

# The scattering matrix

## 1.1 Introduction

In a typical scattering experiment, performed at an accelerator laboratory, a particle from the accelerated beam strikes another particle in the target material (usually a proton) and the result may be the production of several different types of particles, travelling in various directions, as in fig. 1.1. Thus, before the interaction, we have an initial state  $|i\rangle$  composed of two free particles (beam and target), and when the interaction is over, a final state  $|f\rangle$  consisting often of many particles. A complete quantum-mechanical theory of the scattering process, if it existed, would allow us to deduce the probability of achieving any particular final state from the given initial state.

We define the scattering operator,  $S$ , such that its matrix elements between the initial and final states  $\langle f|S|i\rangle$ , give us the probability  $P_{fi}$  that  $|f\rangle$  will be the final state resulting from  $|i\rangle$ , i.e.

$$P_{fi} = |\langle f|S|i\rangle|^2 = \langle i|S^\dagger|f\rangle\langle f|S|i\rangle \quad (1.1.1)$$

where  $S^\dagger$  is the Hermitian adjoint of  $S$ . A knowledge of the full scattering matrix (or  $S$ -matrix for short) containing the matrix elements connecting any conceivable initial state to any conceivable final state would clearly constitute a complete description of all possible particle interactions, which is, of course, our ultimate goal.

Unfortunately, there is as yet no fundamental theory for the strong interactions of elementary particles, so it is not possible to present the subject deductively, but we shall try in this chapter to explain briefly the assumptions on which we will be relying for our subsequent development of Regge theory, i.e. the general principles such as analyticity and crossing, which, though not rigorously verified, have stood the test of time, and will form the basis for our discussion. We shall try to make them plausible by showing how they are incorporated both in non-relativistic potential scattering and quantum field theories, which therefore provide useful sources of intuition.

In a field theory like quantum electrodynamics, these  $S$ -matrix ele-

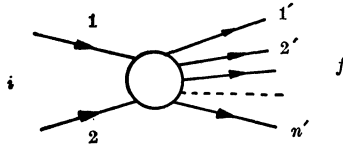


FIG. 1.1 A scattering process with two particles in the initial state and  $n$  in the final state.

ments can be deduced, at least in principle, from the basic Lagrangian describing the interactions of the fundamental particles. But for strong interactions there are many problems with this sort of approach, such as the failure of re-normalization methods and the lack of convergence of the perturbation series. However, the  $S$ -matrix elements themselves are always evaluated between the so-called asymptotic states at times  $t = \pm \infty$ ; or, more accurately, the initial state a long time before the interaction commences, and the final state a long time afterwards (i.e. long compared with the duration of the interaction, typically  $\approx 10^{-22}$  s). What goes on during the interaction is clearly not directly observable. It is thus certainly very useful, and some (see for example Chew (1962)) would claim more in accord with the philosophy of quantum mechanics, to try to develop a theory for the  $S$ -matrix directly. Others still feel that one should start from the interactions of quantized fields, and that our goal should be to obtain for strong interactions something akin to quantum electrodynamics (see for example Bjorken and Drell (1965) for a review of this subject). We are still so far from a complete theory that such disputes seem premature. Here we shall adopt mainly an  $S$ -matrix viewpoint, chiefly because in working with  $S$ -matrix elements one is concerned with (almost) directly measurable quantities, and so the  $S$ -matrix provides an excellent vantage point from which to survey the confrontation of theoretical speculation with experimental fact.

In the following sections we introduce the basic ideas of  $S$ -matrix theory, the unitarity equations and the analyticity properties of scattering amplitudes. We show how these analyticity assumptions allow one to write dispersion relations for the scattering amplitudes, and discuss the ambiguities which such dispersion relations frequently possess because they involve divergent integrals. We also briefly consider Feynman perturbation field theory and Yukawa potential-scattering models, and show how they incorporate many of these features. This will set the stage for the introduction of Regge theory in the next chapter.

We shall employ the usual units for particle physics, in which the velocity of light,  $c$ , and Planck's constant,  $\hbar$ , are both set equal to unity. Energies, momenta and masses are all expressed in electron volts, or more conveniently in  $\text{GeV} \equiv 10^9 \text{eV}$ . This unit can be converted into a time or length using

$$\hbar = 6.58 \times 10^{-25} \text{ GeV s}$$

$$\hbar c = 1.97 \times 10^{-16} \text{ GeV m}$$

A convenient alternative unit of length is the fermi

$$1 \text{ fm} \equiv 10^{-15} \text{ m} = \frac{10}{1.97 \text{ GeV}} \approx 5 \text{ GeV}^{-1}$$

Cross-sections are usually measured in millibarns;  $1 \text{ mb} = 10^{-31} \text{ m}^2$  which may be converted into  $\text{GeV}$  units using

$$\text{GeV}^{-2} = 0.389 \text{ mb.}$$

## 1.2 The $S$ -matrix

$S$ -matrix theory starts from the following basic assumptions.

### *Postulate (i)*

Free particle states, containing any number of particles, satisfy the superposition principle of quantum mechanics, so that if  $|\psi_\alpha\rangle$  and  $|\psi_\beta\rangle$  are physical states so is  $|\psi_\gamma\rangle \equiv a|\psi_\alpha\rangle + b|\psi_\beta\rangle$  where  $a$  and  $b$  are arbitrary complex numbers. (There are in fact superselection rules such as charge and baryon-number conservation which violate this rule but they will not trouble us here; see Martin and Spearman (1970).)

### *Postulate (ii)*

Strong interaction forces are of short range. We know from nuclear physics that the strong interaction is not felt at distances greater than a few times  $10^{-15} \text{ m}$  (a few pion Compton wavelengths). This means that we can regard the particles as free (i.e. non-interacting) except when they are very close together, and so the asymptotic states, before and after an experiment is performed, consist of just free particles. (We regard a bound state such as the deuteron as a single particle.) Clearly this is only justified if we neglect long-range forces such as electromagnetism and gravitation. In fact, they cannot be incorporated into the  $S$ -matrix framework without considerable difficulty

and we shall mainly ignore these weaker interactions and suppose ourselves to be dealing with an idealized world where they have been ‘switched off’.

To define completely a single free-particle state we must first specify all its internal quantum numbers, i.e. its charge  $Q$ , baryon number  $B$ , isospin  $I$ , strangeness  $S$ , parity  $P$  (and for a non-strange meson the  $G$ -parity  $G$ , and charge conjugation  $C_n$ ), and its spin  $\sigma$  (where the eigenvalue of  $\sigma^2$  is  $[\sigma(\sigma + 1)]$ ). (The classification of particles in terms of these quantum numbers is discussed in chapter 5.) We denote these quantum numbers collectively by the ‘particle type’  $T$ . We must also specify the component of its spin along a chosen quantization axis, say,  $\sigma_3$ , and its mass  $m$ , energy  $E$ , and momentum  $\mathbf{p}$ , in some chosen Lorentz frame.

*Postulate (iii)*

The scattering process, and hence the  $S$ -matrix, is invariant under Lorentz transformations. It is thus convenient to regard  $E \equiv p_0$  as the time component of a relativistic four-vector whose space components are  $p_1, p_2$  and  $p_3$ , i.e.

$$p_\mu \equiv (p_0, \mathbf{p}), \quad \mu = 0, 1, 2, 3 \quad (1.2.1)$$

Since we are always concerned with free particles for which the total energy is given by

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4 \quad (1.2.2)$$

where  $m$  is the particle’s rest mass, and as we work in units where  $c \equiv 1$ , the four-momentum satisfies the ‘mass-shell’ constraint

$$\sum_\mu p_\mu p^\mu \equiv p^2 = p_0^2 - \mathbf{p}^2 = E^2 - \mathbf{p}^2 = m^2 \quad (1.2.3)$$

so only three of its four components are independent once the mass is given.

In this book we shall adopt the commonly used convention that the spin quantization axis will be the direction of motion of the particle in the chosen frame of reference. The component of the spin along this axis is called the helicity,  $\lambda$ , and is defined by

$$\lambda \equiv \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \quad (1.2.4)$$

Clearly  $\lambda$  can take any of the  $2\sigma + 1$  possible values,  $\sigma, \sigma - 1, \dots, -\sigma$ .

Thus a single-particle state is denoted by

$$|T, \lambda, \mathbf{p}_\mu\rangle \equiv |P\rangle \quad (1.2.5)$$

and such states are irreducible representations of the Lorentz group (for proof see for example Martin and Spearman (1970)).

Obviously states corresponding to different momenta, different intrinsic quantum numbers, or different helicities must be orthogonal to each other, so their scalar products take the form

$$\langle P' | P \rangle \equiv \langle T', \lambda', p'_\mu | T, \lambda, p_\mu \rangle = N \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{T'T} \delta_{\lambda\lambda} \quad (1.2.6)$$

where  $\delta^3(\mathbf{p}' - \mathbf{p})$  is a short-hand notation for

$$\delta(p'_1 - p_1) \delta(p'_2 - p_2) \delta(p'_3 - p_3),$$

and  $N$  is a normalization factor.

We want to normalize our state vectors in a Lorentz invariant manner. The normalization of the state will tell us the number of particles in a given phase-space volume element  $d^3\mathbf{p}$  about the vector  $\mathbf{p}$ , but this is clearly not a Lorentz invariant quantity because the size of such a volume element  $d^3\mathbf{p}$  is not invariant. However, the volume element  $d^4p\delta(p^2 - m^2)$  is manifestly invariant, while the  $\delta$ -function ensures that the mass-shell constraint (1.2.3) is obeyed. In fact, it can be re-expressed as

$$d^4p\delta(p^2 - m^2) = \frac{d^3\mathbf{p}}{2p_0} \theta(p_0) \quad (1.2.7)$$

because, with the usual rules for manipulating the Dirac  $\delta$ -function, i.e.

$$\delta(ax) = 1/a \delta(x)$$

we find

$$\begin{aligned} \delta(p^2 - m^2) &\equiv \delta(p_0^2 - \mathbf{p}^2 - m^2) = \frac{1}{2p_0} \delta[p_0 - \sqrt{(\mathbf{p}^2 + m^2)}] \\ &\quad - \frac{1}{2p_0} \delta[p_0 + \sqrt{(\mathbf{p}^2 + m^2)}] \quad (1.2.8) \end{aligned}$$

and we shall always restrict our integrations to positive  $p_0$  only. Hence it is convenient to choose  $N$  in (1.2.6) such that

$$\langle P' | P \rangle = (2\pi)^3 2p_0 \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{T'T} \delta_{\lambda\lambda} \quad (1.2.9)$$

The factor  $(2\pi)^3$  is purely a matter of convention, but the presence of  $p_0$  ensures, through (1.2.7), that our normalization remains invariant under Lorentz transformations.

A state consisting of  $n$  free particles may be written as a direct product of single particle states

$$\begin{aligned} |T_1, \lambda_1, p_1; T_2, \lambda_2, p_2; \dots; T_n, \lambda_n, p_n\rangle \\ \equiv |P_1 \dots P_n\rangle = |P_1\rangle \otimes |P_2\rangle \otimes \dots \otimes |P_n\rangle \quad (1.2.10) \end{aligned}$$

and has the normalization, from (1.2.9),

$$\langle P'_1 \dots P'_n | P_1 \dots P_n \rangle = \prod_{i=1}^n (2\pi)^3 2p_{0i} \delta^3(\mathbf{P}'_i - \mathbf{P}_i) \delta_{T'_i T_i} \delta_{\lambda'_i \lambda_i} \delta_{n'n} \quad (1.2.11)$$

*Postulate (iv)*

The scattering matrix is unitary. This follows if the free particle states  $|m\rangle$ ,  $m = 1, 2, \dots$  constitute a complete orthonormal set of basis states satisfying the completeness relation

$$\sum_m |m\rangle \langle m| = 1 \quad (1.2.12)$$

since starting from any given state  $|i\rangle$  the probability that there will be *some* final state must be unity. So from (1.1.1)

$$\begin{aligned} \sum_m P_{mi} &= \sum_m \langle m | S | i \rangle^2 = \sum_m \langle i | S^\dagger | m \rangle \langle m | S | i \rangle \\ &= \langle i | S^\dagger S | i \rangle = 1 \end{aligned} \quad (1.2.13)$$

and as this must be true for any state  $|i\rangle$  we have

$$S^\dagger S = 1 = S S^\dagger \quad (1.2.14)$$

so  $S$  is a unitary matrix.

