varphi))$. Therefore T is exact.

Consider an exact sequence $M \xrightarrow{\varphi} P \xrightarrow{\psi} Q$ in $\mathfrak{N}$. $S(\psi)S(\varphi) = S(\psi\varphi) = S(0) = 0$. If $p = e_{11}p \in S(P)$ and $S(\psi)(p) = 0$ then $\psi(p) = 0$ and $p \in \operatorname{Ker}(\psi) = \operatorname{Im}(\varphi)$. Therefore there exists $m \in M$ such that $\varphi(m) = p$. $S(\varphi)(e_{11}m) = \varphi(e_{11}m) = e_{11}\varphi(m) = e_{11}p = p \in \operatorname{Im}(S(\varphi))$. Therefore S is exact.

PROPOSITION 3. S and T preserve finite generation.

Proof. Let $M \in |\mathfrak{M}|$ and let $\{m_i : i = 1, \ldots, r\}$ be a set of generators for M. Then $\{(m_i, 0, \ldots, 0) : i = 1, \ldots, r\}$ is a set of generators for $T(M)$.

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Let $M \in |\eta|$ and let $\{m_i : i = 1, \ldots, r\}$ be a set of generators for M. Then $\{e_{1j} m_i : i = 1, \ldots, r; j = 1, \ldots, n\}$ is a set of generators for $S(M)$.

Ring Theoretic Properties.

PROPOSITION 4. If R and R' are rings and the functor F is an isomorphism from the category of left R-modules to the category of left R'-modules, then

(i) If $F(M)$ is projective, so is M.

(ii) If $F(M)$ is injective, so is M.

Proof. Suppose $F(M)$ is projective. Let G be an inverse for F and let γ be a natural isomorphism from the identity functor to GF. Let $\varphi : A \to B$ be an R-epimorphism $\psi : M \to B$ an R-homomorphism. Then $F(\varphi) : F(A) \to F(B)$ is an R'-epimorphism and $F(\psi) : F(M) \to F(B)$ is an R'-homomorphism. Since $F(M)$ is projective, there exists an R'-homomorphism $\theta : F(M) \to F(A)$ such that $F(\varphi)\theta = F(\psi)$. $\gamma(A)^{-1} G(\theta)\gamma(M) : M \to A$ is an R-homomorphism. Moreover $\varphi\gamma(A)^{-1} G(\theta)\gamma(M) =$ $\gamma(B)^{-1} GF(\varphi)\gamma(A)\gamma(A)^{-1} G(\theta)\gamma(M) = \gamma(B)^{-1} GF(\varphi) G(\theta)\gamma(M)$ $= \gamma(B)^{-1} G(F(\varphi)\theta)\gamma(M) = \gamma(B)^{-1} GF(\psi)\gamma(M) = \psi$. Therefore M is projective. The injective case may be proved dually.

COROLLARY. If M is projective, so is $F(M)$. If M is injective, so is $F(M)$.

Proof. Notice that G is also an isomorphism, and that $M \cong GF(M)$.

THEOREM 1. A ring R is left self-injective if and only if $R_{(n)}$ is.[1]

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[1] This theorem was proved by Y. Utumi [7] but by an entirely different method. The present proof applies in the more general case of an isomorphism between the categories of left R-modules and left R'-modules for any two rings R and R', and may be simplified somewhat in our case, since $S(R_{(n)}) \cong R^n$.

Proof. $S(R_{(n)})$ is a finitely generated projective left R-module since $R_{(n)}$ is a finitely generated projective left $R_{(n)}$-module. That is, $S(R_{(n)})$ is a direct factor of a direct product of copies of R (since finite sums are the same as finite products). Therefore $S(R_{(n)})$ is injective if and only if R is left self-injective. But by Proposition 4 and the Corollary, $S(R_{(n)})$ is injective if and only if $R_{(n)}$ is left self-injective. Q.E.D.

A ring R will be called left hereditary if every left ideal of R is projective. R is left semi-hereditary if every finitely generated left ideal of R is projective.

THEOREM 2. R is left (semi-) hereditary if and only if $R_{(n)}$ is.[1]

Proof. We use the fact that a ring R is (semi-) hereditary if and only if every (finitely generated) submodule of a projective left R-module is projective.[2]

Let R be (semi-) hereditary and let I be a (finitely generated) left ideal of $R_{(n)}$. Then S(I) is isomorphic to a (finitely generated) submodule of $S(R_{(n)})$, a projective left R-module by the Corollary to Proposition 4. Hence S(I) is projective. Therefore I is projective, and $R_{(n)}$ is left (semi-) hereditary. The proof of the converse is similar.

Let M and N be modules. A homomorphism $\varphi: M \to N$ is called minimal if $\mathrm{Ker}(\varphi)$ is a small submodule of M. A projective cover[3] of a module M is a minimal epimorphism π from a projective module P to M. A ring R is left perfect if every left R-module has a projective cover. R is left semi-perfect if every finitely generated left R-module has a projective cover.

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[1] This theorem is due to L. Levy [5].

[2] Cartan and Eilenberg [3], pp. 14, 15.

[3] These concepts are defined by Bass [1].

