ANZIAM J. 55(2013), 39–54
doi:10.1017/S1446181113000308

ON LAGUERRE–SOBOLEV TYPE ORTHOGONAL POLYNOMIALS: ZEROS AND ELECTROSTATIC INTERPRETATION

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(Received 22 January, 2013; revised 4 June, 2013; first published online 8 October, 2013)

Abstract

We study the sequence of monic polynomials orthogonal with respect to inner product

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)e^{-x}x^\alpha\, dx + Mp(\zeta)q(\zeta) + Np'(\zeta)q'(\zeta),$$

where $\alpha > -1$, $M \geq 0$, $N \geq 0$, $\zeta < 0$, and p and q are polynomials with real coefficients. We deduce some interlacing properties of their zeros and, by using standard methods, we find a second-order linear differential equation satisfied by the polynomials and discuss an electrostatic model of their zeros.

2010 *Mathematics subject classification*: 42C05.

Keywords and phrases: orthogonal polynomials, Laguerre polynomials, Sobolev-type inner products, zeros, electrostatic interpretation.

1. Introduction

Let $\{L_n^\alpha(x)\}_{n\in\mathbb{N}}$ be the classical Laguerre polynomials, orthogonal with respect to inner product

$$\langle p, q\rangle_\alpha = \int_0^\infty p(x)q(x)e^{-x}x^\alpha\, dx.$$

These polynomials are well known in the literature for their representation as hypergeometric functions, the distribution and interlacing properties of their zeros, and their electrostatic interpretation, as well as their characterization as eigenfunctions of a second-order linear differential equation [18, 21, 22]. By using the Laguerre weight, it is possible to introduce another measure by adding a finite discrete part, and express the polynomials orthogonal with respect to the new measure in terms of the classical Laguerre polynomials. In particular, let $\{\widetilde{L}_n^\alpha(x)\}_{n\in\mathbb{N}}$ be the *Laguerre–Sobolev*

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type polynomials, orthogonal with respect to the *Sobolev-type inner product*

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)e^{-x}x^\alpha\,dx + Mp(a)q(a) + Np'(a)q'(a), \tag{1.1}$$

where $M, N \geq 0$, $\alpha > -1$ and $a < 0$. The sequence $\{\widetilde{L}_n^\alpha(x)\}_{n\in\mathbb{N}}$ is a case of *orthogonal polynomials of Laguerre–Sobolev type*. The case $a = 0$ has been extensively analysed in the literature [7, 8, 15, 16, 20]. These papers have established the representation of the corresponding polynomials as a hypergeometric series, properties of the zeros, a holonomic second-order linear differential equation and a higher-order recurrence relation that such polynomials satisfy, as well as asymptotic properties. Concerning (1.1), that is, if $a < 0$, analytical and asymptotic properties, as well as the distribution of the zeros, dependence on the masses and interlacing properties, have been studied in several papers [2, 3, 5, 6, 17, 19].

The main purpose of this paper is to analyse interlacing properties of the zeros of the polynomials associated with the inner product (1.1) and present an electrostatic model for these zeros, based on the work of Ismail [12, 13] concerning electrostatic models for general orthogonal polynomials. In Section 2 we present some auxiliary results on the existence of a sequence of polynomials orthogonal for every positive Borel measure supported on a infinite subset of the real line, as well as some structural and algebraic properties of Laguerre polynomials. We then discuss some interlacing properties of the zeros in Section 3. In Section 4 we deduce, by using standard techniques, a second-order linear differential equation satisfied for every $\widetilde{L}_n^\alpha(x)$ and the electrostatic interpretation of the zeros as equilibrium points with respect to a logarithmic potential under the action of an external field. Numerical experiments to illustrate the results of Section 4 are presented in Section 5.

2. Auxiliary results

Let μ be a positive Borel measure supported on a infinite subset Ω of the real line, and assume that

$$\int_\Omega |x|^n\,d\mu(x) < \infty$$

for every n if Ω is unbounded. We define the inner product

$$\langle p, q\rangle_\mu = \int_\Omega p(x)q(x)\,d\mu(x), \quad \text{for every } p, q \in \mathbf{P},$$

on the space $\mathbf{P}$ of polynomials with real coefficients, and the corresponding norm is given by

$$\|p\|_\mu = \left[\int_\Omega |p(x)|^2\;d\mu(x)\right]^{1/2} \quad \text{for every } p \in \mathbf{P}.$$

Definition 2.1. A sequence of polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ is said to be a *sequence of polynomials orthogonal with respect to* $\langle\cdot,\cdot\rangle_\mu$ if:

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(i) the degree of every $P_n(x)$ is n;
(ii) $\langle P_n, P_m\rangle_\mu = \int_\Omega P_n(x)P_m(x)\,d\mu(x)\{^{\neq 0 \text{ if } m=n}_{=0 \text{ if } m\neq n.}$

Also, if the leading coefficient of $P_n(x)$ is 1 for every n then $\{P_n(x)\}_{n\in\mathbb{N}}$ is said to be a *monic orthogonal polynomial sequence (MOPS)*. Moreover, if $\|P_n\|_\mu = 1$ for every n then $\{P_n(x)\}_{n\in\mathbb{N}}$ is said to be an *orthonormal polynomial sequence*.

Given the measure μ, the existence and uniqueness of a MOPS with respect to $\langle\cdot,\cdot\rangle_\mu$ is guaranteed by the next result [14, 22].

Theorem 2.2. *For every positive Borel measure μ, there exists a unique MOPS.*

Under the above assumptions on μ, and with $\Omega\subseteq\mathbf{R}$ now an interval (finite or infinite), suppose that the measure is perturbed by adding a finite discrete part,

$$\langle f, g\rangle = \int_\Omega fg\,d\mu + M_0 f(\zeta)g(\zeta) + M_1 f'(\zeta)g'(\zeta),$$

where $\zeta\in\mathbb{R}$ and $M_0, M_1\geq 0$, and let $\{P_n(x)\}_{n\in\mathbb{N}}$ be the MOPS with respect to this product. Then we have the following result [3].