For our many-particle states with normalization (1.2.11) the completeness relation (1.2.12) reads

$$\sum_{m=1}^{\infty} \prod_{i=1}^m \sum_{\lambda_i} \sum_{T_i} (2\pi)^{-3} \int \frac{d^3 \mathbf{P}_i}{2p_{0i}} |P_1 \dots P_m\rangle \langle P_1 \dots P_m| = 1 \quad (1.2.15)$$

since the summation must run over all possible numbers, types and helicities of particles, as well as over all their possible momenta. So in terms of these states the unitarity relation (1.2.13) becomes

$$\begin{aligned} \sum_{m=1}^{\infty} \prod_{i=1}^m \sum_{\lambda_i} \sum_{T_i} (2\pi)^{-3} \int \frac{d^3 \mathbf{q}_i}{2q_{0i}} \langle P'_1 \dots P'_n | S | Q_1 \dots Q_m \rangle \\ \times \langle Q_1 \dots Q_m | S^\dagger | P_1 \dots P_n \rangle = \langle P'_1 \dots P'_n | P_1 \dots P_n \rangle \end{aligned} \quad (1.2.16)$$

where  $Q_i \equiv \{T_i, \lambda_i, q_{\mu_i}\}$  is used to label the intermediate-state particles with four-momenta  $q_{\mu_i}$ . Note that in these equations we have treated the particles as non-identical as we shall continue to do below. For identical particles one must sum over the  $n!$  ways of pairing the momenta in (1.2.11), and correspondingly  $(n!)^{-1}$  appears in the completeness relation (1.2.15), and hence in (1.2.16).

This unitarity equation (1.2.16) is of fundamental importance in determining the nature of the  $S$ -matrix. However, it is also rather

complicated, and it becomes much easier to understand, and to utilize, if we represent it diagrammatically in terms of ‘bubble diagrams’. (A more complete account of this subject will be found in Eden *et al.* (1966).)

### 1.3 Bubble diagrams and scattering amplitudes

The summation over different types of particles and their different helicities in (1.2.16) adds unnecessarily to the notational complexity of the equation. For the rest of this chapter we shall only be concerned with the momentum-space properties of the  $S$ -matrix, so we shall cease to refer to  $T$  and  $\lambda$ , and write all our equations as though there existed only a single type of particle of zero spin. Thus an  $n$ -particle state will be written as just  $|p_1 \dots p_n\rangle$ . Each integration over a momentum should therefore be regarded as implying also a summation over all the different types of particles which can contribute, given the restrictions required by quantum number conservation, and over all the  $2\sigma_i + 1$  possible helicities available to a particle of spin  $\sigma_i$ .

We denote each  $S$ -matrix element representing a scattering process by a ‘bubble’ with lines corresponding to the incoming and outgoing particles, viz.

$$\langle p'_1 \dots p'_{n'} | S | p_1 \dots p_n \rangle \equiv \begin{array}{c} \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad (1.3.1)$$

and 
$$\langle p'_1 \dots p'_{n'} | S^\dagger | p_1 \dots p_n \rangle \equiv \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad (1.3.2)$$

The intermediate states appearing in a unitarity equation such as (1.2.16) are denoted by

$$\prod_{i=1}^m \int (2\pi)^{-3} \frac{d^3 q_i}{2q_{0i}} \equiv \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad (1.3.3)$$

the bars on the ends indicating that such lines must be attached to bubbles. The overlap between states (1.2.11) is written

$$\langle p'_1 \dots p'_{n'} | p_1 \dots p_n \rangle = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \times \delta_{n'n} \quad (1.3.4)$$

Because of Lorentz invariance (postulate (iii)) we know that energy and momentum are conserved in a scattering process, and hence an  $S$ -matrix element such as (1.3.1) vanishes unless

$$\sum_{i=1}^n p_{\mu_i} = \sum_{i=1}^{n'} p'_{\mu_i}, \quad \mu = 0, 1, 2, 3 \quad (1.3.5)$$

This implies that for example in (1.2.16) only intermediate states with  $\left(\sum_{i=1}^m m_i\right)^2 \leq \left(\sum_{i=1}^n p_i\right)^2$  contribute to the sum. The equality occurs at the threshold energy for the process  $|p_1 \dots p_n\rangle \rightarrow |q_1 \dots q_m\rangle$ .

Thus suppose we have, as will always be the case in practice, a two-particle initial state, and suppose that for simplicity we take all the hadrons to have the same mass,  $m$ . (This would mean of course that they were all stable as they would have no state of lower mass into which to decay.) Then for  $(2m)^2 \leq (p_1 + p_2)^2 \leq (3m)^2$ , i.e. above the two-particle threshold but below that for three particles, only a two-particle intermediate state, and only a two-particle final state, can occur in the unitarity equation (1.2.16) which becomes

$$\int \prod_{i=1}^2 (2\pi)^{-3} \frac{d^3 q_i}{2q_{0i}} \langle p'_1 p'_2 | S | q_1 q_2 \rangle \langle q_1 q_2 | S^\dagger | p_1 p_2 \rangle = \langle p'_1 p'_2 | p_1 p_2 \rangle \tag{1.3.6}$$

and with the above rules it may be rewritten as

$$\begin{array}{c} p_1 \quad q_1 \quad p'_1 \\ \rightarrow \quad \rightarrow \quad \rightarrow \\ \text{---} \text{---} \text{---} \\ \left( S^\dagger \right) \quad \left( S \right) \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ p_2 \quad q_2 \quad p'_2 \end{array} = \text{---} \tag{1.3.7}$$

But if the energy of the initial state is increased, so

$$(3m)^2 \leq \left(\sum_i p_i\right)^2 \leq (4m)^2,$$

two- or three-particle states are possible for the initial state (in principle) and for the intermediate and final states (in practice), so (1.2.16) gives us the set of unitarity equations.

$$\begin{array}{l} \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} + \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} = \text{---} \\ \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} + \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} = 0 \\ \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} + \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} = 0 \\ \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} + \text{---} \left( S^\dagger \right) \text{---} \left( S \right) \text{---} = \text{---} \end{array} \tag{1.3.8}$$

The generalization to higher energies where even more particles can occur should be obvious.

The finite range of the strong interaction force (postulate (ii)) permits a further development of these equations. For example, the  $S$ -matrix element with two particles in both the initial and final states



can be decomposed as follows:

$$\begin{aligned} \overline{\textcircled{S}} &= \overline{\text{---}} + \overline{\textcircled{+}} \\ &= \langle p'_1, p'_2 | p_1, p_2 \rangle + \langle p'_1, p'_2 | S_c | p_1, p_2 \rangle \end{aligned} \tag{1.3.9}$$

Here the first term applies if the two particles never get close enough to interact, while the second, the so-called ‘connected part’, represents the interaction of the two particles. (The + sign is used for the connected part of  $S$  for reasons which will become apparent below.) These are quite distinct because in the first term each particle has the same energy and momentum in the final state as it had in the initial state, while with the second term only the total energy and total momentum of the two particles need be conserved. Putting in the conservation  $\delta$ -functions of (1.2.11) and four-momentum conservation for  $\overline{\textcircled{+}}$  explicitly, (1.3.9) gives

$$\begin{aligned} \overline{\textcircled{S}} &= (2\pi)^6 4p_{01}p_{02} \delta^3(\mathbf{p}'_1 - \mathbf{p}_1) \delta^3(\mathbf{p}'_2 - \mathbf{p}_2) \\ &\quad + i(2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \langle p'_1 p'_2 | A | p_1 p_2 \rangle \end{aligned} \tag{1.3.10}$$

The factor  $i(2\pi)^4$  is included to give a conventional normalization to the  $A$ -matrix or ‘scattering amplitude’ representing  $\textcircled{+}$ .

On the other hand the  $2 \rightarrow 3$   $S$ -matrix element is only possible if the two particles actually scatter, so

$$\begin{aligned} \overline{\textcircled{S}} &= \begin{matrix} 1 \rightarrow \\ 2 \rightarrow \end{matrix} \textcircled{+} \begin{matrix} \rightarrow 1' \\ \rightarrow 2' \\ \rightarrow 3' \end{matrix} = i(2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2 - p'_3) \\ &\quad \times \langle p'_1 p'_2 p'_3 | A^+ | p_1 p_2 \rangle \end{aligned} \tag{1.3.11}$$

If there are more external lines there may be more disconnected parts, thus

$$\overline{\textcircled{S}} = \overline{\text{---}} + \overline{\textcircled{+}} + \overline{\textcircled{+}} + \overline{\textcircled{+}} + \overline{\textcircled{+}} \tag{1.3.12}$$

For  $S^\dagger$  we write correspondingly

$$\overline{\textcircled{S^\dagger}} = \overline{\text{---}} + (-1) \overline{\textcircled{-}} \tag{1.3.13}$$

where

$$\overline{\textcircled{-}} = i(2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2) \langle p'_1 p'_2 | A^- | p_1 p_2 \rangle \tag{1.3.14}$$


the minus signs again being conventional.

This disconnectedness property allows a considerable further simplification of the unitarity equations. Thus, on substituting



### 1.4 The analyticity properties of scattering amplitudes

We have so far written the scattering amplitudes,  $A^\pm(p_1 \dots p_n; p'_1 \dots p'_n)$  as arbitrary functions of the four-momenta of the particles involved. However, Lorentz invariance implies that  $A$  must be a Lorentz scalar, and hence may be written as a function of Lorentz scalars only. As long as we are neglecting spin this means that  $A$  is a function only of scalar products of the momenta.

Thus for the four-line process  $1+2 \rightarrow 3+4$  the amplitude  $A(p_1, p_2; p_3, p_4)$  will be a function of Lorentz scalars such as  $(p_1 + p_2)^2$ ,  $(p_1 + p_3)^2$ ,  $(p_1 + p_2 + p_3)^2$  etc. (Remember  $p_i^2 = m_i^2$ ,  $i = 1, \dots, 4$ , are not variables.) However, not all these are independent quantities, since, for example  $(p_1 + p_2)^2 = (p_3 + p_4)^2$  by four-momentum conservation. In general for an  $n$ -line process there are  $4n$  variables (the components of the  $n$  four-vectors), but  $n$  mass-shell constraints of the form  $p_i^2 = m_i^2$ , 4 constraints for overall energy and momentum conservation, and 6 constraints for rotational invariance in the four-dimensional Minkowski space, leaving us with  $3n - 10$  independent variables. Thus, if we regard a single particle propagator as a 'scattering process'  $1 \rightarrow 2$ , , we have  $n = 2$  so there are  $-4$  degrees of freedom, i.e. the 4 constraints  $p_{1\mu} = p_{2\mu}$ ,  $\mu = 0, 1, 2, 3$ . For the more realistic process  $1 + 2 \rightarrow 3 + 4$ ,  $n = 4$ , and so there are two independent variables, while  $1 + 2 \rightarrow 3 + 4 + 5$  depends on 5 variables, and so on. We denote these variables by the Lorentz invariants

$$s_{ijk} \dots \equiv (\pm p_i \pm p_j \pm p_k \dots)^2.$$

But what sort of function of these invariants is  $A$ ? This brings us to the next postulate of  $S$ -matrix theory.

