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EXISTENCE OF A WEAK SOLUTION FOR A CLASS OF FRACTIONAL LAPLACIAN EQUATIONS

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Abstract

We study the existence of a weak solution of a nonlocal problem

$$-\mathcal{L}_K u - \mu u g_1 + h(u) g_2 = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

where $\mathcal{L}_k$ is a general nonlocal integrodifferential operator of fractional type, μ is a real parameter and Ω is an open bounded subset of $\mathbb{R}^n$ ($n > 2s$, where $s \in (0,1)$ is fixed) with Lipschitz boundary $\partial\Omega$. Here $f, g_1, g_2 : \Omega \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are functions satisfying suitable hypotheses.

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1. Introduction

Recently, there has been considerable attention paid to the study of fractional and nonlocal operators of elliptic type. Many interesting problems in the standard framework of the Laplacian are widely studied in the literature. Nonlocal operators arise in a natural way in many contexts, such as thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, conservation laws, multiple scattering, minimal surfaces, materials science and water waves. It is natural to enquire about the existence of a solution for a nonlocal framework by extending the corresponding classical results.

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n > 2s$, (where $s \in (0,1)$) with Lipschitz boundary $\partial\Omega$. Let $\mu \in \mathbb{R}$ and $f, g_1, g_2 : \Omega \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be functions satisfying suitable hypotheses. We study the existence of weak solutions for the class of nonlocal problems given by

$$\begin{aligned} -\mathcal{L}_K u - \mu u g_1 + h(u) g_2 = f \quad &\text{in } \Omega, \\ u = 0 \quad &\text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{1.1}$$

where $\mathcal{L}_K$ is the integrodifferential operator defined as

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy \quad x \in \mathbb{R}^n,$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ such that

$$mK \in L^1(\mathbb{R}^n) \quad \text{where } m(x) = \min\{1, |x|^2\}; \tag{1.2}$$

$$\text{there exists } \lambda > 0 \text{ such that } K(x) \geq \lambda |x|^{-(n+2s)} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}; \text{ and} \tag{1.3}$$

$$K(x) = K(-x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

An example for the singular kernel K is given by $K(x) := |x|^{-(n+2s)}$, which gives rise to the fractional Laplace operator $-(-\Delta)^s$ and which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\,dy, \quad x \in \mathbb{R}^n.$$

The homogeneous Dirichlet condition in (1.1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on the boundary $\partial\Omega$, which is consistent with the nonlocal nature of the operator $\mathcal{L}_K$. (We refer to [3] and the references therein for further details on the fractional Laplacian).

Let the functional space X denote the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map

$$(x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}$$

is in $L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (C\Omega \times C\Omega), dx\,dy)$ (here $C\Omega := \mathbb{R}^n \setminus \Omega$). We define X_0 to be the linear subspace of X, where

$$X_0 := \{g \in X : g = 0 \text{ almost everywhere in } \mathbb{R}^n \setminus \Omega\}.$$

The space X_0 is introduced in [16]; also refer to [15]. By a weak solution of (1.1), we mean a solution $u \in X_0$ of the equation

$$\begin{aligned}\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)\,dx\,dy - \mu \int_\Omega g_1(x)u(x)\varphi(x)\,dx \\ + \int_\Omega h(u(x))g_2(x)\varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx \quad \forall \varphi \in X_0.\end{aligned} \tag{1.5}$$

Equivalently, (1.5) represents the weak formulation of (1.1). The weak solution of (1.1) lies in a functional space X_0, but this is not equivalent to the usual fractional Sobolev space. The choice of the space X_0 allows us to correctly encode the Dirichlet boundary datum in the weak formulation of (1.1). In Section 2, we recall the definitions of weak solutions of (1.1) and of the functional space X_0, to make the present paper self-contained.