Theorem 2.3. *Let $M_1 > 0$. If $n > 2$ then $P_n(x)$ has at least $n-2$ zeros with odd multiplicity in* $\operatorname{int}(\Omega)$. *Also, for every n, the zeros of $P_n(x)$ are real and simple.*

A sequence of orthogonal polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ is said to be *classical* if for every n, $P_n(x)$ satisfies the second-order linear differential equation

$$\sigma(x)\phi''(x) + \tau(x)\phi'(x) + \lambda_n\phi(x) = 0,$$

where $\sigma(x)$ is a polynomial of degree at most two, $\tau(x)$ is a polynomial of degree one, and λ_n is a real number. As a particular and well-known case, the Laguerre monic orthogonal polynomials $\{L_n^\alpha(x)\}_{n\in\mathbb{N}}$ are the polynomials orthogonal with respect to the inner product

$$\langle p, q\rangle_\alpha = \int_0^\infty p(x)q(x)e^{-x}x^\alpha\,dx \quad \text{with } \alpha > -1.$$

These polynomials satisfy some important properties that we use in the following. The details of the proof can be found in several texts [1, 4, 14, 22].

Proposition 2.4. *Let $\{L_n^\alpha(x)\}_{n\in\mathbb{N}}$ be the sequence of Laguerre monic orthogonal polynomials.*

(i) *We have the following three-term recurrence relation: for every $n\in\mathbb{N}$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)L_n^\alpha(x) + n(n+\alpha)L_{n-1}^\alpha(x), \tag{2.1}$$

with $L_0^\alpha(x) = 1$ and $L_1^\alpha(x) = x-(\alpha+1)$.

(ii) *For every $n\in\mathbb{N}$,*

$$\|L_n^\alpha\|_\alpha^2 = n!\,\Gamma(n+\alpha+1).$$

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(iii) *For every* $n \in \mathbb{N}$,

$$(L_n^\alpha(x))' = nL_{n-1}^{\alpha+1}(x).$$

(iv) *For every* $n \in \mathbb{N}$,

$$x(L_n^\alpha(x))' = nL_n^\alpha(x) + n(n+\alpha)L_{n-1}^\alpha(x). \tag{2.2}$$

(v) *For every* $n \in \mathbb{N}$, $L_n^\alpha(x)$ *satisfies the differential equation*

$$xy'' + (\alpha + 1 - x)\, y' = -ny.$$

(vi) *For every* $n \in \mathbb{N}$,

$$L_n^\alpha(x) = (-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!} =: {}_1F_1(-n, \alpha+1 \mid x),$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, *with* $n \geq 1$ *and* $(a)_0 = 1$, *is the Pochhammer symbol.*

(vii) *If*

$$K_n(x,y) = \sum_{k=0}^{n} \frac{L_k^\alpha(x)L_k^\alpha(y)}{\|L_k^\alpha\|_\alpha^2}$$

denotes the nth kernel polynomial, then for every $n \in \mathbb{N}$, *we have the Christoffel–Darboux formula,*

$$K_n(x,y) = \frac{L_{n+1}^\alpha(x)L_n^\alpha(y) - L_{n+1}^\alpha(y)L_n^\alpha(x)}{\|L_n^\alpha\|_\alpha^2(x-y)}.$$

Let $\{\widetilde{L}_n^\alpha(x)\}_{n\in\mathbb{N}}$ be a sequence of monic orthogonal polynomials with respect to (1.1). It is well known [3] that every $\widetilde{L}_n^\alpha(x)$ can be expressed in terms of classical Laguerre polynomials by

$$(x-a)^2\widetilde{L}_n^\alpha(x) = p(x;n)L_n^\alpha(x) + q(x;n)L_{n-1}^\alpha(x), \tag{2.3}$$

where $p(x;n)$ and $q(x;n)$ are polynomials with degree two and one, respectively, and their representation is given by

$$p(x;n) = (x-a)^2 + A_{n-1}(x-a) + B_{n-1},$$

where

$$A_{n-1} = \frac{-N(\widetilde{L}_n^\alpha)'(a)(L_{n-1}^\alpha)'(a) - M\widetilde{L}_n^\alpha(a)L_{n-1}^\alpha(a)}{\|L_{n-1}^\alpha\|_\alpha^2}, \quad B_{n-1} = \frac{-N(\widetilde{L}_n^\alpha)'(a)L_{n-1}^\alpha(a)}{\|L_{n-1}^\alpha\|_\alpha^2}$$

and

$$q(x;n) = C_{n-1}(x-a) + D_{n-1} \tag{2.4}$$

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with

$$C_{n-1} = \frac{M\widetilde{L}_n^\alpha(a)L_n^\alpha(a) + N(\widetilde{L}_n^\alpha)'(a)(L_n^\alpha)'(a)}{||L_{n-1}^\alpha||_\alpha^2}, \quad D_{n-1} = \frac{N(\widetilde{L}_n^\alpha)'(a)L_n^\alpha(a)}{||L_{n-1}^\alpha||_\alpha^2}.$$