*Postulate (v): Maximal analyticity of the first kind*

The scattering amplitudes are the real boundary values of analytic functions of the invariants  $s_{ijk} \dots$  regarded as complex variables, with only such singularities as are demanded by the unitarity equations.

Thus although obviously only real values of the  $s_{ijk} \dots$  make physical sense we are going to treat them as complex variables, and suppose that the amplitudes are analytic functions of the  $s_{ijk}$ , so that we can obtain the physical scattering amplitude by taking the limit  $s \rightarrow \text{real}$ .

A simple understanding of why the amplitudes may plausibly be expected to have such analyticity properties can be obtained from the following argument. Consider the scattering of a wave packet

travelling initially along the  $z$  axis with velocity  $v$ ,

$$\psi_{\text{in}}(z, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} d\omega \phi(\omega) e^{-i\omega(t-z/v)} \quad (1.4.1)$$

where  $\omega$  is the energy ( $\hbar \equiv 1$ ), and, taking the Fourier inverse,

$$\phi(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dt \psi(0, t) e^{i\omega t} \quad (1.4.2)$$

To make physical sense this integral must converge for real  $\omega$ , but it defines  $\phi(\omega)$  for all complex values of  $\omega$ . If the wave packet does not reach  $z = 0$  until  $t = 0$  then  $\psi(0, t) = 0$  for  $t < 0$  so

$$\phi(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} dt \psi(0, t) e^{i\omega t} \quad (1.4.3)$$

This means that  $\phi(\omega)$  is an analytic function of  $\omega$  regular in the upper-half plane (i.e. for  $\text{Im}\{\omega\} > 0$ ) since in this region the integral (1.4.3) must certainly converge (because it exists for real  $\omega$ , and we get even better convergence from  $e^{-(\text{Im}\{\omega\})t}$  for  $\text{Im}\{\omega\} > 0$ ). Similarly for the scattered wave we have

$$\psi_{\text{out}}(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{r} \int_{-\infty}^{\infty} d\omega A(\omega) \phi(\omega) e^{-i\omega(t-r/v)} \quad (1.4.4)$$

where, by definition,  $A(\omega)$  is the scattering amplitude for scattering at a given energy (see for example Schiff (1968)). If the scattering process is causal the scattered wave cannot have reached a distance  $r$  from the scattering centre until time  $t = r/v$  has elapsed so

$$\psi_{\text{out}}(\mathbf{r}, t) = 0 \quad \text{for } t < r/v,$$

which from the Fourier inverse of (1.4.4), with repetition of the argument (1.4.1) to (1.4.3), implies that  $A(\omega)$  is also an analytic function of  $\omega$  in the upper-half plane.

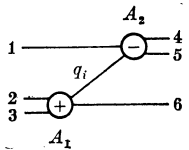
The difficulty with an argument such as this is of course that it assumes that it makes sense to talk about the precise distribution of the wave packet in time despite the fact that we are also assuming that the energy is known with precision, so it is not obvious how far this concept of microscopic causality makes sense. Clearly, no quantum-mechanical measurement could establish what the time distribution of a wave packet is, even in principle. However, we shall see below that we only seem to require micro-causality in the classical limit.

Attempts have been made to deduce the analyticity properties (and singularities) of scattering amplitudes from axiomatic field theory (see for example Goldberger and Watson (1964)), and axiomatic  $S$ -matrix theory (see Eden *et al.* 1966), but there are many difficulties

in discovering how to continue round the various singularities. Only for physical-region singularities is the situation reasonably clear (Bloxxam, Olive and Polkinghorne 1969). If the scattering amplitude can be written as a perturbation series (a sum of Feynman diagrams) the analyticity properties of the individual terms in the series can be found (at least for the lower orders), but of course we are concerned with strong interactions where such a perturbation series is not expected to converge. However, since  $S$ -matrix theory and perturbation theory seem to possess similar singularity structures it is often useful to employ Feynman-diagram models (see section 1.12). Here we shall simply assume that the singularity structure which can be deduced heuristically from the  $S$ -matrix postulates is in fact correct.

### 1.5 The singularity structure

The most important type of singularity which can be identified in the unitarity equations is a simple pole which corresponds to the exchange of a physical particle. The occurrence of such poles can be deduced from the  $3 \rightarrow 3$  unitarity equations (1.3.8), for example, in which we find the term

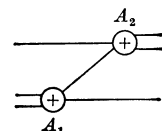


$$\begin{aligned} & \sim A_1^+ (-2\pi i \delta(q_i^2 - m_i^2)) A_2^-, \quad q_i = p_2 + p_3 - p_6 \\ & = p_4 + p_5 - p_1 \end{aligned} \quad (1.5.1)$$

The  $\delta$ -function occurs because of course it is only precisely when  $(p_2 + p_3 - p_6)^2 = m_i^2$  that particle  $i$  can be exchanged between the bubbles. Now since

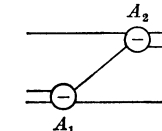
$$\frac{1}{q_i^2 - m_i^2 \pm i\epsilon} = P \frac{1}{q_i^2 - m_i^2} \pm \pi i \delta(q_i^2 - m_i^2)$$

(where  $P$  = principal part), the amplitudes  $\Xi(\pm)$  must contain pole contributions of the form



$$= A_1^+ \frac{1}{q_i^2 - m_i^2 + i\epsilon} A_2^+,$$

and



$$= A_1^- \frac{1}{q_i^2 - m_i^2 - i\epsilon} A_2^- \quad (1.5.2)$$

so that  $\Xi_{\oplus} - \Xi_{\ominus}$  contains the  $\delta$ -function of (1.5.1) in the limit  $\epsilon \rightarrow 0$ . This result is not unexpected because in perturbation theory the Feynman propagator for a spinless particle takes the form of a pole  $(q_i^2 - m_i^2 + i\epsilon)^{-1}$  (see section 1.12 below). Also, we are familiar in nuclear physics with unstable particles (or resonances) which give rise to amplitudes of the Breit–Wigner form  $\sim (q_i^2 - m_i^2 + im_i\Gamma_i)^{-1}$  where  $\Gamma_i$  is the width of the resonance, giving a complex pole at  $q_i^2 = m_i^2 - im_i\Gamma_i$ .

The additional feature which we can observe in (1.5.2) is that the residue of the pole at  $q_i^2 = m_i^2$  can be ‘factorized’ into the amplitudes for the two separate scatterings involving particle  $i$ , viz.  $1 + i \rightarrow 4 + 5$  and  $2 + 3 \rightarrow i + 6$ . It is sometimes said that this factorization is a consequence of unitarity, but really it stems from the disconnectedness postulate (ii) since (1.5.2) can represent successive scattering processes which are completely independent of each other and occurring at two well separated places ( $\gg 1$  fm).

We thus find that the exchange of a particle gives a pole in  $q^2$  in the  $S$ -matrix; and vice versa the presence of a pole in  $q^2$  indicates the presence of a particle, stable if it occurs for real  $q^2$ , unstable if it occurs for complex  $q^2$ , as in the Breit–Wigner formula.

The next-simplest singularity is due to the exchange of two particles, as in (1.3.21). This gives rise to a branch point at the threshold  $(p_1 + p_2)^2 = (2m)^2$ . Transforming the integration variable  $q \rightarrow q - p_1$  we get

$$A^+ - A^- = \frac{i}{(2\pi)^3} \int d^4q \delta(q^2 - m^2) \delta((p_1 + p_2 - q)^2 - m^2) A^+ A^- \quad (1.5.3)$$

In the centre-of-mass system  $p_1 = (p_{01}, \mathbf{p})$  and  $p_2 = (p_{02}, -\mathbf{p})$ , so

$$(p_1 + p_2) = (p_{01} + p_{02}, \mathbf{0}) \equiv (\sqrt{s}, \mathbf{0}) \quad (1.5.4)$$

where we have defined  $\sqrt{s}$  to be the total energy in the centre-of-mass system. Putting  $q = (q_0, \mathbf{q})$ , the argument of the second  $\delta$ -function in (1.5.3) becomes

$$(p_1 + p_2 - q)^2 - m^2 = s - 2(\sqrt{s})q_0 + q^2 - m^2 = s - 2(\sqrt{s})q_0 \quad (1.5.5)$$

since the first  $\delta$ -function gives  $q^2 = m^2$ . So

$$\begin{aligned} A^+ A^- &= \frac{i}{(2\pi)^2} \int d^4q \delta(q^2 - m^2) \delta(s - 2(\sqrt{s})q_0) A^+ A^- \\ &= \frac{i}{(2\pi)^2 2\sqrt{s}} \int dq_0 d^3\mathbf{q} \delta(q_0^2 - |\mathbf{q}|^2 - m^2) \delta(\frac{1}{2}\sqrt{s} - q_0) A^+ A^- \\ &= \frac{i}{(2\pi)^2 2\sqrt{s}} \int d^3\mathbf{q} \delta(\frac{1}{4}s - |\mathbf{q}|^2 - m^2) A^+ A^- \end{aligned} \quad (1.5.6)$$

Putting  $d^3\mathbf{q} = \frac{1}{2} \int |\mathbf{q}| d|\mathbf{q}|^2 d\Omega$ , where  $d\Omega$  is the element of solid angle associated with the direction of  $\mathbf{q}$ , this gives

$$A^+ - A^- = i \frac{\sqrt{(\frac{1}{4}s - m^2)}}{(4\pi)^2 \sqrt{s}} \int d\Omega A^+ A^- \tag{1.5.7}$$

Below the threshold the unitarity equation can be extended to read

$$\text{---} \bigcirc \text{---} \text{---} - \text{---} \bigcirc \text{---} \text{---} = 0 \quad \text{or} \quad A^+ - A^- = 0 \tag{1.5.8}$$

so  $A^+$  and  $A^-$  can be regarded as the same function  $A(s \pm i\epsilon, \dots)$  analytically continued above or below the two-particle threshold at  $s \equiv (p_1 + p_2)^2 = 4m^2$  where there is a branch point, the discontinuity across the square-root branch cut being given by (1.5.7) (see fig. 1.2). The physical amplitude is of course to be evaluated with  $s$  real, but we have a choice of approaching the real axis from above or below. We choose (by convention) the  $+i\epsilon$  prescription for  $A^+$  to the effect that

$$\text{Physical } A^+(s, \dots) = \lim_{\epsilon \rightarrow 0} A^+(s + i\epsilon, \dots) \tag{1.5.9}$$

and draw the branch cut along the real  $s$ -axis as shown in fig. 1.2. The sheet of the  $s$  plane exhibited in fig. 1.2 is called the ‘physical sheet’.

Since  $A$  is real below threshold it is clear from the Schwarz reflection principle (Titchmarsh 1939) that  $A(s^*, \dots) = A^*(s, \dots)$ , and that  $A^-$  is just the complex conjugate of  $A^+$ , and so

$$\text{Physical } A^-(s, \dots) = \lim_{\epsilon \rightarrow 0} A(s - i\epsilon, \dots) \tag{1.5.10}$$

An amplitude satisfying this reflection relation is said to be ‘Hermitian analytic’, or ‘real analytic’.