In the current literature [1, 2, 4–6, 10–15, 17–19], many authors have studied nonlocal fractional Laplacian equations with superlinear, subcritical, asymptotically linear and critical nonlinearities. In [15], Servadei and Valdinoci studied the existence of nontrivial solutions of the problem

$$\begin{aligned} -\mathcal{L}_K u = f(x,u) \quad &\text{in } \Omega, \\ u = 0 \quad &\text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{1.6}$$

using the mountain pass theorem. The function $f(x,u)$ in (1.6) satisfies some suitable hypotheses and one of the conditions is that

$$\lim_{|t|\to 0} \frac{f(x,t)}{|t|} = 0 \quad \text{uniformly in } x \in \Omega.$$

The function $\tilde{f}(x,u) = \mu u g_1 - h(u) g_2 + f$, considered in (1.1), is not a special case of the function $f(x,u)$ in [15]. To study the existence of a solution of (1.1), the main tools we used are a result due to Hess [7] on linear demicontinuous operators and results on embedding theorems in suitable fractional Sobolev spaces. The study is inspired by a semilinear elliptic Dirichlet boundary value problem (BVP) on a bounded domain given in the book by Zeidler [21], where h is a real-valued bounded continuous function defined on $\mathbb{R}$ and $\mu > 0$ is a real number with certain restrictions.

The paper is organized as follows. Section 2 deals with preliminaries and some basic results. Section 3 is concerned with the main result, namely, the existence of a weak solution of (1.1) in a suitable fractional Sobolev space. Finally, Section 4 deals with an extension to a class of continuous functions h that are not necessarily bounded.

2. Preliminaries

Throughout, let Ω be an open bounded subset of $\mathbb{R}^n$, $n > 2s$ (where $s \in (0,1)$ is fixed) with Lipschitz boundary $\partial\Omega$. We briefly recall the definition of the functional space X_0, as introduced in [16].

Let X denote the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map

$$(x,y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}$$

is in $L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (C\Omega \times C\Omega), dx\,dy)$ (here $C\Omega := \mathbb{R}^n \setminus \Omega$). We define X_0 to be the linear subspace of X given by

$$X_0 := \{g \in X : g = 0 \text{ almost everywhere in } \mathbb{R}^n \setminus \Omega\}.$$

We know that X and X_0 are nonempty, since $C_0^2(\Omega) \subseteq X_0$, by [16, Lemma 5.1]. We define the norm on the space X as

$$\|g\|_X := \|g\|_{2,\Omega} + \left(\int_Q |g(x) - g(y)|^2 K(x-y)\,dx\,dy\right)^{1/2},$$

where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} = (C\Omega) \times (C\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$. We can easily verify that $\|\cdot\|_X$ is a norm on X (for a proof, we refer the reader to [15]). $\|\cdot\|_{p,\Omega}$ and $\|\cdot\|_{\infty,\Omega}$ denote the standard norms in $L^p(\Omega)$ and $L^\infty(\Omega)$, respectively.

For $v \in X_0$, by [15, Lemmas 6 and 7], in the subsequent work, we define the function

$$v \mapsto \|v\|_{X_0} = \Big(\int_Q |v(x) - v(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2} \tag{2.1}$$

as the norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (refer to [15, Lemmas 7]), with inner product

$$\langle u, v\rangle_{X_0} := \int_Q (u(x) - u(y))(v(x) - v(y))\,K(x-y)\,dx\,dy.$$

Since $v \in X_0$ (and so $v = 0$ almost everywhere in $\mathbb{R}^n \setminus \Omega$), we note that, in (2.1) (and in the related scalar product), the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$. While, for a general kernel K satisfying conditions (1.2)–(1.4), $X_0 \subset H^s(\mathbb{R}^n)$, in the case $K(x) := |x|^{-(n+2s)}$, the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ that vanish almost everywhere outside Ω (refer to [18, Lemma 7]).

We define $H^s(\mathbb{R}^n)$ to be the usual fractional Sobolev space endowed with the norm (the *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{2,\mathbb{R}^n} + \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}.$$

We recall the embedding properties of X_0 into the usual Lebesgue spaces (refer to [15, Lemma 8]), for the sake of completeness.

Lemma 2.1. *The embedding $j : X_0 \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for any $q \in [1, 2^*]$, while the embedding is compact whenever $q \in [1, 2^*)$, where $2^* = 2n/(n-2s)$. Hence, for any $q \in [1, 2^*]$, there exists a positive constant C(depending on q) such that*

$$\|v\|_{q,\mathbb{R}^n} \le C\|v\|_{X_0} \tag{2.2}$$

for any $v \in X_0$.