Moreover, the expressions for the every $\widetilde{L}_n^\alpha(a)$ and $(\widetilde{L}_n^\alpha)'(a)$ are

$$\widetilde{L}_n^\alpha(a) = \frac{L_n^\alpha(a)(1 + NK_{n-1}^{(1,1)}(a,a)) - (L_n^\alpha)'(a)NK_{n-1}^{(0,1)}(a,a)}{(1 + MK_{n-1}(a,a))(1 + NK_{n-1}^{(1,1)}(a,a)) - MN(K_{n-1}^{(1,0)})^2(a,a)},$$

$$(\widetilde{L}_n^\alpha)'(a) = \frac{(1 + MK_{n-1}(a,a))(L_n^\alpha)'(a) - MK_{n-1}^{(1,0)}(a,a)L_n^\alpha(a)}{(1 + MK_{n-1}(a,a))(1 + NK_{n-1}^{(1,1)}(a,a)) - MN(K_{n-1}^{(1,0)})^2(a,a)},$$

and, as consequence of Cauchy–Bunyakovskii–Schwarz inequality,

$$(1 + MK_{n-1}(a,a))(1 + NK_{n-1}^{(1,1)}(a,a)) - MN(K_{n-1}^{(1,0)})^2(a,a) \geq 1.$$

REMARK 2.5. We can give other formulae for $\widetilde{L}_n^\alpha(x)$ in terms of polynomials orthogonal with respect to the weights $(x-a)^2e^{-x}x^\alpha$ and $(x-a)^4e^{-x}x^\alpha$. These perturbations on the classical Laguerre measure are known in the literature as iterated Christoffel perturbations [3, 19].

3. Interlacing of zeros

When $N = 0$, the zeros of $\widetilde{L}_n^\alpha(x)$ interlace with the zeros of $L_n^\alpha(x)$ [5]. When the two masses are positive, under certain conditions, it can be shown that this property is satisfied. However, in general, the zeros of $\widetilde{L}_n^\alpha(x)$ and $L_n^\alpha(x)$ do not interlace. We show that the interlacing depends of the location of the smallest zero of $\widetilde{L}_n^\alpha(x)$.

Let $\{\widetilde{x}_{n,k}\}_{k=1}^n$ be the zeros of $\widetilde{L}_n^\alpha(x)$, in increasing order. To prove the interlacing properties, we need the following lemmas.

LEMMA 3.1.

$$\operatorname{sign}(\widetilde{L}_n^\alpha(a) \times (\widetilde{L}_n^\alpha)'(a)) = \begin{cases} 1 & \text{if } \widetilde{x}_{n,1} < a \\ -1 & \text{if } \widetilde{x}_{n,1} > a. \end{cases}$$

PROOF. Let $\widetilde{x}_{n,1} < a$ and assume that $\widetilde{L}_n^\alpha(a) > 0$. We suppose that $(\widetilde{L}_n^\alpha)'(a) < 0$ (the case $\widetilde{L}_n^\alpha(a) < 0$, $(\widetilde{L}_n^\alpha)'(a) > 0$ is similar) in order to obtain a contradiction. Let us denote

$$\phi(x) = \frac{\widetilde{L}_n^\alpha(x)}{x - \widetilde{x}_{n,1}}.$$

Notice that $\phi(a) > 0$. Moreover, $\phi(x)\widetilde{L}_n^\alpha(x) \geq 0$ for every $x \geq 0$. Using the orthogonality of $\{\widetilde{L}_n^\alpha(x)\}_{n\in\mathbb{N}}$, we get

$$\langle \phi(x), \widetilde{L}_n^\alpha(x) \rangle = \int_0^\infty \phi(x)\widetilde{L}_n^\alpha(x)e^{-x}x^\alpha\, dx + M\phi(a)\widetilde{L}_n^\alpha(a) + N\phi'(a)(\widetilde{L}_n^\alpha)'(a) = 0.$$

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Given that $\int_0^\infty \phi(x)\widetilde{L}_n^\alpha(x)e^{-x}x^\alpha\,dx \geq 0$ and $M\phi(a)\widetilde{L}_n^\alpha(a) > 0$, $N\phi'(a)(\widetilde{L}_n^\alpha)'(a)$ must be negative, but

$$\phi'(a) = \frac{(\widetilde{L}_n^\alpha)'(a)(a-\widetilde{x}_{n,1}) - \widetilde{L}_n^\alpha(a)}{(a-\widetilde{x}_{n,1})^2} < 0,$$

and then $N\phi'(a)(\widetilde{L}_n^\alpha)'(a) > 0$, which is a contradiction. Now if $\widetilde{x}_{n,1} > a$, take into account the representation

$$\widetilde{L}_n^\alpha(x) = \prod_{k=1}^n (x-\widetilde{x}_{nk})$$

and its derivative

$$(\widetilde{L}_n^\alpha)'(x) = \sum_{j=1}^n \prod_{k=1,k\neq j}^n (x-\widetilde{x}_{nk}).$$

Given that $a-\widetilde{x}_{nk} < 0$ for $k = 1, 2, 3, \ldots, n$, if n is even we find that $\widetilde{L}_n^\alpha(a)$ >0 and $(\widetilde{L}_n^\alpha)'(a) < 0$. Similarly, if n is odd, $\widetilde{L}_n^\alpha(a) < 0$ and $(\widetilde{L}_n^\alpha)'(a) > 0$. As a consequence, $\operatorname{sign}(\widetilde{L}_n^\alpha(a) \times (\widetilde{L}_n^\alpha)'(a)) = -1$. □

Lemma 3.2. *For every $k = 1, 2, 3, \ldots, n$, the zeros $\{x_{n,k}\}_{k=1}^n$ of $L_n^\alpha(x)$ satisfy*

$$\frac{(L_n^\alpha)'(a)}{L_n^\alpha(a)} < \frac{1}{a-x_{n,k}}. \tag{3.1}$$

Proof. If $\{y_{n-1,k}\}$, $k = 1, 2, 3, \ldots, n-1$, are the zeros of $L_{n-1}^{\alpha+1}(x)$, and $x \neq x_{n,k}$, then

$$\frac{(L_n^\alpha)'(x)}{L_n^\alpha(x)} = \frac{nL_{n-1}^{\alpha+1}(x)}{L_n^\alpha(x)} = n\frac{\prod_{k=1}^{n-1}(x-y_{n-1,k})}{\prod_{k=1}^{n}(x-x_{n,k})}.$$