These results may be generalized to give us the discontinuity across the branch cut associated with an arbitrary number of particles, 1 up to  $n$ , in the intermediate state (fig. 1.3) which according to Cutkosky (1960, 1961) is

$$\text{Disc } \{A\} = \int \prod_{l=1}^{n-1} \frac{id^4k_l}{(2\pi)^4} \prod_{i=1}^n [-2\pi i \delta(q_i^2 - m_i^2)] A_1^+ A_2^- \tag{1.5.11}$$

where the integration is over the  $n - 1$  independent loops  $l$  which are formed by the  $n$  intermediate lines. Since

$$\frac{1}{q_i^2 - m_i^2 \pm i\epsilon} = P \frac{1}{q_i^2 - m_i^2} \pm \pi i \delta(q_i^2 - m_i^2) \tag{1.5.12}$$

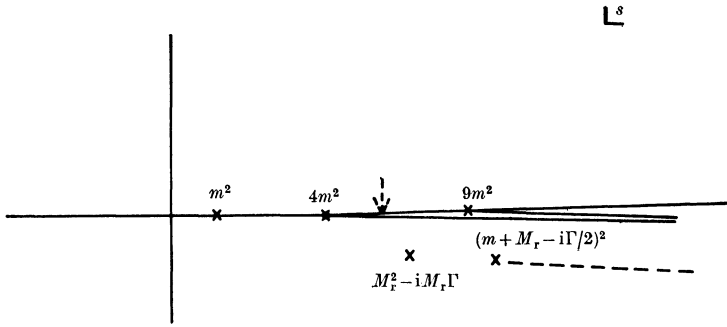


FIG. 1.2 Singularities of the scattering amplitude in the complex  $s$  plane, showing the pole at  $s = m^2$ , the threshold branch points at  $s = 4m^2, 9m^2, \dots$ , a resonance pole at  $s = M_r^2 - iM_r\Gamma$  on the unphysical sheet reached through the branch cut, and the  $m + M_r$  threshold branch cut. The physical value for  $A^+$  is obtained by approaching the real axis from above, as shown by the arrow.

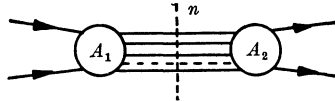


FIG. 1.3 The discontinuity across an  $n$ -particle intermediate state.

(where  $P \equiv$  principal part) it proves possible to rewrite (1.5.11) as

$$\text{Disc}\{A\} = \text{Disc} \int \prod_{l=1}^{n-1} \frac{id^4 k_l}{(2\pi)^4} \prod_{i=1}^n \frac{1}{(q_i^2 - m_i^2)} A_1^\dagger A_2 \quad (1.5.13)$$

This is in fact the same as the discontinuity obtained using Feynman propagators for the intermediate-state particles (see section 1.12 below).

The singularities of integrals like (1.5.13) have been investigated in detail (see Eden *et al.* 1966) and their positions are given by the Landau rules (Landau (1959); see section 1.12 below):

- (i)  $q_i^2 = m_i^2$  for all  $i = 1, \dots, n$ ; (1.5.14)
- (ii)  $\sum_{\text{loop } l} \alpha_l q_l = 0$  for some constants  $\alpha_i$ , the summation going right round each closed loop, and  $\alpha_i \neq 0$  for any  $i$  in the loop.

It is thus possible to identify all the singularities of an amplitude by drawing all the (infinite number of) different intermediate states composed of all the various particles in the theory which can take us from the initial state to the final state. We shall consider some further examples below. The positions and discontinuities across the cuts are



all calculable (in principle) from these Landau and Cutkosky rules once we know the particle poles.

These singularities include the poles on the real axis due to the stable particles, and branch points also on the real axis due to the various stable-particle thresholds. We have also noted that an unstable particle or resonance gives rise to a pole below the real axis at  $q_i^2 = m_i^2 - im_i\Gamma_i$  where  $\Gamma_i$  is its decay width. Since the real part of the resonance mass must obviously be greater than the threshold energy of the channel into which the particle can decay, this pole will not be on the physical sheet, but on the sheet reached by going down through the threshold branch point. Branch cuts involving such particles will also be off the physical sheet (see fig. 1.2).

We have mentioned that these singularities are supposed to stem from causality. Coleman and Norton (1965) have shown that in the physical region the Landau equations (1.5.14) correspond to the kinematic conditions for the event represented by the given diagram to occur classically. That is to say, if we regard each internal propagator as representing a pointlike particle having momentum  $q_i$ , then the vertices where the particle is emitted and absorbed can be regarded as having a space-time separation

$$\Delta_i = q_i \alpha_i$$

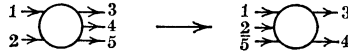
where  $\alpha_i$  is the proper time elapsing between emission and absorption. If  $\alpha_i = 0$  these two points are coincident. For it to be possible for a particle to pass round a closed loop we clearly need  $\sum_{\text{loop}} \Delta_i = 0$  which is just (1.5.14) (ii). And (1.5.14) (i) is just the mass-shell condition for the four-momentum. Hence a physical region singularity occurs only when the relevant Feynman diagram can represent a real physical process for pointlike, classical relativistic particles. Micro-causality thus seems to be needed in  $S$ -matrix theory only in the correspondence-principle limit when quantum mechanics approaches classical mechanics.

## 1.6 Crossing

A very important result of the above analyticity property is a relation it implies between otherwise quite separate scattering processes. This relation is known as 'crossing'.

If we consider the amplitude for  $1 + 2 \rightarrow 3 + 4 + 5$  it is intuitively rather obvious that it will have the same set of singularities as the amplitude for  $1 + 2 + \bar{5} \rightarrow 3 + 4$ , where  $\bar{5}$  is the anti-particle of 5, since

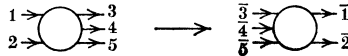
all we have to do is reverse the direction of the line corresponding to particle 5, i.e. we cross it over, viz.



The intermediate states in these two bubbles will be exactly similar.

It is clear that  $\bar{5}$  has to be the anti-particle of 5 because it must have the opposite sign for all the additive quantum numbers if both processes are to be possible. Of course these two processes occur for different regions of the variables since the first requires (*inter alia*)  $\sqrt{s_{12}} \geq \sqrt{s_{34}} + m_5$  while the second needs  $\sqrt{s_{34}} \geq \sqrt{s_{12}} + m_{\bar{5}}$ . However, since the two amplitudes have the same singularities it should, in principle, be possible to obtain one from the other by analytic continuation.

Furthermore, if we rotate all the legs



we get back to the same region of the variables, and so the amplitudes for  $1 + 2 \rightarrow 3 + 4 + 5$  and  $\bar{3} + \bar{4} + \bar{5} \rightarrow \bar{1} + \bar{2}$  should be identical. This is an example of *TCP* invariance since it requires that the *S*-matrix be unchanged by the combined operations of time reversal *T*, charge conjugation *C*, and parity inversion *P* (which is obviously what we need to get the anti-particles going backwards in space and time).

Unfortunately, it is not possible to prove the above results as we cannot be sure that analytic continuation from the physical region of one process will necessarily take us onto the physical sheet of the other process. We have to assume that the continuations can be made without leaving the physical sheet of the *s* variables. However, such results do hold in perturbation theory, and seem very plausible also in particle physics.

### 1.7 The 2 → 2 amplitude

As an example, which will be of considerable use to us later, we consider in some detail the kinematics and singularities of the scattering process  $1 + 2 \rightarrow 3 + 4$  (fig. 1.4 (a)). The channels are named after their respective energy invariants, to be introduced below.

By crossing and the *TCP* theorem all the six processes

$$\left. \begin{array}{ll}
 1 + 2 \rightarrow 3 + 4 & \bar{3} + \bar{4} \rightarrow \bar{1} + \bar{2} \quad (s\text{-channel}) \\
 1 + \bar{3} \rightarrow \bar{2} + 4 & 2 + \bar{4} \rightarrow \bar{1} + 3 \quad (t\text{-channel}) \\
 1 + \bar{4} \rightarrow \bar{2} + 3 & 2 + \bar{3} \rightarrow \bar{1} + 4 \quad (u\text{-channel})
 \end{array} \right\} \quad (1.7.1)$$

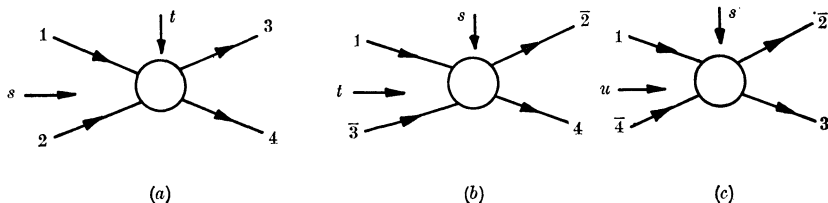


FIG. 1.4 The scattering processes in the  $s$ ,  $t$  and  $u$  channels of (1.7.1).

will share the same scattering amplitude, but the pairs of channels labelled  $s$ ,  $t$  and  $u$  will occupy different regions of the variables.

In the centre-of-mass system for particles 1 and 2 we write their four-momenta as

$$p_1 = (E_1, \mathbf{q}_{s12}), \quad p_2 = (E_2, -\mathbf{q}_{s12}) \tag{1.7.2}$$

$\mathbf{q}_{s12}$  being the three-momentum, equal but opposite for the two particles. Similarly for the final state

$$p_3 = (E_3, \mathbf{q}_{s34}), \quad p_4 = (E_4, -\mathbf{q}_{s34}) \tag{1.7.3}$$

Since the initial and final states involve only free particles the mass-shell constraints must be satisfied:

$$\left. \begin{aligned} p_1^2 &= E_1^2 - q_{s12}^2 = m_1^2 \\ p_2^2 &= E_2^2 - q_{s12}^2 = m_2^2 \\ p_3^2 &= E_3^2 - q_{s34}^2 = m_3^2 \\ p_4^2 &= E_4^2 - q_{s34}^2 = m_4^2 \end{aligned} \right\} \tag{1.7.4}$$

We define the invariant

$$\left. \begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ &= (E_1 + E_2)^2 = (E_3 + E_4)^2 \end{aligned} \right\} \tag{1.7.5}$$

which is the square of the total centre-of-mass energy for the  $s$ -channel processes. Now combining (1.7.5) and (1.7.4)

$$s = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 \tag{1.7.6}$$

where the dot denotes a four-vector product. Similarly

$$p_1 \cdot (p_1 + p_2) = m_1^2 + p_1 \cdot p_2 = E_1 \sqrt{s} \tag{1.7.7}$$

using (1.7.2) and (1.7.5). Then combining (1.7.6) and (1.7.7) we get

$$E_1 = \frac{1}{2\sqrt{s}} (s + m_1^2 - m_2^2) \tag{1.7.8}$$

for the centre-of-mass energy of particle 1 in terms of  $s$ . Likewise we find

$$\left. \begin{aligned} E_2 &= \frac{1}{2\sqrt{s}}(s + m_2^2 - m_1^2) \\ E_3 &= \frac{1}{2\sqrt{s}}(s + m_3^2 - m_4^2) \\ E_4 &= \frac{1}{2\sqrt{s}}(s + m_4^2 - m_3^2) \end{aligned} \right\} \quad (1.7.9)$$

Then from (1.7.8) and (1.7.4) we get

$$q_{s12}^2 = E_1^2 - m_1^2 = \frac{1}{4s} [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2] \quad (1.7.10)$$

It is convenient to introduce the 'triangle function'