For further details on fractional Sobolev spaces, we refer the reader to [3] and to the references therein, while for other details on X and X_0, we refer to [16].

Let Y^* denote the dual of the real Banach space Y. $\|\cdot\|$ and $\|\cdot\|_{Y^*}$ denote the norms on Banach space Y and dual space Y^*, respectively. For $x \in Y$ and $f \in Y^*$, let $(f|x)$ denote the evaluation of linear functional f at x. At each step, a generic constant is denoted by c or k_0, in order to avoid too many suffices.

Definition 2.2. Let $B : Y \to Y^*$ be an operator on the real separable reflexive Banach space Y. Then:

(i) $B + N$ is asymptotically linear if B is linear and

$$\frac{\|Nu\|}{\|u\|} \to 0 \quad \text{as } \|u\| \to \infty; \text{ and}$$

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(ii) B satisfies condition (S) if

$$u_n \rightharpoonup u \quad \text{and} \quad \lim_{n\to\infty}(Bu_n - Bu|u_n - u) = 0 \quad \text{implies } u_n \to u.$$

We say that B satisfying condition (S) is a (S)-operator.

The following describes a real Gårding form G (compare with [20, page 364]).

Definition 2.3. Let X and Z be Hilbert spaces over $\mathbb{R}$ with the continuous embedding $X \subseteq Z$. Then $G : X \times X \to \mathbb{R}$ is called a *Gårding form* if and only if G is bilinear and bounded and there is a constant $c > 0$ and a real constant C such that

$$G(u, u) \geq c\|u\|_X^2 - C\|u\|_Z^2 \quad \text{for all } u \in X. \tag{2.3}$$

Equation (2.3) is called the Gårding inequality. If $C = 0$, then G is called a *strict Gårding form*. The Gårding form G is called *regular* if and only if the embedding $X \subseteq Z$ is compact.

In Section 3, we need the following result.

Proposition 2.4. *Let $B, N : Y \to Y^*$ be operators on the real separable reflexive Banach space Y. Then:*

(i) *the operator $B : Y \to Y^*$ is linear and continuous;*
(ii) *the operator $N : Y \to Y^*$ is demicontinuous and bounded;*
(iii) *$B + N$ is asymptotically linear; and*
(iv) *for each $T \in Y^*$ and for each $t \in [0, 1]$, the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S) in Y.*

If $Bu = 0$ implies $u = 0$ then, for each $T \in Y^*$, the equation $Bu + Nu = T$ has a solution in Y.

For a detailed proof of the above theorem, we refer to [7] or to [21, Theorem 29.C].
We need the following hypotheses for further study.

(H_1) Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded (that is, $|h(t)| \leq A, t \in \mathbb{R}, A > 0$) and continuous function.
(H_2) Assume that $g_1 \in L^\infty(\Omega)$, $g_2 \in L^2(\Omega)$ and $f \in L^2(\Omega)$.

We define the functionals $B_1, B_2 : X_0 \times X_0 \to \mathbb{R}$ as

$$B_1(u, \phi) = \int_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x - y)\,dx\,dy - \int_\Omega \mu u(x)g_1(x)\varphi(x)\,dx$$

$$B_2(u, \varphi) = \int_\Omega h(u(x))g_2(x)\varphi(x)\,dx.$$

Also define $T : X_0 \to \mathbb{R}$ as

$$T(\varphi) = \int_\Omega f(x)\varphi(x)\,dx.$$

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A function $u \in X_0$ is a solution of (1.1) if

$$B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi), \quad \forall \varphi \in X_0.$$

We note that

$$\begin{aligned}
|B_1(u,\varphi)| &\le \int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy + \mu \int_\Omega |g_1(x)||u(x)||\varphi(x)|\,dx \\
&\le \Big(\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2}\Big(\int_Q |\varphi(x)-\varphi(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2} \\
&\qquad + |\mu|\,\|g_1\|_{\infty,\Omega}\Big(\int_\Omega |u(x)|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\varphi(x)|^2\,dx\Big)^{1/2} \\
&= \|u\|_{X_0}\|\varphi\|_{X_0} + |\mu|\,\|g_1\|_{\infty,\Omega}\|u\|_{2,\Omega}\|\varphi\|_{2,\Omega} \\
&\le (1 + C|\mu|\,\|g_1\|_{\infty,\Omega})\|u\|_{X_0}\|\varphi\|_{X_0},
\end{aligned} \tag{2.4}$$

where C is a constant arising out of the inequality (2.2) in Lemma 2.1.