In particular, evaluating at $x = a$,

$$\frac{(L_n^\alpha)'(a)}{L_n^\alpha(a)} < \frac{\prod_{k=1}^{n-1}(a-y_{n-1,k})}{\prod_{k=1}^{n}(a-x_{n,k})}. \tag{3.2}$$

Given that $x_{n,i} < y_{n-1,i} < x_{n,i+1}$ [4], we obtain

$$\frac{a-y_{n-1,i}}{a-x_{n,i+1}} < 1 \quad \text{for } i = 1, 2, 3, \ldots, n-1,$$

and by using (3.2) we have

$$\frac{(L_n^\alpha)'(a)}{L_n^\alpha(a)} < \frac{1}{a-x_{n,1}}.$$

Given that $x_{n,1} \leq x_{n,k}$, for $k = 1, 2, 3, \ldots, n-1$,

$$\frac{1}{a-x_{n,1}} \leq \frac{1}{a-x_{n,k}},$$

and as a conclusion we obtain (3.1). □

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In the next proposition we present the main result of this section, concerning the interlacing of the zeros of $\widetilde{L}_n^\alpha(x)$ and $L_n^\alpha(x)$.

Proposition 3.3. *If $\widetilde{x}_{n,k} > a$ for $k = 1, 2, 3, \ldots, n$, the zeros of $\widetilde{L}_n^\alpha(x)$ and $L_n^\alpha(x)$ interlace in the following way:*

$$\widetilde{x}_{n,1} < x_{n,1} < \widetilde{x}_{n,2} < x_{n,2} < \cdots < \widetilde{x}_{n,n} < x_{n,n}.$$

Proof. Let $x_{n,k}$ be any zero of $L_n^\alpha(x)$. We evaluate (2.3) at $x = x_{n,k}$:

$$(x_{n,k} - a)^2 \widetilde{L}_n^\alpha(x_{n,k}) = q(x_{n,k}; n) L_{n-1}^\alpha(x_{n,k}).$$

It is enough to prove that $q(x_{n,k}; n) > 0$. Indeed, by using (2.4),

$$q(x_{n,k}; n) = \frac{M\widetilde{L}_n^\alpha(a)L_n^\alpha(a)(x_{n,k} - a) + N(\widetilde{L}_n^\alpha)'(a)[(L_n^\alpha)'(a)(x_{n,k} - a) + L_n^\alpha(a)]}{\|L_{n-1}^\alpha\|_\alpha^2}.$$

If $(\widetilde{L}_n^\alpha)'(a) > 0$ (which implies that $L_n^\alpha(a) < 0$; see Lemma 3.1), we want to prove that

$$(L_n^\alpha)'(a)(x_{n,k} - a) + L_n^\alpha(a) > 0,$$

or

$$\frac{(L_n^\alpha)'(a)}{L_n^\alpha(a)} < \frac{1}{a - x_{n,k}},$$

but this follows from (3.1). In the same way, if $(\widetilde{L}_n^\alpha)'(a) < 0$ (which implies that $L_n^\alpha(a) > 0$), we prove that

$$(L_n^\alpha)'(a)(x_{n,k} - a) + L_n^\alpha(a) < 0,$$

or

$$\frac{(L_n^\alpha)'(a)}{L_n^\alpha(a)} < \frac{1}{a - x_{n,k}}.$$

Thus

$$M\widetilde{L}_n^\alpha(a)L_n^\alpha(a)(x_{n,k} - a) + N(\widetilde{L}_n^\alpha)'(a)[(L_n^\alpha)'(a)(x_{n,k} - a) + L_n^\alpha(a)] > 0,$$

or

$$q(x_{n,k}; n) > 0,$$

which implies that $\widetilde{L}_n^\alpha(x_{n,k})$ and $L_{n-1}^\alpha(x_{n,k})$ have the same sign, and since the zeros of $L_{n-1}^\alpha(x)$ interlace with the zeros of $L_n^\alpha(x)$, the positive zeros of $\widetilde{L}_n^\alpha(x)$ interlace with the zeros of $L_n^\alpha(x)$. □

We now show some results for the case where one of the two masses in (1.1) is zero. We denote by $\{L_k^{[\alpha,2]}(x)\}_{k\in\mathbb{N}}$ the sequence of monic polynomials orthogonal with respect to the weight function $(x - a)^2 e^{-x} x^\alpha$, and by $\{x_{n,k}^{[2]}\}_{k=1}^n$ the

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zeros of $L_n^{[\alpha,2]}(x)$ (see the paper by Fejzullahu and Zejnullahu [9] and the book by Szegö [22] for more information concerning canonical perturbations of Christoffel type). Also, $\{\widetilde{L}_n^{\alpha,M}(x)\}_{k\in\mathbb{N}}$ and $\{\widetilde{L}_n^{\alpha,N}(x)\}_{k\in\mathbb{N}}$ denote, respectively, the sequences of monic polynomials orthogonal with respect to the inner product (1.1) when $N = 0$ and $M = 0$. Let $\{x_{n,k}^M\}_{k=1}^n$ be the zeros of $\widetilde{L}_n^{\alpha,M}(x)$ and $\{x_{n,k}^N\}_{k=1}^n$ the zeros of $\widetilde{L}_n^{\alpha,N}(x)$.

We use the next result, the proof of which is given by Huertas et al. [19].

Proposition 3.4. *Let $M \geq 0$ and $N > 0$, and denote by $\{\widetilde{x}_{n,k}\}_{k=1}^n$ the zeros of $\widetilde{L}_n^{\alpha}(x)$. Assume that $\widetilde{x}_{n,1} < a$. Then the inequalities*

$$a < \widetilde{x}_{n,2} < x_{n-1,1}^{[2]} < \cdots < \widetilde{x}_{n,n} < x_{n-1,n-1}^{[2]}$$

hold for every $n \in \mathbb{N}$.