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \quad (1.7.11)$$

so that 
$$q_{s12}^2 = \frac{1}{4s} \lambda(s, m_1^2, m_2^2) \quad (1.7.12)$$

and similarly we find 
$$q_{s34}^2 = \frac{1}{4s} \lambda(s, m_3^2, m_4^2)$$

We next introduce the invariant

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2 \quad (1.7.13)$$

This is evidently the square of the total centre-of-mass energy in the  $t$  channel, remembering that we have to change the sign of  $p_3$  and  $p_2$  on crossing. For this process we have

$$E_1 = \frac{1}{2\sqrt{t}}(t + m_1^2 - m_3^2) \quad (1.7.14)$$

$$q_{t13}^2 = \frac{1}{4t} \lambda(t, m_1^2, m_3^2) \quad \text{etc.} \quad (1.7.15)$$

and the threshold occurs at  $t = (m_1 + m_3)^2$ . However, as far as the  $s$  channel is concerned  $t$  represents the momentum transferred in the scattering process, i.e. the difference between the momenta of particles 1 and 3. So from (1.7.13), using (1.7.2) and (1.7.3)

$$\begin{aligned} t &= m_1^2 + m_3^2 - 2p_1 \cdot p_3 \\ &= m_1^2 + m_3^2 - 2E_1 E_3 + 2q_{s12} \cdot q_{s34} \\ &= m_1^2 + m_3^2 - 2E_1 E_3 + 2q_{s12} q_{s34} \cos \theta_s \end{aligned} \quad (1.7.16)$$

where  $\theta_s$  is the scattering angle between the directions of motion of particles 1 and 3 in the  $s$ -channel centre-of-mass system (fig. 1.4(a)). And on substituting (1.7.8) and (1.7.9) we get

$$\begin{aligned} z_s \equiv \cos \theta_s &= \frac{s^2 + s(2t - \Sigma) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{4sq_{s12}q_{s34}} \\ &= \frac{s^2 + s(2t - \Sigma) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)\lambda^{\frac{1}{2}}(s, m_3^2, m_4^2)} \end{aligned} \tag{1.7.17}$$

from (1.7.12), (1.7.13), where we have defined

$$\Sigma = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{1.7.18}$$

Similarly, as far as the  $t$ -channel is concerned  $s$  represents the momentum transfer and we find

$$\begin{aligned} z_t \equiv \cos \theta_t &= \frac{t^2 + (2s - \Sigma) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{4tq_{t13}q_{t24}} \\ &= \frac{t^2 + t(2s - \Sigma) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{\lambda^{\frac{1}{2}}(t, m_1^2, m_3^2)\lambda^{\frac{1}{2}}(t, m_2^2, m_4^2)} \end{aligned} \tag{1.7.19}$$

Finally, for the  $u$ -channel process the centre-of-mass energy squared is  $u \equiv (p_1 - p_4)^2 = (p_3 - p_2)^2 = m_1^2 + m_4^2 - 2p_1 \cdot p_4$  (1.7.20)

and we can write down similar expressions for the energies, momenta and scattering angle of the particles in this channel.

However, we know from section 1.4 that the four-line amplitude depends only on two independent invariants, so there must be a relation between  $s$ ,  $t$  and  $u$ . In fact, combining (1.7.6), (1.7.16) and (1.7.20) we find

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2m_1^2 + 2p_1 \cdot (p_2 - p_3 - p_4)$$

but momentum conservation requires  $p_1 + p_2 = p_3 + p_4$ , and using (1.7.4), (1.7.18) we get

$$s + t + u = \Sigma \tag{1.7.21}$$

We shall usually work with  $s$  and  $t$  as the independent variables.

These formulae greatly simplify for equal-mass scattering  $m_1 = m_2 = m_3 = m_4$  since

$$\lambda^{\frac{1}{2}}(s, m^2, m^2) = [s(s - 4m^2)]^{\frac{1}{2}}$$

giving

$$\left. \begin{aligned} q_{s12}^2 = q_{s34}^2 &= \frac{s - 4m^2}{4}; & z_s &= 1 + \frac{2t}{s - 4m^2} = -1 - \frac{2u}{s - 4m^2} \\ q_{t13}^2 = q_{t24}^2 &= \frac{t - 4m^2}{4}; & z_t &= 1 + \frac{2s}{t - 4m^2} = -1 - \frac{2u}{t - 4m^2} \end{aligned} \right\} \tag{1.7.22}$$

The physical region for the  $s$  channel is given by

$$s \geq \max \{ (m_1 + m_2)^2, (m_3 + m_4)^2 \}$$

(i.e. the threshold for the process) and  $-1 \leq \cos \theta_s \leq 1$ . This boundary is conveniently expressed by the function

$$\phi(s, t) \equiv 4sq_{s12}^2 q_{s34}^2 \sin^2 \theta_s = 0 \tag{1.7.23}$$

which using (1.7.12), (1.7.13), (1.7.17) and a little algebra gives

$$\begin{aligned} \phi(s, t) = stu - s(m_1^2 - m_3^2)(m_2^2 - m_4^2) - t(m_1^2 - m_2^2)(m_3^2 - m_4^2) \\ - (m_1^2 m_4^2 - m_3^2 m_2^2)(m_1^2 + m_4^2 - m_3^2 - m_2^2) = 0 \end{aligned} \tag{1.7.24}$$

or

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & m_2^2 & t & m_1^2 \\ 1 & m_2^2 & 0 & m_3^2 & s \\ 1 & t & m_3^2 & 0 & m_4^2 \\ 1 & m_1^2 & s & m_4^2 & 0 \end{vmatrix} = 0 \tag{1.7.25}$$

Despite the unsymmetrical appearance of equation (1.7.24), we also find

$$\phi(s, t) = 4tq_{t13}^2 q_{t24}^2 \sin^2 \theta_t = 4uq_{u14}^2 q_{u23}^2 \sin^2 \theta_u \tag{1.7.26}$$

and so  $\phi(s, t) = 0$  gives the boundaries of the physical regions for the  $s$ ,  $t$  and  $u$  channels. For equal-mass scattering (1.7.24) reduces to  $stu = 0$ , so the boundaries are just the lines  $s = 0$ ,  $t = 0$  and  $u = 0$ . For unequal masses the boundary curves become asymptotic to these lines. Some examples are shown in fig. 1.5 where  $s$ ,  $t$  and  $u$  are plotted subject to the constraint (1.7.21).

The various singularities may also be plotted on the Mandelstam diagram. Thus, if all the masses are equal we may expect bound state poles at  $s = m^2$ ,  $t = m^2$  and  $u = m^2$ , the two-particle branch point at  $s$ ,  $t$  or  $u = 4m^2$ , and further thresholds at  $9m^2$ ,  $16m^2$  etc. due to 3, 4 and more particle intermediate states. For the more realistic  $\pi N \rightarrow \pi N$  scattering we show in fig. 1.5(b) the nucleon pole and various resonances (ignoring isospin complications).

Because of the crossing property the nearby singularities in the  $t$  and  $u$  channels may be expected to control the behaviour of the  $s$ -channel scattering amplitude near the forward and backward directions ( $z_s = \pm 1$  respectively). Thus in  $\pi N$  scattering there is a forward peak at  $t = 0$  due to the  $\pi\pi$  threshold branch cut, and in particular due to the dominant resonances,  $\rho$ ,  $f$  etc., which occur in the  $\pi\pi$  channel, and a backward peak for  $u \approx 0$  due to the exchange of  $N$ ,  $\Delta$  and other

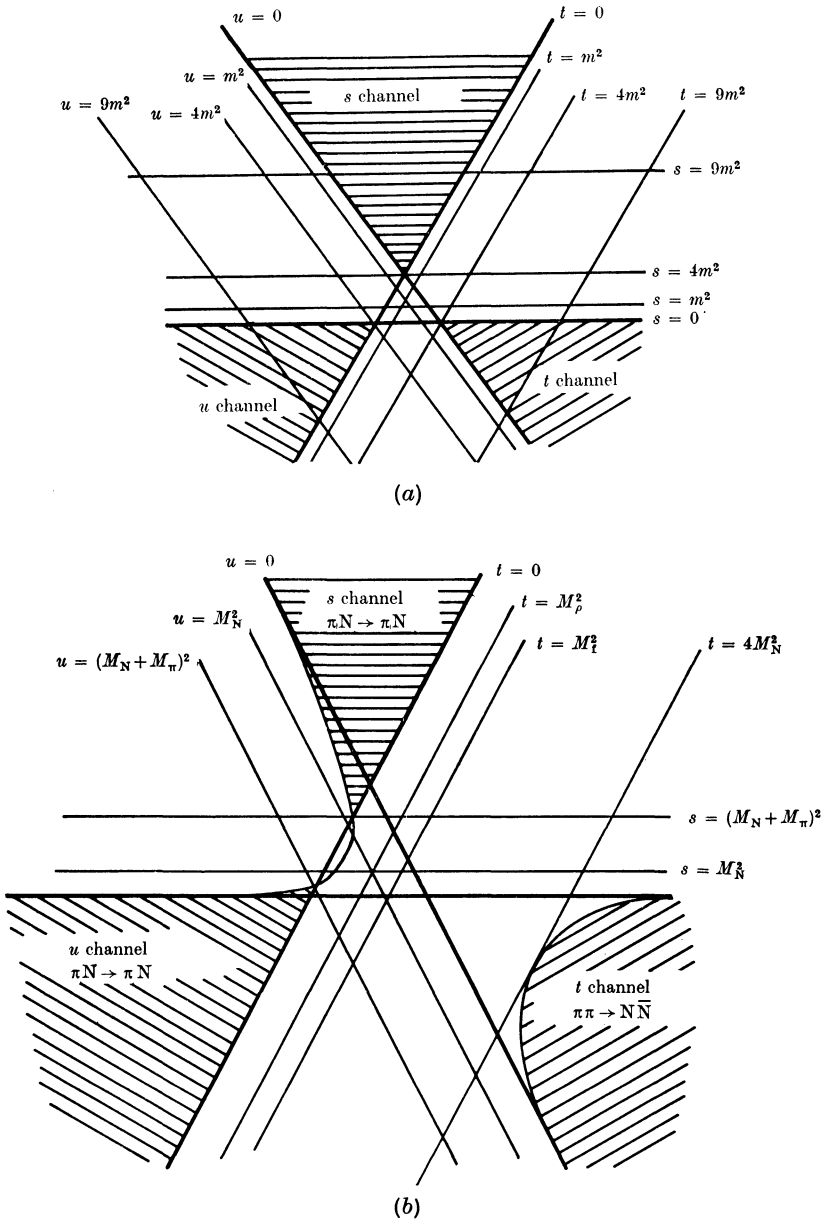


FIG. 1.5 (a) The Mandelstam  $s-t-u$  plot for equal mass-scattering, showing the positions of the pole at  $m^2$ , and the branch points at  $4m^2, 9m^2, \dots$  in each channel. The three physical regions are shown shaded. (b) The Mandelstam plot for  $\pi N$  scattering (ignoring isospin), showing the physical regions and some of the nearest singularities, the nucleon poles in the  $s$  and  $u$  channels, and the  $\rho$  and  $f$  poles in the  $t$  channel (not to scale).

baryon resonance poles. This dominance of exchanged poles will be an important aspect of Regge theory.

Although it is always most convenient theoretically to work in the centre-of-mass frame, experiments (except those using colliding beams such as the CERN-ISR) are performed in the so-called laboratory frame in which the target particle is at rest. If we call 1 the beam particle, and 2 the target, we have

$$p_{1L} = (E_L, \mathbf{p}_L), \quad p_{2L} = (m_2, 0) \quad (1.7.27)$$

where  $E_L$  is the energy and  $\mathbf{p}_L$  the three-momentum of the beam particle in the laboratory frame. The mass-shell condition (1.2.3) requires

$$E_L^2 = \mathbf{p}_L^2 + m_1^2 \quad (1.7.28)$$

so that the invariant  $s$  can be expressed in terms of laboratory quantities as

$$\begin{aligned} s &= (p_{1L} + p_{2L})^2 = (E_L + m_2, \mathbf{p}_L)^2 = m_1^2 + m_2^2 + 2m_2 E_L \\ &= m_1^2 + m_2^2 + 2m_2 \sqrt{(\mathbf{p}_L^2 + m_1^2)} \end{aligned} \quad (1.7.29)$$

For energies very much greater than the masses this becomes

$$s \approx 2m_2 E_L \approx 2m_2 p_L \quad (1.7.30)$$

Similarly from (1.7.13), if  $E_{4L}$  is the energy of the final-state particle, 4, in the laboratory frame we find

$$t = m_3^2 + m_4^2 - 2m_2 E_{4L} \quad (1.7.31)$$

## 1.8 Experimental observables

The scattering amplitudes which we have introduced in section 1.3 are, of course, not directly measurable. What are actually determined in a scattering experiment are (ideally) the momenta, energies and spin polarizations of all the  $n$  particles which are produced in a given two-particle collision  $1 + 2 \rightarrow n$ , and the aim of theory is to determine the probability of a given final state emerging from the given initial state.