By hypotheses (H_1) and (H_2) and Holder's inequality,

$$\begin{aligned}
|B_2(u,\varphi)| &\le \int_\Omega |h(u(x))||\varphi(x)||g_2|\,dx \\
&\le A\int_\Omega |\varphi(x)||g_2(x)|\,dx \\
&\le A\|\varphi\|_{2,\Omega}\|g_2\|_{2,\Omega} \le AC\|\varphi\|_{X_0}\|g_2\|_{2,\Omega}.
\end{aligned} \tag{2.5}$$

Also,

$$|T(\varphi)| \le \int_\Omega |f(x)||\varphi(x)|\,dx \le \|f\|_{2,\Omega}\|\varphi\|_{2,\Omega} \le C\|f\|_{2,\Omega}\|\varphi\|_{X_0}, \tag{2.6}$$

where C is a constant arising out of the inequality (2.2). Now $B_1(u,\cdot), B_2(u,\cdot)$ is linear and bounded. We define the operators

$$B, N : X_0 \to [X_0]^*$$

as

$$(Bu|\varphi) = B_1(u,\varphi), \quad (Nu|\varphi) = B_2(u,\varphi) \quad \text{for } u,\varphi \in X_0.$$

Then (1.1) is equivalent to the operator equation $Bu + Nu = T$, $u \in X_0$.

3. Main results

In this section, we study the existence of a weak solution for (1.1).

THEOREM 3.1. *Assume (H_1) and (H_2). Let $\mu > 0$ not be an eigenvalue of*

$$\begin{aligned}
-\mathcal{L}_K u - \mu u(x) &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \mathbb{R}^n \setminus \Omega
\end{aligned} \tag{3.1}$$

and, in addition, let

$$1 > \mu C \|g_1\|_{\infty,\Omega}, \tag{3.2}$$

where C is a constant arising out of the inequality (2.2). Then the BVP (1.1) has a weak solution $u \in X_0$. Moreover, every (weak) solution u of (1.1) satisfies

$$\|u\|_{X_0} \leq \frac{C\{A\|g_2\|_{2,\Omega} + \|f\|_{2,\Omega}\}}{(1 - C\mu\|g_1\|_{\infty,\Omega})},$$

where A is a constant, from hypotheses (H_1).

Proof. We give a brief sketch of the proof. First we write the BVP (1.1) as operator equation

$$u \in X_0 : Bu + Nu = T \quad \text{ in } [X_0]^*,$$

where $T \in [X_0]^*$, $B, N : X_0 \to [X_0]^*$ satisfies all the conditions given in Proposition 2.4. For convenience, we divide the proof into five steps.

Step 1. From the previous section, we know that the operator B is linear and continuous. By Lemma 2.1, the embedding of $X_0 \hookrightarrow\hookrightarrow L^2(\Omega)$ is compact which shows that $B_1(\cdot\,,\cdot)$ is a strict regular Gårding form [20, page 364]. In fact, we obtain

$$\begin{aligned} B_1(u,u) &= \int_{\mathbb{R}^{2n}} (u(x) - u(y))(u(x) - u(y))K(x-y)\,dx\,dy - \int_\Omega \mu u^2(x) g_1(x)\,dx \\ &\geq \int_Q (u(x) - u(y))^2 K(x-y)\,dx\,dy - \mu\|g_1\|_{\infty,\Omega} \int_\Omega u^2\,dx \\ &= \|u\|^2_{X_0} - \mu\|g_1\|_{\infty,\Omega}\|u\|^2_{2,\Omega}. \end{aligned} \tag{3.3}$$