The previous result reads

$$a < x_{n,2}^N < x_{n-1,1}^{[2]} < \cdots < x_{n,n}^N < x_{n-1,n-1}^{[2]}. \tag{3.3}$$

On the other hand, in (1.1), if $N = 0$ and $M > 0$, it is possible to show [5, 11] the interlacing property

$$a < x_{n,1}^M < x_{n,1} < x_{n-1,1}^{[2]} < x_{n,2}^M < x_{n,2} < \cdots < x_{n-1,n-1}^{[2]} < x_{n,n}^M < x_{n,n}, \tag{3.4}$$

where $\{x_{n,k}\}_{k=1}^n$ are the zeros of $L_n^\alpha(x)$.

Now we show that the zeros of $L_n^\alpha(x)$ and $\widetilde{L}_n^{\alpha,N}(x)$ are interlaced.

Proposition 3.5. *For every $N > 0$, the zeros of $L_n^\alpha(x)$ and $\widetilde{L}_n^{\alpha,N}(x)$ satisfy*

$$x_{n,1}^N < x_{n,1} < x_{n,2}^N < x_{n,2} < \cdots < x_{n,n}^N < x_{n,n}. \tag{3.5}$$

Proof. According to Dueñas et al. [6], the polynomial $\widetilde{L}_n^{\alpha,N}(x)$ is represented by the formula

$$(x-a)^2\widetilde{L}_n^{\alpha,N}(x) = L_n^\alpha(x)\Big[(x-a)^2 - \frac{N(L_n^\alpha)'(a)}{1+NK_{n-1}^{(1,1)}(a,a)}\,\frac{L_{n-1}^\alpha(a)+(x-a)(L_{n-1}^\alpha)'(a)}{\|L_{n-1}^\alpha\|_\alpha^2}\Big] + L_{n-1}^\alpha(x)\frac{N(L_n^\alpha)'(a)}{1+NK_{n-1}^{(1,1)}(a,a)}\,\frac{L_n^\alpha(a)+(x-a)(L_n^\alpha)'(a)}{\|L_{n-1}^\alpha\|_\alpha^2}.$$

Substituting $x = x_{n,k}$,

$$(x_{n,k}-a)^2\widetilde{L}_n^{\alpha,N}(x_{n,k}) = L_{n-1}^\alpha(x_{n,k})\frac{N(L_n^\alpha)'(a)}{1+NK_{n-1}^{(1,1)}(a,a)}\,\frac{L_n^\alpha(a)+(x_{n,k}-a)(L_n^\alpha)'(a)}{\|L_{n-1}^\alpha\|_\alpha^2}.$$

We show that

$$L_n^\alpha(a)(L_n^\alpha)'(a) + (x_{n,k}-a)[(L_n^\alpha)'(a)]^2 > 0. \tag{3.6}$$

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We assume that $L_n^\alpha(a) > 0$ and n is even (the procedure is similar if n is odd). Taking into account that $a - x_{n,k}$ is negative, we multiply (3.1) by $L_n^\alpha(a)$, $(L_n^\alpha)'(a)$ and N, and obtain (3.6). Thus

$$\frac{N(L_n^\alpha)'(a)}{1 + NK_{n-1}^{(1,1)}(a,a)} \frac{L_n^\alpha(a) + (x_{n,k} - a)(L_n^\alpha)'(a)}{\|L_{n-1}^\alpha\|_\alpha^2} > 0,$$

and then $\widetilde{L}_n^{\alpha,N}(x_{n,k})$ and $L_{n-1}^\alpha(x_{n,k})$ have the same sign, and since the zeros of $L_{n-1}^\alpha(x)$ interlace with the zeros of $L_n^\alpha(x)$, the positive zeros of $\widetilde{L}_n^{\alpha,N}(x)$ interlace in the same way with the zeros of $L_n^\alpha(x)$. □

Finally, using (3.3)–(3.5), we obtain the following corollary.

COROLLARY 3.6. *If $M > 0$, $N > 0$ and $x_{n,1}^N < a$ then*

$$x_{n,1}^N < a < x_{n,1}^M < x_{n,2}^N < x_{n,2}^M < \cdots < x_{n,n}^N < x_{n,n}^M.$$

4. Holonomic equation and electrostatic interpretation

The connection formula (2.3) can be written as

$$\widetilde{L}_n^\alpha(x) = \phi(x;n)L_n^\alpha(x) + \psi(x;n)L_{n-1}^\alpha(x), \tag{4.1}$$

where

$$\phi(x;n) = \frac{p(x;n)}{(x-a)^2} \quad \text{and} \quad \psi(x;n) = \frac{q(x;n)}{(x-a)^2}.$$

(Henceforth, for the sake of brevity, we write $\phi(x)$ and $\psi(x)$ instead of $\phi(x;n)$ and $\psi(x;n)$, respectively.) Taking the derivative in (4.1) and multiplying by x, we get

$$\begin{aligned} x(\widetilde{L}_n^\alpha)'(x) = \phi(x)[x(L_n^\alpha)'(x)] + x\phi'(x)L_n^\alpha(x) \\ + \psi(x)[x(L_{n-1}^\alpha)'(x)] + x\psi'(x)L_{n-1}^\alpha(x). \end{aligned}$$