LEMMA 1. Let P be a projective module. An epimorphism $\pi:P \to M$ is a projective cover for M if and only if for any proper monomorphism (i.e. one which is not an isomorphism) θ into P, $\pi\theta$ is not an epimorphism.

Proof. Suppose $\theta:S \to P$ is a proper monomorphism and $\pi\theta$ is an epimorphism. Then for all $p \in P$ there exists $s \in S$ such that $\pi\theta(s) = \pi(p)$. $p - \theta(s) \in \mathrm{Ker}(\pi)$ and $p = \theta(s) + (p - \theta(s))$. Therefore $P = \mathrm{Im}(\theta) + \mathrm{Ker}(\pi)$. Since $\mathrm{Im}(\theta)$ is a proper submodule of P, π is not a projective cover of M.

Suppose π is not a projective cover of M. Then there exists a proper submodule S of P such that $S + \mathrm{Ker}(\pi) = P$. Let $i:S \to P$ be the inclusion map, a proper monomorphism. Let $m \in M$. Since π is an epimorphism, there exists $p \in P$ such that $\pi(p) = m$. Since $S + \mathrm{Ker}(\pi) = P$, there exist $s \in S$ and $x \in \mathrm{Ker}(\pi)$ such that $p = i(s) + x$. $\pi i(s) = \pi(p - x) = \pi(p) - \pi(x) = m - 0 = m$. Therefore πi is an epimorphism.

THEOREM 3. R is left (semi-) perfect if and only if $R_{(n)}$ is.[1]

Proof. Suppose R is left (semi-) perfect and M is a (finitely generated) left $R_{(n)}$-module. Then $S(M)$ has a projective cover $\pi:P \to S(M)$. $T(P)$ is a projective $R_{(n)}$-module and $\nu(M)^{-1}T(\pi):T(P) \to M$ is an epimorphism. Let $\theta:N \to T(P)$ be a proper monomorphism. Then $\mu(P)^{-1}S(\theta):S(N) \to P$ is a proper monomorphism. Therefore $\pi\mu(P)^{-1}S(\theta)$ is not an epimorphism. But $\pi\mu(P)^{-1}S(\theta) = \mu(S(M))^{-1}ST(\pi)\mu(P)\mu(P)^{-1}S(\theta) = \mu(S(M))^{-1}S(T(\pi)\theta)$. Since $\mu(S(M))^{-1}$ is an isomorphism, $S(T(\pi)\theta)$ is not an epimorphism. Therefore $T(\pi)\theta$ is not an epimorphism, and since $\nu(M)^{-1}$ is an isomorphism, $\nu(M)^{-1}T(\pi)\theta$ is not an epimorphism. Therefore $\nu(M)^{-1}T(\pi)$ is a projective cover for M, and $R_{(n)}$ is left (semi-) perfect. The proof of the converse is similar.

It is well-known that a ring R is (Von Neumann) regular

[1] This theorem was stated for perfect rings by H. Bass [1].

if and only if every left R-module is flat.[1] We shall use this characterization to show that R is regular if and only if $R_{(n)}$ is. We define functors S^o from the category of right $R_{(n)}$-modules to the category of right R-modules and T^o in the opposite direction analogously to S and T. μ^o and ν^o will denote the natural isomorphisms.

PROPOSITION 5. Let M be a right $R_{(n)}$-module, N a left $R_{(n)}$-module. Then $M \otimes_{R_{(n)}} N \cong S^o(M) \otimes_R S(N)$ naturally.

Proof. $S^o(M) = Me_{11}$ and $S(N) = e_{11}N$. Let $M \times N$ be the Cartesian product of M and N. Define $\theta : M \times N \to Me_{11} \otimes_R e_{11}N$ by $\theta(m, n) = \sum_{i=1}^{n} (me_{i1} \otimes_R e_{1i}n)$. θ is clearly linear in m and n. For all $r \in R$ and all e_{ij}, $\theta(mre_{ij}, n) = \sum_k mre_{ij}e_{k1} \otimes_R e_{1k}n =$
$mre_{ij}e_{j1} \otimes_R e_{1j}n = mre_{i1} \otimes_R e_{1i}e_{ij}n = me_{i1} \otimes_R e_{1i}re_{ij}n =$
$\sum_k me_{k1} \otimes_R e_{1k}re_{ij}n = \theta(m, re_{ij}n)$. Thus θ is $R_{(n)}$-bilinear.

Let $\chi(M, N) : M \otimes_{R_{(n)}} N \to S^o(M) \otimes_R S(N)$ be the map induced by θ. $\chi(M, N)$ is onto since if $me_{11} \in S^o(M)$ and $e_{11}n \in S(N)$ then $me_{11} \otimes_R e_{11}n = \chi(M, N)(me_{11} \otimes_{R_{(n)}} e_{11}n)$. To see that $\chi(M, N)$ is 1-1, notice that $\sum_k m_k \otimes_{R_{(n)}} n_k = \sum_k \sum_{i=1}^{n} m_k e_{i1} \otimes_{R_{(n)}} e_{1i}n_k$, which is 0 if $\chi(M, N)(\sum_k m_k \otimes_{R_{(n)}} n_k) = \sum_k \sum_{i=1}^{n} m_k e_{i1} \otimes_R e_{1i}n_k$ is, since $Me_{11} \subset M$, $e_{11}N \subset N$, and $R \subset R_{(n)}$.