Let $u_k \rightharpoonup u$ weakly in X_0 and

$$\lim_{k\to\infty} (Bu_k - Bu|u_k - u) = 0. \tag{3.4}$$

Claim. $u_k \to u$ strongly in X_0 or B satisfies condition (S). Since B is linear, as in (3.3),

$$\begin{aligned} (Bu_k - Bu|u_k - u) = (B(u_k - u)|u_k - u) &= B_1(u_k - u, u_k - u) \\ &\geq \|u_k - u\|^2_{X_0} - \mu\|g_1\|_{\infty,\Omega}\|u_k - u\|^2_{2,\Omega} \\ &\geq (1 - C\mu\|g_1\|_{\infty,\Omega})\|u_k - u\|^2_{X_0}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we note that

$$0 \leq (1 - C\mu\|g_1\|_{\infty,\Omega}) \lim_{k\to\infty} \|u_k - u\|^2_{X_0} \leq \lim_{k\to\infty} (Bu_k - Bu|u_k - u) = 0.$$

Since $(1 - C\mu\|g_1\|_{\infty,\Omega}) > 0$, $\|u_k - u\|^2_{X_0} \to 0$ as $k \to \infty$; otherwise, $\|u_k - u\|_{X_0} \to 0$, as $k \to \infty$, which implies that B satisfies condition (S).

Step 2.

Claim. $B + N$ is asymptotically linear. By the boundedness of h, we observe that,

$$|(Nu|\varphi)| = |B_2(u,\varphi)| \leq AC\|g_2\|_{2,\Omega}\|\varphi\|_{X_0} \quad \forall u \in X_0,$$

which implies that

$$\|Nu\|_{X_0^*} \leq C',$$

where $C' = AC\|g_2\|_{2,\Omega}$ is a constant depending on Ω. Consequently,

$$\frac{\|Nu\|_{X_0^*}}{\|u\|_{X_0}} \to 0 \quad \text{as } \|u\|_{X_0} \to \infty,$$

which shows that $B + N$ is asymptotically linear and that the operator N is strongly continuous (see [21, Corollary 26.14, page 572]).

Step 3. From Step 2, we note that the operator B satisfies condition (S). Since N is strongly continuous, we note that $t(Nu - T)$ is strongly continuous for $t \in [0, 1]$. For each $t \in [0, 1]$, the operator $A_t(u) = Bu + t(Nu - T)$ is a strongly continuous perturbation of the (S)-operator B. So the operator $A_t(u)$ satisfies condition (S) (compare with [21, Proposition 27.12, page 595]).

Step 4. Now $Bu = 0$ implies that

$$\int_{\mathbb{R}^{2n}} (u(x) - u(y))^2 K(x-y)\,dx\,dy - \int_G \mu u^2(x) g_1(x)\,dx = 0.$$

Consequently,

$$(1 - C\mu\|g_1\|_{\infty,\Omega})\|u\|_{X_0}^2 \leq 0,$$

which shows that $u = 0$ (since $1 - C\mu\|g_1\|_{\infty,\Omega} > 0$).

By Proposition 2.4, $Bu + Nu = T$ has a solution $u \in X_0$ which, equivalently, shows that the BVP (1.1) has a solution $u \in X_0$.

Step 5. As in (3.5) (with the help of embedding in Lemma 2.1),

$$B_1(u,u) \geq (1 - C\mu\|g_1\|_{\infty,\Omega})\|u\|_{X_0}^2.$$

Since $1 > C\mu\|g_1\|_{\infty,\Omega}$, we obtain

$$\|u\|_{X_0}^2 \leq \left(\frac{1}{1 - C\mu\|g_1\|_{\infty,\Omega}}\right) B_1(u,u). \tag{3.6}$$

Also, we note that

$$|B_1(u,u)| \leq C\{A\|g_2\|_{2,\Omega} + \|f\|_{2,\Omega}\}\|u\|_{X_0}. \tag{3.7}$$

By (3.6) and (3.7),

$$\|u\|_{X_0} \leq \frac{C\{A\|g_2\|_{2,\Omega} + \|f\|_{2,\Omega}\}}{(1 - C\mu\|g_1\|_{\infty,\Omega})}. \qquad \square$$

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Next we dispense with the condition (3.2) when g_1 does not change sign. The two results are related to the cases when $g_1 \geq 0$ with $\mu \leq 0$ and $g_1 \leq 0$ with $\mu > 0$. These results are similar to Theorem 3.1, but with suitable changes.