Using (2.2) and (2.1) respectively, we obtain

$$\begin{aligned} x(\widetilde{L}_n^\alpha)'(x) &= L_n^\alpha(x)[\phi(x)n + x\phi'(x)] \\ &\quad + L_{n-1}^\alpha(x)[\phi(x)n(n+\alpha) + \psi(x)(n-1) + x\psi'(x)] \\ &\quad + \psi(x)\,(n-1)\,(n-1+\alpha)L_{n-2}^\alpha(x), \\ x(\widetilde{L}_n^\alpha)'(x) &= L_n^\alpha(x)[\phi(x)n + x\phi'(x) - \psi(x)] + L_{n-1}^\alpha(x)[\phi(x)n(n+\alpha) \\ &\quad + \psi(x)(n-1) + x\psi'(x) + \psi(x)(x - (2n-1+\alpha))]. \end{aligned}$$

After some algebraic manipulations, we obtain

$$x(\widetilde{L}_n^\alpha)'(x) = \widehat{A}(x,n)L_n^\alpha(x) + \widehat{B}(x,n)L_{n-1}^\alpha(x),$$

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with

$$\begin{aligned}\widehat{A}(x,n) &= \phi(x)n + x\phi'(x) - \psi(x)\\ &= \frac{1}{(x-a)^4}[p(x-a)^2 n + xp'(x-a)^2 - 2p(x-a) - (x-a)^2 q], \end{aligned} \tag{4.2}$$

$$\begin{aligned}\widehat{B}(x,n) &= \phi(x)n(n+\alpha) + \psi(x)(n-1) + x\psi'(x) + \psi(x)(x-(2n-1+\alpha))\\ &= \frac{1}{(x-a)^4}[p(x-a)^2 n(n+\alpha) + q(x-a)^2(x-(n+\alpha))\\ &\qquad + x[q'(x-a)^2 - 2q(x-a)]]. \end{aligned} \tag{4.3}$$

Again, by using (4.1), we get a linear system of two equations in the unknowns $L_n^\alpha(x)$ and $L_{n-1}^\alpha(x)$:

$$\begin{gathered}\phi(x)L_n^\alpha(x) + \psi(x)L_{n-1}^\alpha(x) = \widetilde{L}_n^\alpha(x),\\ \widehat{A}(x,n)L_n^\alpha(x) + \widehat{B}(x,n)L_{n-1}^\alpha(x) = x(\widetilde{L}_n^\alpha)'(x).\end{gathered}$$

Then

$$L_n^\alpha(x) = \frac{\widehat{B}(x,n)\widetilde{L}_n^\alpha(x) - x\psi(x)(\widetilde{L}_n^\alpha)'(x)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}, \tag{4.4}$$

$$L_{n-1}^\alpha(x) = \frac{x\phi(x)(\widetilde{L}_n^\alpha)'(x) - \widetilde{L}_n^\alpha(x)\widehat{A}(x,n)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}, \tag{4.5}$$

and replacing in (2.2) yields

$$\begin{aligned} x\bigg(\frac{\widehat{B}(x,n)\widetilde{L}_n^\alpha(x) - x\psi(x)(\widetilde{L}_n^\alpha)'(x)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}\bigg)' &= n\bigg(\frac{\widehat{B}(x,n)\widetilde{L}_n^\alpha(x) - x\psi(x)(\widetilde{L}_n^\alpha)'(x)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}\bigg)\\ &\quad + n(n+\alpha)\bigg(\frac{x\phi(x)(\widetilde{L}_n^\alpha)'(x) - \widetilde{L}_n^\alpha(x)\widehat{A}(x,n)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}\bigg), \end{aligned} \tag{4.6}$$

where

$$\Phi(x,n) = \phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n).$$

By using (4.2) and (4.3), we get

$$\begin{aligned}\Phi(x,n) &= \frac{1}{(x-a)^6}[p^2(x-a)^2 n(n+\alpha) + pq(x-a)^2(x-(n+\alpha))\\ &\qquad + pxq'(x-a)^2 - pq(x-a)^2 n - xqp'(x-a)^2 + (x-a)^2 q^2]\\ &= \frac{1}{(x-a)^4}[p^2 n(n+\alpha) + pq[x-2n-\alpha] + pxq' - xqp' + q^2],\end{aligned}$$

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and, as a consequence,

$$\Phi(x,n) = \frac{1}{(x-a)^4}\Delta(x,n),$$

where

$$\Delta(x,n) = p^2 n(n+\alpha) + pq[x-2n-\alpha] + x[pq' - qp'] + q^2.$$

REMARK 4.1. Notice that the degree of $\Delta(x,n)$ is four, and that this polynomial does not vanish at the zeros of $\widetilde{L}_n^\alpha(x)$, given that the zeros of $\widetilde{L}_n^\alpha(x)$ are simple (multiplicity 1), and by using equations (4.4) and (4.5). The behaviour of $\Delta(x,n)$ for n large constitutes a open problem.

Taking the first derivative in (4.6), we get

$$\begin{aligned} n&\bigg(\frac{\widehat{B}(x,n)\widetilde{L}_n^\alpha(x) - x\psi(x)(\widetilde{L}_n^\alpha)'(x)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}\bigg) + n(n+\alpha)\bigg(\frac{x\phi(x)(\widetilde{L}_n^\alpha)'(x) - \widetilde{L}_n^\alpha(x)\widehat{A}(x,n)}{\phi(x)\widehat{B}(x,n) - \psi(x)\widehat{A}(x,n)}\bigg) \\ &= x\frac{[(\widehat{B}(x,n))'\widetilde{L}_n^\alpha(x) + \widehat{B}(x,n)(\widetilde{L}_n^\alpha)'(x) - (x\psi(x))'(\widetilde{L}_n^\alpha)'(x) - x\psi(x)(\widetilde{L}_n^\alpha)''(x)]}{\Phi(x,n)} \\ &\quad + x\frac{\Phi'(x,n)[x\psi(x)(\widetilde{L}_n^\alpha)'(x) - \widehat{B}(x,n)\widetilde{L}_n^\alpha(x)]}{\Phi^2(x,n)}, \end{aligned}$$