From (1.1.1), and the definition of the scattering amplitude (1.3.10), (1.3.11) etc., the probability per unit time per unit volume that from the given initial state

$$|i\rangle = |P_1, P_2\rangle$$

we shall get the final state  $|f_n\rangle = |P'_1 \dots P'_n\rangle$  is the transition rate

$$R_{fi} = (2\pi)^4 \delta^4(p_f - p_i) |\langle f_n | A | i \rangle|^2 \quad (1.8.1)$$



The scattering cross-section,  $\sigma_{12 \rightarrow n}$ , for this process is defined as the total transition rate per unit incident flux. The flux of incident particles,  $F$ , i.e. the number incident per unit area per unit time, is just given by the relative velocity of the two particles,  $|\mathbf{v}_1 - \mathbf{v}_2|$ , divided by the invariant normalization volume  $V$ , i.e. the volume of phase space occupied by the two single particles, which from (1.2.11) is just

$$V = (2p_{01} 2p_{02})^{-1}$$

So in the centre-of-mass system we have

$$F = 4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2| \tag{1.8.2}$$

The centre-of-mass velocities are, from (1.7.2),

$$\mathbf{v}_1 = \frac{\mathbf{q}_{s12}}{E_1}, \quad \mathbf{v}_2 = -\frac{\mathbf{q}_{s12}}{E_2} \tag{1.8.3}$$

so 
$$F = 4E_1 E_2 \left( \frac{q_{s12}}{E_1} + \frac{q_{s12}}{E_2} \right) = 4(E_1 + E_2) q_{s12} = 4(\sqrt{s}) q_{s12} \tag{1.8.4}$$

which is, of course, invariant. To obtain the total transition rate we have to sum over all the possible final states  $|f_n\rangle$  which contain the  $n$  particles, so

$$\begin{aligned} \sigma_{12 \rightarrow n} &= \sum_f \frac{R_{fi}}{F} = \frac{1}{4q_{s12}\sqrt{s}} \sum_f (2\pi)^4 \delta(p_f - p_i) |\langle f_n | A | i \rangle|^2 \\ &= \frac{1}{4q_{s12}\sqrt{s}} \int \prod_{i=1}^n \frac{d^4 p'_i}{(2\pi)^3} \delta(p_i'^2 - m_i^2) (2\pi)^4 \delta^4 \left( \sum_{i=1}^n p'_i - p_1 - p_2 \right) \\ &\quad \times \sum_{\text{spins}} |\langle P'_i \dots P'_n | A | P_1 P_2 \rangle|^2 \end{aligned} \tag{1.8.5}$$

where we have integrated over all possible momenta of the  $n$  final-state particles remembering the normalization (1.2.11), and (1.2.7). For the time being we shall continue to deal only with spinless particles, and drop the  $\sum_{\text{spin}}$  and replace the  $P'_i$  by  $p'_i$ . The factor

$$\begin{aligned} d\Phi_n &\equiv \prod_{i=1}^n \left( \frac{d^4 p'_i}{(2\pi)^3} \delta(p_i'^2 - m_i^2) \right) (2\pi)^4 \delta^4 \left( \sum_{i=1}^n p'_i - p_1 - p_2 \right) \\ &= \prod_{i=1}^n \left( \frac{d^3 p'_i}{2p_{i0} (2\pi)^3} \right) (2\pi)^4 \delta^4 \left( \sum_{i=1}^n p'_i - p_1 - p_2 \right) \end{aligned} \tag{1.8.6}$$

represents the volume of phase space available to the  $n$  final-state particles, and the integral in (1.8.5) is over this volume.

The total scattering cross-section for particles 1 and 2 is obtained by summing (1.8.5) over all possible final states containing different numbers of particles, viz.

$$\sigma_{12}^{\text{tot}} = \sum_{n=2}^{\infty} \sigma_{12 \rightarrow n} \quad (1.8.7)$$

If there are only two particles, 3 and 4, in the final state, with centre-of-mass four-momenta given by (1.7.3) we have, from (1.8.5),

$$\sigma_{12 \rightarrow 34} = \frac{1}{4q_{s12}(\sqrt{s})(2\pi)^2} \int \frac{d^3\mathbf{p}_3 d^3\mathbf{p}_4}{2E_3 \cdot 2E_4} |\langle p_3 p_4 | A | p_1 p_2 \rangle|^2 \delta^4(p_3 + p_4 - p_1 - p_2) \quad (1.8.8)$$

Since the three-momenta of the particles are equal and opposite in (1.7.3) we can use the  $\delta$ -function in (1.8.8) to perform one of the integrations, leaving

$$\sigma_{12 \rightarrow 34} = \frac{1}{4q_{s12}(\sqrt{s})(2\pi)^2} \int \frac{d^3\mathbf{q}_{s34}}{2E_3 \cdot 2E_4} \delta(E_3 + E_4 - \sqrt{s}) |\langle p_3 p_4 | A | p_1 p_2 \rangle|^2 \quad (1.8.9)$$

We can express the momentum volume element in polar coordinates  $d^3\mathbf{q}_{s34} = q_{s34}^2 dq_{s34} d\Omega$ , where  $d\Omega = \sin\theta_s d\theta_s d\phi$  is the element of solid angle associated with the direction of particle 3, say, the polar angles being defined with respect to the beam direction, the  $z$  axis. Then defining

$$E = E_3 + E_4 = (m_3^2 + q_{s34}^2)^{\frac{1}{2}} + (m_4^2 + q_{s34}^2)^{\frac{1}{2}} \quad (1.8.10)$$

gives 
$$dE = \left( \frac{q_{s34}}{E_3} + \frac{q_{s34}}{E_4} \right) dq_{s34} = \frac{q_{s34} E}{E_3 E_4} dq_{s34} \quad (1.8.11)$$

and so 
$$\int \frac{q_{s34}^2 dq_{s34}}{E_3 E_4} \delta(E - \sqrt{s}) = \int \frac{q_{s34} dE}{E} \delta(E - \sqrt{s}) = \frac{q_{s34}}{\sqrt{s}} \quad (1.8.12)$$

and we end up with

$$\sigma_{12 \rightarrow 34} = \frac{q_{s34}}{64\pi^2 s q_{s12}} \int |\langle p_3 p_4 | A | p_1 p_2 \rangle|^2 d\Omega \quad (1.8.13)$$

It is therefore useful to introduce the 'differential cross-section'

$$\frac{d\sigma}{d\Omega} \equiv \frac{q_{s34}}{64\pi^2 s q_{s12}} |\langle p_3 p_4 | A | p_1 p_2 \rangle|^2 \quad (1.8.14)$$

which gives the probability of particle 3 being scattered into  $d\Omega$ , per unit incident flux.

As we are at the moment only considering spinless particles the scattering probability will always be independent of the azimuthal

angle  $\phi$ , as there is nothing to select any particular direction perpendicular to the beam, and from (1.7.16) at fixed  $s$

$$d\Omega = d(\cos \theta_s) d\phi = \frac{dt}{2q_{s12}q_{s34}} d\phi \tag{1.8.15}$$

so, since  $\int d\phi = 2\pi$ , we can more conveniently take as the differential cross-section

$$\frac{d\sigma}{dt} = \frac{1}{64\pi q_{s12}^2 s} |A(s, t)|^2 \tag{1.8.16}$$

In general we can obtain the partial (or differential) cross-sections with respect to any invariant simply by inserting a  $\delta$ -function into (1.8.5). Thus defining  $t' \equiv (p_1 - p_i)^2$  we have

$$\begin{aligned} \frac{d\sigma}{dt'} &= \frac{1}{4q_{s12}\sqrt{s}} \int \prod_{i=1}^n \frac{d^4 p'_i}{(2\pi)^3} \delta(p_i'^2 - m_i^2) (2\pi)^4 \delta^4(\Sigma p'_i - p_1 - p_2) \\ &\quad \times \delta(t' - (p_1 - p_i')^2) \sum_{\text{spins}} |\langle P'_i \dots P'_n | A | P_1 P_2 \rangle|^2 \end{aligned} \tag{1.8.17}$$

and clearly this can be repeated to give the partial cross-section with respect to any number of independent invariants.

### 1.9 The optical theorem

The total cross-section (1.8.7) satisfies a remarkable unitarity relation called the ‘optical theorem’ of which we shall make frequent use below.

The unitarity equation (1.2.14) reads, for a particular initial and final state,

$$(SS^\dagger)_{fi} = \sum_n S_{fn} S_{ni}^\dagger = \delta_{fi} \tag{1.9.1}$$

For elastic scattering  $1 + 2 \rightarrow 1 + 2$  we have from (1.3.10)

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i) \langle f | A | i \rangle \tag{1.9.2}$$

which with (1.3.13) gives us, from (1.9.1),

$$i(\langle f | A^+ | i \rangle - \langle f | A^- | i \rangle) = -(2\pi)^4 \sum_n \delta^4(p_n - p_i) \langle f | A^+ | n \rangle \langle n | A^- | i \rangle \tag{1.9.3}$$

and if the initial and final states are identical we get (remembering (1.5.10))

$$2 \text{Im} \{ \langle i | A | i \rangle \} = (2\pi)^4 \sum_n \delta^4(p_n - p_i) |\langle n | A^+ | i \rangle|^2 \tag{1.9.4}$$

But the right-hand side is the same as (1.8.7) with (1.8.5) apart from the flux factor so we obtain the relation

$$\sigma_{12}^{\text{tot}} = \frac{1}{2q_{s12}\sqrt{s}} \text{Im} \{ \langle i | A | i \rangle \} \tag{1.9.5}$$

Since the final state must be identical with the initial state,  $\langle i | A | i \rangle$  is the forward elastic scattering amplitude ( $1 + 2 \rightarrow 1 + 2$ ) with the directions of motion of the particles unchanged, i.e.  $\theta_s = 0$ , which means (from (1.7.16) with  $m_3 = m_1, m_2 = m_4$ ) that  $t = 0$ , so

$$\sigma_{12}^{\text{tot}} = \frac{1}{2q_{s12}\sqrt{s}} \text{Im} \{A^{\text{el}}(s, t = 0)\} \quad (1.9.6)$$

This optical theorem is well known in non-relativistic potential scattering (see for example Schiff (1968)) where it tells us that because of the conservation of probability the magnitude of the wave function in the 'shadow' behind the target at ( $\theta_s = 0$ ) must be reduced relative to the incoming wave by an amount equal to the total scattering in all directions. Equation (1.9.5) is just this same conservation requirement extended to the relativistic situation where particle creation can also occur. Note that it is only the elastic amplitude for  $1 + 2 \rightarrow 1 + 2$  which appears on the right-hand side, but the total cross-section for  $1 + 2 \rightarrow \text{anything}$  is on the left-hand side.

We can understand how this relation occurs diagrammatically from fig. 1.6, where the last step follows from (1.5.11) since we are taking the discontinuity of  $\text{---}\oplus\text{---}$  across the  $n$ -particle cut and summing over all possible intermediate states (compatible with four-momentum conservation). The real analyticity of  $A$  implies that

$$\text{Disc} \{A\} = \text{Im} \{A\}.$$

This optical theorem is one of the most useful constraints which unitarity imposes on a scattering amplitude. We shall also consider some generalizations in chapter 10.