THEOREM 3.2. *Suppose that* (H_1) *and* (H_2) *hold. Let* $g_1 \geq 0$ *and* $\mu \leq 0$. *Then the BVP (1.1) has a solution* $u \in X_0$ *and*

$$\|u\|_{X_0} \leq C\{A\|g_2\|_{2,\Omega} + \|f\|_{2,\Omega}\},$$

where C *is a constant arising out of the inequality (2.2).*

PROOF. As in Theorem 3.1, the basic idea is to reduce the problem (1.1) to an operator equation $Bu + Nu = T$ and study the existence of a solution with the help of Proposition 2.4. To proceed, we define B, N and T, as in Theorem 3.1, and , by a similar argument to that used for estimates (2.4), (2.5) and (2.6),

$$\begin{aligned}
|B_1(u, \varphi)| &\leq (1 + C|\mu|\|g_1\|_{\infty,\Omega})\|u\|_{X_0}\|\varphi\|_{X_0} \\
|B_2(u, \varphi)| &\leq CA\|g_2\|_{2,\Omega}\|\varphi\|_{X_0} \\
|T(\varphi)| &\leq C\|f\|_{2,\Omega}\|\varphi\|_{X_0},
\end{aligned}$$

where the constant C comes from Lemma 2.1. The compact embedding of $X_0 \hookrightarrow\hookrightarrow L^2(\Omega)$ shows that $B_1(\cdot\,,\cdot)$ is a strict regular Gårding form. Also, $\mu \leq 0$ and $g_1 \geq 0$ yield

$$B_1(u, u) = \int_{\mathbb{R}^{2n}} (u(x) - u(y))^2 K(x - y)\, dx\, dy - \int_{\Omega} \mu u^2(x) g_1(x)\, dx \geq \|u\|^2_{X_0}. \tag{3.8}$$

Let $u_k \rightharpoonup u$ weakly in X_0 and

$$\lim_{k\to\infty} (Bu_k - Bu|u_k - u) = 0. \tag{3.9}$$

We claim that $u_k \to u$ strongly in X_0 or B satisfies condition (S). Since B is linear, as in (3.8),

$$\begin{aligned}
(Bu_k - Bu|u_k - u) &= (B(u_k - u)|u_k - u) \\
&= B_1(u_k - u, u_k - u) \geq \|u_k - u\|^2_{X_0}.
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we note that

$$0 \leq \lim_{k\to\infty} \|u_k - u\|^2_{X_0} \leq \lim_{k\to\infty} (Bu_k - Bu|u_k - u) = 0,$$

which implies that $\|u_k - u\|_{X_0} \to 0$ as $k \to \infty$ and, consequently, B satisfies condition (S). Next, we also show that $B + N$ is asymptotically linear and that N is strongly continuous. The proof is similar to that of Theorem 3.1 and we omit it for brevity. Since $\mu \leq 0$, we get that $Bu = 0$ implies that $u = 0$ and hence we note that $\mu \leq 0$ is not an eigenvalue of (3.1). By Proposition 2.4, $Bu + Nu = T$ has a solution $u \in X_0$, which equivalently shows that the BVP (1.1) has a solution $u \in X_0$. Since $\mu \leq 0$ and $g_1 \geq 0$, we get (as in (3.8)) $B_1(u, u) \geq \|u\|^2_{X_0}$. Then, by a similar argument to that in Theorem 3.1,

$$\|u\|_{X_0} \leq C\{A\|g_2\|_{2,\Omega} + \|f\|_{2,\Omega}\},$$

where C is a constant arising out of the inequality (2.2). □

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With suitable modifications in the proof of Theorem 3.2, we arrive at the following result.

THEOREM 3.3. *Suppose that* (H_1) *and* (H_2) *hold. Let* $g_1 \leq 0$ *and* $\mu > 0$. *Then (1.1) has a weak solution* $u \in X_0$ *and there is a constant* k_0 *such that* $\|u\|_{X_0} \leq k_0$ *for every (weak) solution* u.