or

$$A(x,n)(\widetilde{L}_n^\alpha)''(x) + B(x,n)(\widetilde{L}_n^\alpha)'(x) + C(x,n)\widetilde{L}_n^\alpha(x) = 0, \tag{4.7}$$

where

$$\begin{aligned} A(x,n) &= \frac{x^2\psi(x)}{\Phi(x,n)}, \\ B(x,n) &= \frac{x(x\psi(x))' - x\widehat{B}(x,n) - nx\psi(x) + n(n+\alpha)x\phi(x)}{\Phi(x,n)} - \frac{x^2\psi(x)\Phi'(x,n)}{\Phi^2(x,n)}, \\ C(x,n) &= \frac{x\widehat{B}(x,n)\Phi'(x,n)}{\Phi^2(x,n)} - \frac{n\widehat{B}(x,n) - n(n+\alpha)\widehat{A}(x,n) - x(\widehat{B}(x,n))'}{\Phi(x,n)}. \end{aligned}$$

Taking into account (4.3),

$$x(x\psi(x))' - x\widehat{B}(x,n) - nx\psi(x) + n(n+\alpha)x\phi(x) = x\psi(x)[\alpha - x + 1].$$

Then the coefficient $B(x,n)$ can be written as

$$B(x,n) = \frac{x\psi(x)[(\alpha+1) - x]}{\Phi(x,n)} - \frac{x^2\psi(x)\Phi'(x,n)}{\Phi^2(x,n)}.$$

We summarize the above results in the next theorem.

THEOREM 4.2. *Let $\{L_n^\alpha(x)\}_{n\in\mathbb{N}}$ be the sequence of classical Laguerre orthogonal polynomials and let $\{\widetilde{L}_n^\alpha(x)\}_{n\in\mathbb{N}}$ be the sequence of polynomials orthogonal with respect to the inner product (1.1). For every $n \in \mathbb{N}$, $\widetilde{L}_n^\alpha(x)$ satisfies the differential equation*

$$A(x,n)\rho''(x) + B(x,n)\rho'(x) + C(x,n)\rho(x) = 0,$$

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where

$$A(x,n)=\frac{x^2\psi(x)}{\Phi(x,n)},$$
$$B(x,n)=\frac{x\psi(x)[(\alpha+1)-x]}{\Phi(x,n)}-\frac{x^2\psi(x)\Phi'(x,n)}{\Phi^2(x,n)},$$
$$C(x,n)=\frac{x\widehat{B}(x,n)\Phi'(x,n)}{\Phi^2(x,n)}-\frac{n\widehat{B}(x,n)-n(n+\alpha)\widehat{A}(x,n)-x(\widehat{B}(x,n))'}{\Phi(x,n)},$$

with

$$\Phi(x,n)=\frac{1}{(x-a)^4}\Delta(x,n),$$

where

$$\Delta(x,n)=p(x;n)^2n(n+\alpha)+p(x;n)q(x;n)[x-2n-\alpha] + x[p(x;n)q(x;n)'-q(x;n)p(x;n)']+q(x;n)^2.$$

We now deduce an electrostatic interpretation for the zeros of $\widetilde{L}_n^\alpha(x)$. We direct the reader to the work of Heine [10] and Stieltjes [21] on the electrostatic interpretation of the zeros of classical orthogonal polynomials, and the work of Ismail [12, 13] on electrostatic models for general orthogonal polynomials.

Let $\widetilde{x}_{n,k}$ be the kth zero of $\widetilde{L}_n^\alpha(x)$. Substitution in (4.7) yields

$$\frac{(\widetilde{L}_n^\alpha)''(\widetilde{x}_{n,k})}{(\widetilde{L}_n^\alpha)'(\widetilde{x}_{n,k})}=-\frac{B(\widetilde{x}_{n,k},n)}{A(\widetilde{x}_{n,k},n)}=1-\frac{(1+\alpha)}{\widetilde{x}_{n,k}}+D[\ln(\Phi(\widetilde{x}_{n,k},n))].$$

Then

$$\frac{(\widetilde{L}_n^\alpha)''(\widetilde{x}_{n,k})}{(\widetilde{L}_n^\alpha)'(\widetilde{x}_{n,k})}=1-\frac{1+\alpha}{\widetilde{x}_{n,k}}+D[\ln(\Phi(\widetilde{x}_{n,k},n))]. \tag{4.8}$$

On the other hand, if we consider the representation of the $\widetilde{L}_n^\alpha(x)$ in terms of their zeros,

$$\widetilde{L}_n^\alpha(x)=(x-\widetilde{x}_{n,1})(x-\widetilde{x}_{n,2})\cdots(x-\widetilde{x}_{n,n}),$$

then at any root $\widetilde{x}_{n,k}$ of $\widetilde{L}_n^\alpha(x)$ we have

$$\frac{(\widetilde{L}_n^\alpha)''(\widetilde{x}_{n,k})}{(\widetilde{L}_n^\alpha)'(\widetilde{x}_{n,k})}=-2\sum_{j=1,j\neq k}^{n}\frac{1}{\widetilde{x}_{n,j}-\widetilde{x}_{n,k}}=2\sum_{j=1,j\neq k}^{n}\frac{1}{\widetilde{x}_{n,k}-\widetilde{x}_{n,j}}.$$

Using this property and (4.8), we get

$$-2\sum_{j=1,j\neq k}^{n}\frac{1}{\widetilde{x}_{n,k}-\widetilde{x}_{n,j}}+1-\frac{1+\alpha}{\widetilde{x}_{n,k}}+D[\ln(\Phi(x,n))]\Big|_{\widetilde{x}_{n,k}}=0, \tag{4.9}$$

for $k=1,2,\ldots,n$.