### 1.10 Single-variable dispersion relations

According to our discussion in section 1.5 the only singularities which appear on the physical sheet are believed to be the poles corresponding to stable particles, and the threshold branch points. Thus, if we consider equal-mass scattering, and if we hold  $t$  fixed at some small, real, negative value (see fig. 1.5) in the  $s$  plane we find the singularities shown in fig. 1.7. On the right-hand side, for  $\text{Re}\{s\} > 0$ , we have the  $s$ -channel bound-state pole and the various  $s$ -channel thresholds. On the left, for  $\text{Re}\{s\} < 0$ , we meet the  $u$ -channel pole and the  $u$ -channel thresholds. The spacing between the two clearly depends on the relation (1.7.21)

$$s = 4m^2 - t - u \quad (1.10.1)$$



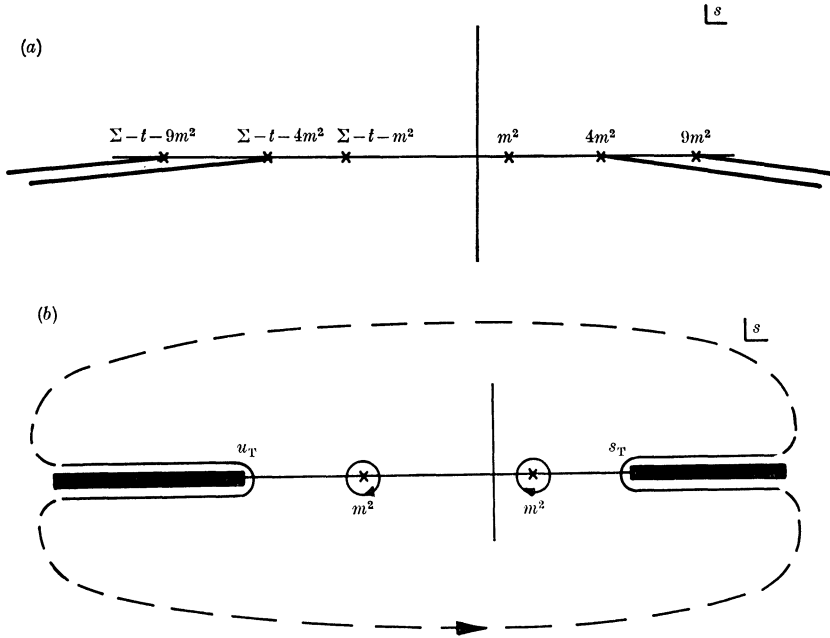


FIG. 1.7 (a) The physical-sheet singularities in  $s$  for fixed  $t$  ( $\Sigma = 4m^2$ ). (b) The integration contour in the complex  $s$  plane, expanded to infinity but enclosing the cuts and poles on the real axis.

where the integral is evaluated over any closed anti-clockwise contour in the complex  $s$  plane enclosing the point  $s$  such that  $A(s, t, u)$  is regular (holomorphic) inside and on that contour (fig. 1.7(b)). We then expand the contour so that it encircles the poles and encloses the branch cuts, as shown, giving

$$A(s, t, u) = \frac{g_s(t)}{m^2 - s} + \frac{g_u(t)}{m^2 - u} + \frac{1}{2\pi i} \int_C \frac{ds'}{s' - s} A(s', t, u) \quad (1.10.5)$$

(Remember  $s'$  and  $u'$  are related by  $s' + t + u' = 4m^2$ .) Then if

$$|A(s, t, u)| \xrightarrow{s \rightarrow \infty} |s|^{-\epsilon}, \quad \epsilon > 0 \quad (1.10.6)$$

the contribution from the circle at infinity will vanish, and we end up with

$$A(s, t, u) = \frac{g_s(t)}{m^2 - s} + \frac{g_u(t)}{m^2 - u} + \frac{1}{\pi} \int_{s_T}^{\infty} \frac{D_s(s', t)}{s' - s} ds' + \frac{1}{\pi} \int_{u_T}^{\infty} \frac{D_u(u', t)}{u' - u} du' \quad (1.10.7)$$

where  $s_T$  and  $u_T$  are the  $s$ - and  $u$ -channel thresholds, respectively.

Such dispersion relations were originally derived for the scattering of light by free electrons by Kramers (1927) and Kronig (1926), and provide the crucial test of the analyticity assumptions which we introduced in section 1.5. They agree with experiment within the accuracy of the available experimental data (see for example Eden (1971)). Theoretically, they are of great importance because we have found that once we are given the particle poles all the other singularities of the scattering amplitudes and their discontinuities can be found from the unitarity equations (at least in principle). So the unitarity equations give us  $\text{Im}\{A\}$ , but not  $\text{Re}\{A\}$ . But, once we know all the discontinuities of an amplitude, by using dispersion relations we can determine the real part of the amplitude too, and so unitarity plus analyticity determines the amplitudes completely, given the particle poles.

However, the convergence requirement (1.10.6) is frequently not satisfied, in which case we have to resort to subtractions. Thus if we have (neglecting the other terms in (1.10.7) for simplicity)

$$A(s, t, u) = \frac{1}{\pi} \int_{s_{\pi}}^{\infty} \frac{D_s(s', t)}{s' - s} ds' \tag{1.10.8}$$

but the integrand diverges as  $s' \rightarrow \infty$ , we write instead a dispersion relation for  $A(s, t, u) [(s - s_1)(s - s_2) \dots (s - s_n)]^{-1}$  including sufficient terms in the bracket to ensure convergence (assuming a finite number will suffice). So

$$A(s, t, u) \prod_{i=1}^n (s - s_i)^{-1} = \sum_{j=1}^n \frac{A(s_j, t, u_j)}{(s - s_j)} \prod_{\substack{i=1 \\ i \neq j}}^n (s_j - s_i)^{-1} + \frac{1}{\pi} \int_{s_{\pi}}^{\infty} \frac{D_s(s', t)}{(s' - s_1) \dots (s' - s_n)(s' - s)} ds' \tag{1.10.9}$$

since we pick up an extra contribution from each of the poles at  $s = s_1, s_2, \dots, s_n$ . Hence

$$A(s, t, u) = F_{n-1}(s, t) + \frac{1}{\pi} \prod_{i=1}^n (s - s_i) \int_{s_{\pi}}^{\infty} \frac{D_s(s', t)}{(s' - s_1) \dots (s' - s_n)(s' - s)} ds' \tag{1.10.10}$$

where  $F_{n-1}(s, t)$  is an arbitrary polynomial in  $s$  of degree  $n - 1$ , but now the integral converges if  $D_s(s, t) \xrightarrow{s \rightarrow \infty} s^{n-\epsilon}$ ,  $\epsilon > 0$ . Thus the divergence problem is solved at the expense of introducing an arbitrary polynomial which is not determined (at least directly) by the unitarity

equations. One of the main purposes of Regge theory is to close this gap by determining the subtractions.

A particularly useful form of these dispersion relations is for forward elastic scattering, such as  $\pi N \rightarrow \pi N$ , at  $t = 0$  where  $u = \Sigma - s$ . From the optical theorem (1.9.5)

$$\left. \begin{aligned} D_s(s, 0) &= \text{Im} \{A^{\text{el}}(s, 0)\} = 2q_{s12}(\sqrt{s}) \sigma_{12}^{\text{tot}}(s) \\ D_u(u, 0) &= \text{Im} \{A^{\text{el}}(u, 0)\} = 2q_{u14}(\sqrt{u}) \sigma_{14}^{\text{tot}}(u) \end{aligned} \right\} \quad (1.10.11)$$

and these cross-sections will be identical if particles 2 and  $\bar{2}$  ( $= 4$ ) are the same. It can be shown (section 2.4) that  $\sigma_{12}^{\text{tot}}(s) \xrightarrow{s \rightarrow \infty} \text{constant}$  (modulo possible  $\log s$  factors) so only two subtractions are needed in (1.10.7). So making the subtractions at  $s = 0$  we get (neglecting any pole contributions) for real  $s$  above the  $s$ -channel threshold

$$\begin{aligned} \text{Re} \{A^{\text{el}}(s, 0)\} &= a_0 + a_1 s + \frac{s^2}{\pi} P \int_{s_T}^{\infty} ds' (\sqrt{s'}) q'_{s12} \sigma_{12}^{\text{tot}}(s') \\ &\quad \times \left( \frac{1}{s'^2(s'-s)} + \frac{1}{(s'-\Sigma)^2(s'+s-\Sigma)} \right) \end{aligned} \quad (1.10.12)$$

(where  $P \equiv$  principal value – see (1.5.2)). Thus a knowledge of the total cross-section (with guesses as to its behaviour for very large  $s$  where it has not been measured) allows us to find  $\text{Re} \{A(s, 0)\}$  in terms of just two unknowns, the subtraction constants  $a_0$  and  $a_1$ . Since  $\text{Re} \{A(s, 0)\}$  can also be determined directly by Coulomb interference experiments (see for example Eden (1967)) the validity of these forward dispersion relations can be tested.

### 1.11 The Mandelstam representation

The single-variable dispersion relations were obtained by keeping one invariant fixed ( $t$  fixed in (1.10.4)) and representing the amplitude as a contour integral round the singularities in the other invariant ( $s$ ). But  $D_s(s, t)$  will have singularities in  $t$ , corresponding to the  $t$ -channel thresholds etc. Thus in fig. 1.8(a) we display these  $t$ -channel exchanges in the  $s$ -channel unitarity equation. It will also have  $u$ -channel threshold branch points, but of course  $u$  is not an independent variable, through (1.7.21), and so at fixed positive  $s$  these will appear at negative  $t$  values (see fig. 1.5).

One expects these singularities to lie on the real  $t$  axis, and so one can write a dispersion relation for  $D_s(s, t)$  similar to that for  $A(s, t, u)$



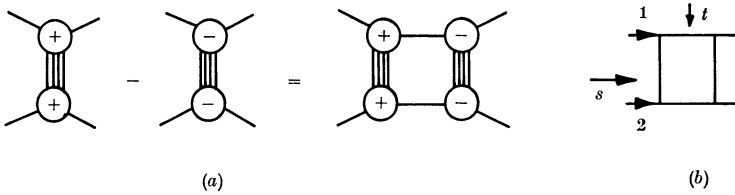


FIG. 1.8 (a) The contribution of  $t$ -channel intermediate states to the  $s$ -channel two-body unitary equation. (b) The ‘box’ diagram, the simplest diagram contributing to  $\rho_{st}(s, t)$ .

itself. We define the discontinuity of  $D_s(s, t)$  across the  $t$  thresholds as

$$\rho_{st}(s, t) = \frac{1}{2i} (D_s(s, t_+) - D_s(s, t_-)), \quad t > b_1(s) > 0 \quad (1.11.1)$$

and across the  $u$  thresholds as

$$\rho_{su}(s, u) = \frac{1}{2i} (D_s(s, u_+) - D_s(s, u_-)), \quad u > b_2(s) > 0 \quad (1.11.2)$$

The boundary functions  $b_{1,2}(s)$  are given by the position of the singularity of the lowest order diagram which contributes to  $\rho$ , usually the box diagram fig. 1.8(b). We shall find in the next section that

$$b_1(s) = b_2(s) = 4m^2 + \frac{4m^4}{s - 4m^2} \quad (1.11.3)$$

for equal-mass kinematics, giving the boundaries shown in fig. 1.9. Hence we can write a dispersion relation at fixed  $s$ ,

$$D_s(s, t) = \frac{1}{\pi} \int_{b_1(s)}^{\infty} \frac{\rho_{st}(s, t'')}{t'' - t} dt'' + \frac{1}{\pi} \int_{b_2(s)}^{\infty} \frac{\rho_{su}(s, u'')}{u'' - u} du'' \quad (1.11.4)$$

Similarly the  $u$ -discontinuity has branch cuts corresponding to the  $s$ - and  $t$ -thresholds, so we can write

$$D_u(u, t) = \frac{1}{\pi} \int_{b_1(u)}^{\infty} \frac{\rho_{tu}(u, t'')}{t'' - t} dt'' + \frac{1}{\pi} \int_{b_2(u)}^{\infty} \frac{\rho_{su}(s'', u)}{s'' - s} ds'' \quad (1.11.5)$$

If these expressions are substituted into (1.10.7) (neglecting the pole terms for simplicity) we end up with

$$\begin{aligned} A(s, t, u) = & \frac{1}{\pi^2} \iint \frac{\rho_{st}(s', t'')}{(s' - s)(t'' - t)} ds' dt'' + \frac{1}{\pi^2} \iint \frac{\rho_{su}(s', u'')}{(s' - s)(u'' - u')} ds' du'' \\ & + \frac{1}{\pi^2} \iint \frac{\rho_{tu}(u', t'')}{(u' - u)(t'' - t)} du' dt'' + \frac{1}{\pi^2} \iint \frac{\rho_{su}(s'', u')}{(u' - u)(s'' - s')} du' ds'' \end{aligned} \quad (1.11.6)$$

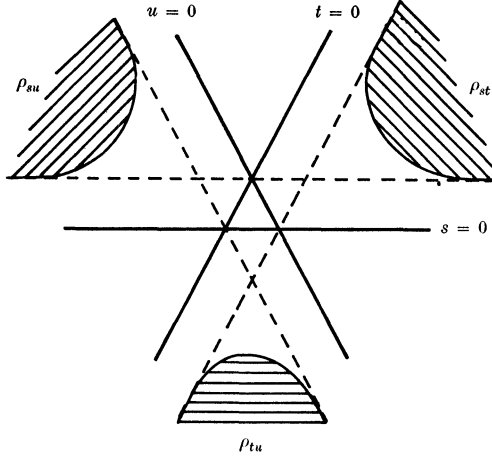


FIG. 1.9 The Mandelstam plot for equal-mass scattering (cf. fig. 1.5(a)), showing the double spectral functions (shaded areas). The boundary of  $\rho_{st}$  is given by (1.11.3).