4. Extensions

In Section 3, the nonlinearity h is assumed to be continuous and bounded. In this section, we extend these results to a class of functions h that are continuous only. The generalized form of Hölder's inequality comes in handy for getting suitable estimates. We establish the existence of a weak solution for (1.1), where $h : \mathbb{R} \to \mathbb{R}$ is required to be continuous and to satisfy $|h(t)| \leq |t|^\epsilon, 0 < \epsilon < 1$ for all $t \in \mathbb{R}$. Again, we consider the cases $\mu \leq 0$ and $\mu > 0$ separately. The proofs are similar to the ones in Section 3, so we restrict ourselves to sketching the differences, whenever needed. The result in [21] is not applicable here since h is not bounded. We collect the common hypotheses for convenience.

(H_1') Suppose that $h : \mathbb{R} \to \mathbb{R}$, defined by $|h(t)| \leq |t|^\epsilon, t \in \mathbb{R}, 0 < \epsilon < 1$.
(H_2') $g_1 \in L^\infty(\Omega)$, $g_2 \in L^{2/(1-\epsilon)}(\Omega), 0 < \epsilon < 1$ and $f \in L^2(\Omega)$.

THEOREM 4.1. *Let the hypotheses* (H_1') *and* (H_2') *hold. Let* $g_1 \geq 0$ *and* $\mu \leq 0$. *Then (1.1) has a weak solution* $u \in X_0$ *and there is a constant* k_0 *such that* $\|u\|_{X_0} \leq k_0$ *for every (weak) solution* u.

PROOF. We give only a sketch of the proof since it is similar to the proof of Theorem 3.2. For $u \in X_0$, from the hypotheses and by Lemma 2.1, we note that

$$|B_1(u, \varphi)| \leq (1 + C|\mu| \|g_1\|_{\infty,\Omega}) \|u\|_{X_0} \|\varphi\|_{X_0},$$
$$|T(\varphi)| \leq C\|f\|_{2,\Omega} \|\varphi\|_{X_0},$$

where the constant C comes from Lemma 2.1. Again, by Lemma 2.1 and the generalized form of Hölder's inequality [9, page 67],

$$|B_2(u, \varphi)| \leq \int_\Omega |h(u(x))||\varphi(x)||g_2|\, dx \leq \|u\|^\epsilon_{2,\Omega} \|\varphi\|_{2,\Omega} \|g_2\|_{2/(1-\epsilon),\Omega}.$$

We also observe that B_1 satisfies condition (S) by a similar argument to that of Theorem 3.2 (also refer to [21, Proposition 27.12]). We observe that

$$|(Nu|\varphi)| = |B_2(u, \varphi)| \leq C\|u\|^\epsilon_{X_0} \|\varphi\|_{X_0} \|g_2\|_{2/(1-\epsilon),\Omega},$$

which implies that

$$\|Nu\|_{X_0^*} \leq C\|u\|^\epsilon_{X_0} \|g_2\|_{2/(1-\epsilon),\Omega} = c\|u\|^\epsilon_{X_0},$$

where $c = C\|g_2\|_{2/(1-\epsilon),\Omega}$ is a constant. So

$$\frac{\|Nu\|_{X_0^*}}{\|u\|_{X_0}} \leq \frac{c\|u\|^\epsilon_{X_0}}{\|u\|_{X_0}} \to 0 \quad \text{as } \|u\|_{X_0} \to \infty. \tag{4.1}$$

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This shows that $B + N$ is asymptotically linear. Also, $u \in L^2(\Omega)$ implies that $h(u) \in L^{2/\epsilon}(\Omega)$ and we define the Nemytskii operator

$$F : L^2(\Omega) \to L^{2/\epsilon}(\Omega)$$

by $(Fu)(x) = h(u(x))$; so F is continuous (by [8, Theorem 2.1]). By hypotheses (H_1') and (H_2') and the generalized Hölder inequality, we note that

$$\begin{aligned}|(Nu_n|\varphi) - (Nu|\varphi)| &\le \int_\Omega |h(u_n) - h(u)||g_2||\varphi|\,dx\\ &\le C\|h(u_n) - h(u)\|_{2/\epsilon,\Omega}\|g_2\|_{2/(1-\epsilon),\Omega}\|\varphi\|_{X_0}.\end{aligned}$$

Let $u_n \rightharpoonup u$ weakly in X_0. Then, by the compact embedding $X_0 \hookrightarrow\hookrightarrow L^2(\Omega)$ and since F is continuous in $L^{2/\epsilon}(\Omega)$,

$$\|Nu_n - Nu\|_{X_0^*} \to 0 \quad \text{as } n \to \infty.$$

By a similar argument to that of Theorem 3.1, the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S). If $\mu \le 0$, then $Bu = 0$ implies $u = 0$ and $\mu \le 0$ is not an eigenvalue of the linear problem (3.1). By Proposition 2.4, the operator equation $Bu + Nu = T$ and, consequently, (1.1) has a solution $u \in X_0$, which completes the proof of the existence of a solution.