To show that χ is natural, it is sufficient to show naturality on the basis elements since all maps concerned preserve sums. Let $\varphi : M \to M'$ be a right $R_{(n)}$-homomorphism, and $\psi : N \to N'$ a left $R_{(n)}$-homomorphism.

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[1] See, for example, Lambek [4], p. 134.

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$$\chi(M', N')(\varphi \otimes_{R_{(n)}} \psi)(m \otimes_{R_{(n)}} n) = \chi(M', N')(\varphi(m) \otimes_{R_{(n)}} \psi(n))$$

$$= \sum_i (\varphi(m)e_{i1} \otimes_R e_{1i} \psi(n)) = \sum_i (\varphi(me_{i1}) \otimes_R \psi(e_{1i} n))$$

$$= \sum_i (S^o(\varphi)(me_{i1}) \otimes_R S(\psi)(e_{1i} n)) = \sum_i (S^o(\varphi) \otimes_R S(\psi))(me_{i1} \otimes_R e_{1i} n)$$

$$= (S^o(\varphi) \otimes_R S(\psi))(\sum_i me_{i1} \otimes_R e_{1i} n)$$

$$= (S^o(\varphi) \otimes_R S(\psi)) \chi(M, N)(m \otimes_{R_{(n)}} n) .$$

Therefore $\chi(M', N')(\varphi \otimes_{R_{(n)}} \psi) = (S^o(\varphi) \otimes_R S(\psi)) \chi(M, N)$.

THEOREM 4. R is (Von Neumann) regular if and only if $R_{(n)}$ is.

<u>Proof.</u> Suppose R is (Von Neumann) regular. Let N be a left $R_{(n)}$-module and $\varphi : M \to M'$ a right $R_{(n)}$-monomorphism. Then $S^o(\varphi) : S^o(M) \to S^o(M')$ is a right R-monomorphism. Since R is regular, $S(N)$ is flat. Therefore $S^o(\varphi) \otimes_R S(1_N) = S^o(\varphi) \otimes_R 1_{S(N)}$ is a monomorphism of abelian groups. Therefore $\varphi \otimes_{R_{(n)}} 1_N = \chi(M', N')^{-1}(S^o(\varphi) \otimes_R S(1_N)) \chi(M, N)$ is a monomorphism, and N is flat. Therefore $R_{(n)}$ is regular. The converse is trivial.

<u>Properties Involving Ideals.</u> There is a well-known 1-1 correspondence between the ideals of R and the ideals of $R_{(n)}$ given by $I \to I_{(n)}$.[1] It follows immediately that R is simple

[1] In the general case (see footnote, p. 368) the correspondence between ideals of R and R' may be defined in terms of the isomorphisms between the categories, but this is unnecessary here.

(0 is a maximal 2-sided ideal), semi-simple (the intersection of all maximal ideals is 0)[1], prime (0 is a prime ideal), or semi-prime (the intersection of all prime ideals is 0) if and only if $R_{(n)}$ has the same property. We shall show that R is primitive (0 is a primitive ideal) or semi-primitive (the intersection of all primitive ideals is 0) if and only if $R_{(n)}$ is. Recall that a primitive ideal is the annihilator of a simple left module (one whose only proper submodule is 0).

LEMMA 2. Let I be an ideal of R and M a left R-module. Then $T(IM) = I_{(n)}T(M)$.

Proof. This is trivial.

THEOREM 5. R is (semi-) primitive if and only if $R_{(n)}$ is.

Proof. We show that I is a primitive ideal of R if and only if $I_{(n)}$ is a primitive ideal of $R_{(n)}$. Let $I = \mathrm{Ann}(M)$ where M is a simple left R-module. T(M) is a simple left $R_{(n)}$-module, and $I_{(n)}T(M) = T(IM) = T(0) = 0$. Let $J_{(n)} = \mathrm{Ann}(T(M))$. Then $I_{(n)} \subset J_{(n)}$. $T(JM) = J_{(n)}T(M) = 0$ and hence $JM = 0$. Therefore $J \subset I$ and so $J_{(n)} = I_{(n)}$ is a primitive ideal of $R_{(n)}$.

Let $I_{(n)} = \mathrm{Ann}(N)$ where N is a simple left $R_{(n)}$-module. S(N) is a simple left R-module. $T(IS(N)) = I_{(n)}TS(N) \cong I_{(n)}N = 0$. If $JS(N) = 0$ then $J_{(n)}TS(N) = T(JS(N)) = T(0) = 0$ and hence $J_{(n)} \subset I_{(n)}$. Therefore $J \subset I$ and $I = \mathrm{Ann}(S(N))$.

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[1] Note that our definition of a semi-simple ring differs from both Bourbaki and Jacobson. Bourbaki defines semi-simple to mean Artinian semi-simple. Jacobson defines semi-simple to mean semi-primitive.

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