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Using equation (4.8), we can consider the two external fields

$$V_1(x) = x - (1+\alpha)\ln|x| - (x-a)^4 \quad \text{and} \quad V_2(x) = \ln(\Delta(x,n)),$$

and also the external potential

$$\begin{aligned} V(x) &= x - (1+\alpha)\ln|x| + \ln(\Phi(x,n)) \\ &= \ln(\Delta(x,n)) - \ln((x-a)^4 e^{-x} x^{\alpha+1}). \end{aligned}$$

Then, given n movable unit charges in the real line, in the presence of the external potential $V(x)$, and letting $\mathbf{x} = (x_1, \ldots, x_n)$ be the vector that represents the positions of these n particles, the total energy of the system, $E(x)$, is

$$E(x) = \sum_{k=1}^{n} V(x_k) - 2 \sum_{1 \le j < k \le n}^{n} \ln|x_j - x_k|.$$

We show that the vector $\mathbf{x} = (\widetilde{x}_{n,1}, \widetilde{x}_{n,2}, \ldots, \widetilde{x}_{n,n})$ is a critical point of the scalar field $E(x)$. Indeed,

$$\begin{aligned} \frac{\partial(E(x))}{\partial x_k}\bigg|_{x=(\widetilde{x}_{n,1},\widetilde{x}_{n,1},\ldots,\widetilde{x}_{n,n})} &= 1 - \frac{1+\alpha}{\widetilde{x}_{n,k}} + D[\ln(\Phi(x,n))]\big|_{\widetilde{x}_{n,k}} + 2\sum_{j=1, j\neq k}^{n} \frac{1}{\widetilde{x}_{n,j} - \widetilde{x}_{n,k}} \\ &= 1 - \frac{1+\alpha}{\widetilde{x}_{n,k}} + D[\ln(\Phi(x,n))]\big|_{\widetilde{x}_{n,k}} - 2\sum_{j=1, j\neq k}^{n} \frac{1}{\widetilde{x}_{n,k} - \widetilde{x}_{n,j}}, \end{aligned}$$

and according to (4.9),

$$\frac{\partial(E(x))}{\partial x_k}\bigg|_{x=(\widetilde{x}_{n,1},\widetilde{x}_{n,2},\ldots,\widetilde{x}_{n,n})} = 0.$$

This shows that $\mathbf{x} = (\widetilde{x}_{n,1}, \widetilde{x}_{n,2}, \ldots, \widetilde{x}_{n,n})$ is an equilibrium point, that is, the zeros are the components of the critical point of the gradient of the total energy.

According to Ismail [12], the term

$$v_{\text{short}}(x) = \ln(\Delta(x,n))$$

represents a *short range potential*, and the term

$$v_{\text{long}}(x) = -\ln((x-a)^4 e^{-x} x^{\alpha+1})$$

represents a *long range potential* which is associated with the perturbation of the classical Laguerre measure.

5. Numerical experiments

In this section we present some numerical experiments conducted using Matlab. We focus on experimenting with the position of the zeros of the polynomial $\widetilde{L}_n^{\alpha}(x)$ and the

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TABLE 1. Zeros of $\widetilde{L}_4^{\alpha}(x)$ and $\Delta(x, 4)$.

	1st	2nd	3rd	4th
$\widetilde{L}_4^{\alpha}(x)$	−2.35226	−0.16692	3.37526	8.603665
$\Delta(x, 4)$	−2.2566	$-2-(9\times10^{-13})$	−1.26271	−0.08764

TABLE 2. Zeros of $\widetilde{L}_{10}^{\alpha}(x)$ and $\Delta(x, 10)$.

	1st	2nd	3rd	4th
$\widetilde{L}_{10}^{\alpha}(x)$	−2.06068	−1.93186	0.891036	2.5545
$\Delta(x, 10)$	−2.004	$-2-(6\times10^{11})$	−1.95278	−1.1589

TABLE 3. Zeros of $\widetilde{L}_{14}^{\alpha}(x)$ and $\Delta(x, 14)$.

	1st	2nd	3rd	4th
$\widetilde{L}_{14}^{\alpha}(x)$	−2.0145	−1.985	0.566	1.6999
$\Delta(x, 14)$	−2.010333	$-2-(6\times10^{-9})$	−1.98946	−1.2787

TABLE 4. Zeros of $\widetilde{L}_{18}^{\alpha}(x)$ and $\Delta(x, 18)$.

	1st	2nd	3rd	4th
$\widetilde{L}_{18}^{\alpha}(x)$	−2.00401	−1.99595	0.4052	1.2531
$\Delta(x, 18)$	−2.00284	$-2-(4\times10^{-8})$	−1.9971	−1.35769

real zeros of the polynomial $\Delta(x, n)$, as n increases. In order to obtain the zeros of $\widetilde{L}_n^{\alpha}(x)$, we use $M = N = 1$, $\alpha = 1$ and $\zeta = -2$, whereas n varies. In Tables 1 and 2 we use $n = 4$ and $n = 10$.

As is well known, if n increases, ζ attracts two zeros of $\widetilde{L}_n^{\alpha}(x)$; moreover, the zeros of $\Delta(x, n)$ are still negative. In Tables 3 and 4 we see the behaviour of the zeros of $\widetilde{L}_n^{\alpha}(x)$ and $\Delta(x, n)$ for degrees $n = 14$ and $n = 18$. We see that two zeros of $\Delta(x, n)$ are less than the mass point while others are larger, but these are close to the mass point position in so far as n increases. We also see that, as expected, the polynomial $\Delta(x, n)$ never vanishes at the zeros of the polynomial $\widetilde{L}_n^{\alpha}(x)$.

Acknowledgements

The author thanks the referee for his careful reading and suggestions, and also Professor Herbert Dueñas for his valuable guidance and his patience.

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