Now, as in (3.8),

$$B_1(u, u) \ge \|u\|_{X_0}^2. \tag{4.2}$$

Also, we note that

$$|B_1(u, u)| \le C\{\|u\|_{X_0}^{\epsilon}\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega}\}\|u\|_{X_0}. \tag{4.3}$$

By (4.2) and (4.3),

$$\|u\|_{X_0} \le C\{\|u\|_{X_0}^{\epsilon}\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega}\}. \tag{4.4}$$

If $\|u\|_{X_0} \ge 1$, from (4.4),

$$\|u\|_{X_0} \le C(\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega})\|u\|_{X_0}^{\epsilon},$$

which implies that

$$\|u\|_{X_0}^{1-\epsilon} \le c \quad 0 < \epsilon < 1,$$

(where $c = C(\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega})$)

$$\text{or} \quad \|u\|_{X_0} \le c^{1/(1-\epsilon)} \quad 0 < \epsilon < 1.$$

If $\|u\|_{X_0} \le 1$, we have nothing to prove. Let $k_0 = \max\{1, c^{1/(1-\epsilon)}\}$. Hence

$$\|u\|_{X_0} \le k_0. \qquad \square$$

REMARK. Theorem 4.1 holds if $g_1 \le 0$ and $\mu > 0$ with the other conditions remaining intact. But when $\mu > 0$ and g_1 change sign, we need additional conditions on μ and g_1 (stated below), as in Theorem 3.1. We state these results below in Theorem 4.2 and we give a sketch of the proof. We note that, in (4.1), the required asymptotic linearity of $B + N$ is a consequence of ϵ lying between zero and one.

THEOREM 4.2. *Let the hypotheses* (H_1') *and* (H_2') *hold. Also, let* $\mu > 0$ *not be an eigenvalue of* (3.1) *and, in addition, let* $1 > C\mu\|g_1\|_{\infty,\Omega}$. *Then the BVP* (1.1) *has a weak solution* $u \in X_0$ *and there is a constant* k_0 *such that* $\|u\|_{X_0} \le k_0$ *for every (weak) solution* u.

PROOF. The proof of the existence of a weak solution $u \in X_0$ for (1.1) is similar to the arguments in Theorem 4.1 and Theorem 3.1 and hence is omitted. As in Theorem 3.1, we note that

$$(1 - C\mu\|g_1\|_{\infty,\Omega})\|u\|_{X_0}^2 \le C\{\|u\|_{X_0}^{\epsilon}\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega}\}\|u\|_{X_0},$$

where C is a constant. Since $1 > C\mu\|g_1\|_{\infty,\Omega}$, we obtain

$$\|u\|_{X_0} \le \frac{C(\|u\|_{X_0}^{\epsilon}\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega})}{(1 - C\mu\|g_1\|_{\infty,\Omega})}. \tag{4.5}$$

If $\|u\|_{X_0} \ge 1$, from (4.5),

$$\|u\|_{X_0} \le \frac{C(\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega})\|u\|_{X_0}^{\epsilon}}{(1 - C\mu\|g_1\|_{\infty,\Omega})},$$

which implies that

$$\|u\|_{X_0}^{1-\epsilon} \le c \quad 0 < \epsilon < 1,$$

(where $c = (C(\|g_2\|_{2/(1-\epsilon),\Omega} + \|f\|_{2,\Omega}))/((1 - C\mu\|g_1\|_{\infty,\Omega}))$)

$$\text{or} \quad \|u\|_{X_0} \le c^{1/(1-\epsilon)}.$$

If $\|u\|_{X_0} \le 1$, we have nothing to prove. Let $k_0 = \max\{1, c^{1/(1-\epsilon)}\}$. Then

$$\|u\|_{X_0} \le k_0.$$

□

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