It must be remembered that this relation, like (1.10.7), is written at fixed  $t$ , so that in the second and fourth terms we have to make use of the relations

$$s + t + u = s' + t + u' = \Sigma \tag{1.11.7}$$

in introducing primes into the variables which come from the denominators in (1.11.4) and (1.11.5). The primed variables are, of course, dummy variables of integration, so we are free to interchange primes in the fourth term, and then add it to the second term giving

$$\iint^\infty \rho_{su}(s', u'') \left( \frac{1}{(s' - s)(u'' - u')} + \frac{1}{(u'' - u)(s' - s'')} \right) ds' du'' \tag{1.11.8}$$

which can be rewritten, using (1.11.7), as

$$\iint^\infty \frac{\rho_{su}(s', u'')}{(s' - s)(u'' - u)} ds' du''$$

so (1.11.6) becomes

$$A(s, t, u) = \frac{1}{\pi^2} \iint^\infty \frac{\rho_{st}(s', t'')}{(s' - s)(t'' - t)} ds' dt'' + \frac{1}{\pi^2} \iint^\infty \frac{\rho_{su}(s', u'')}{(s' - s)(u'' - u)} ds' du'' + \frac{1}{\pi^2} \iint^\infty \frac{\rho_{tu}(u', t'')}{(u' - u)(t'' - t)} du' dt'' \tag{1.11.9}$$

The functions  $\rho_{st}$ ,  $\rho_{su}$ ,  $\rho_{tu}$  are called ‘double spectral functions’, and (1.11.9) is a double dispersion relation. This representation of

the scattering amplitude in terms of its double spectral functions is called the 'Mandelstam representation' (Mandelstam 1958, 1959). We do not know enough about the singularities of the scattering amplitude to be sure that such dispersion relations are valid. In particular we do not know that all the physical sheet singularities lie on the real axis. Indeed, it has been found that with diagrams where the masses of the intermediate states are smaller than those of the external states, anomalous thresholds appear at complex positions on the physical sheet, and the integration contour would have to make an excursion into the complex plane to include them. (A discussion of this problem may be found in Eden *et al.* (1966).) But it seems likely that (1.11.9) will at least be a good approximation for most practical purposes.

We chose to derive (1.11.9) from the fixed- $t$  dispersion relation (1.10.7). However, the final result is symmetrical in the three variables  $s$ ,  $t$  and  $u$ , and could equally well have been obtained starting from fixed- $s$  or fixed- $u$  dispersion relations. This is because the double spectral function is, from (1.11.1) and (1.10.2),

$$\begin{aligned} \rho_{st}(s, t) &= \frac{1}{2i} \left[ \frac{1}{2i} (A(s_+, t_+) - A(s_-, t_+)) - \frac{1}{2i} (A(s_+, t_-) - A(s_-, t_-)) \right] \\ &= -\frac{1}{4} (A(s_+, t_+) + A(s_-, t_-) - A(s_-, t_+) - A(s_+, t_-)) \end{aligned} \quad (1.11.10)$$

which can be taken to be

$$\frac{1}{2i} (D_t(s_+, t) - D_t(s_-, t)) \quad \text{or} \quad \frac{1}{2i} (D_s(s, t_+) - D_s(s, t_-)) \quad (1.11.11)$$

There are two complications about the use of (1.11.9). There is the rather trivial point that we have omitted bound-state poles which may occur in any of the three channels,  $s$ ,  $t$  or  $u$ . These should simply be added as necessary, as in (1.10.7). The more serious problem concerns the possible divergence of the integrand, as  $s'$ ,  $t''$  etc. tend to infinity. Like (1.10.7), (1.11.9) is only defined up to the various subtractions which may be needed to make the integrals converge. We may thus be forced to introduce apparently arbitrary subtractions into the Mandelstam representation. However we shall find in the next chapter that the hypothesis of analytic continuation in angular momentum enables us to determine these subtractions too.

### 1.12 The singularities of Feynman integrals

We have remarked in section 1.5 that the unitarity equations imply that scattering amplitudes have similar singularities to the Feynman diagrams of perturbation quantum field theory. This is not surprising because such field theories give Lorentz invariant scattering amplitudes with the same sort of connectedness properties, and they also satisfy unitarity at least perturbatively. Of course, we do not expect such a perturbation approach to be valid for strong interactions where, since the couplings are not small, the perturbation series will not converge, and where we cannot apply the usual re-normalization techniques. However, one can hope to gain some insight into the form of strong interaction amplitudes from field-theoretical analogies.

The spin properties of the particles will not be very important for our purposes so we shall only consider spinless scalar mesons of mass  $m$  interacting through a Lagrangian  $\mathcal{L}_{\text{int}} = g\phi^3$ . The Feynman rules for such particles are very simple (see Bjorken and Drell 1965). For a given diagram we include a factor  $i[(2\pi)^4(q^2 - m^2 + i\epsilon)]^{-1}$  for each internal line of momentum  $q$ , a factor  $g$  for each vertex, a factor  $(2\pi)^4 \delta^4(q_1 + q_2 - q_3)$  for momentum conservation at each vertex  $1 + 2 \rightarrow 3$ , and we integrate over the four-momenta of each internal line. The  $\delta$ -functions mean that only closed loops have free momenta, however, and one  $\delta$ -function of over-all energy-momentum conservation can be factored out in the definition of the scattering amplitude, as in (1.3.10).

Hence the contribution to the amplitude of the single particle exchange Born diagram fig. 1.10(a) is just

$$\frac{g^2}{q^2 - m^2 + i\epsilon}, \quad q^2 = (p_1 + p_2)^2 \quad (1.12.1)$$

while that of the box diagram. fig. 1.10(b) is

$$-i \frac{g^4}{(2\pi)^4} \int d^4k \{ [(k + p_1)^2 - m^2 + i\epsilon] [(k - p_2) - m^2 + i\epsilon] \\ \times [(k + p_1 - p_3)^2 - m^2 + i\epsilon] [k^2 - m^2 + i\epsilon] \}^{-1} \quad (1.12.2)$$

And an arbitrary diagram gives (neglecting the normalization factors)

$$A \propto \int \frac{d^4k_1 \dots d^4k_l}{\prod_{i=1}^n (q_i^2 - m_i^2 + i\epsilon)} \quad (1.12.3)$$

where the  $k_i$  are the independent loop momenta, and the  $q$ 's are

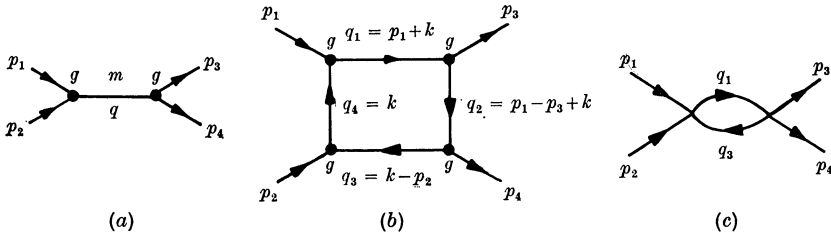


Fig. 1.10 (a) The Feynman diagram for single particle exchange in the  $s$ -channel. (b) The box diagram. (c) The contracted box diagram when the lines  $q_2$  and  $q_4$  are short-circuited by setting  $\alpha_2 = \alpha_4 = 0$ .

constrained by the  $\delta$ -functions at each vertex. Using the Feynman relation

$$\frac{1}{u_1 u_2 \dots u_n} = (n-1)! \int_0^1 d\alpha_1 \dots d\alpha_n \frac{\delta\left(1 - \sum_{i=1}^n \alpha_i\right)}{\left[\sum_{i=1}^n \alpha_i u_i\right]^n} \tag{1.12.4}$$

we can rewrite (1.12.3) as

$$A \propto \int_0^1 d\alpha_1 \dots d\alpha_n \int d^4 k_1 \dots d^4 k_l \frac{\delta(1 - \sum \alpha_i)}{[\sum \alpha_i (q_i^2 - m_i^2) + i\epsilon]^n} \tag{1.12.5}$$

The singularities of such integrals are studied in detail in Eden *et al.* (1966). If a function  $F(x)$  is represented by an integral such as

$$F(x) = \int_a^b f(x, z) dz \tag{1.12.6}$$

it will not necessarily have a singularity just because  $f(x, z)$  does, since the contour of integration can be displaced in the complex  $z$  plane to avoid the singularity, and by Cauchy's theorem all such continuations are equivalent. Singularities arise for two reasons. (i) The singularity in  $f(x, z)$  occurs at an end point of integration,  $a$  or  $b$ , so the contour cannot be deformed to avoid it. Thus

$$F(x) = \int_a^b \frac{1}{z-x} dz = \log\left(\frac{b-x}{a-x}\right) \tag{1.12.7}$$

is singular at  $x = a$  or  $b$ . (ii) Two or more singularities of  $f$  approach the contour from different sides (or a singularity moves off to infinity), thus pinching the contour so that it cannot avoid them. Thus

$$F(x) = \int_a^b \frac{dz}{(z-x)(z-x_0)} = \frac{1}{(x-x_0)} \log\left[\frac{(b-x_0)(b-x)}{(a-x_0)(a-x)}\right] \tag{1.12.8}$$

is singular at  $x = x_0$  where two singularities coincide, as well as at  $x = a, b$  as before. These two types of singularity are known as 'end-point' and 'pinch' respectively.

The generalization to multiple integrals is quite complicated because of the number of variables involved, but it is found that the singularities of the integrand (1.12.5) at  $q_i^2 = m_i^2$  result in singularities of the scattering amplitude if either

$$q_i^2 = m_i^2 \quad \text{or} \quad \alpha_i = 0, \quad \text{for all } i = 1, \dots, n,$$

and 
$$\frac{\partial}{\partial k_j} \sum_{i=1}^n \alpha_i (q_i^2 - m_i^2) = 0 \quad \text{for } j = 1, \dots, l$$

But since (see for example (1.12.2)) each  $q$  is linear in the  $k$ 's the latter condition is equivalent to  $\sum_i \alpha_i q_i = 0$  for each loop  $j$ . These are the Landau equations (1.5.14).

Thus for the box diagram fig. 1.10(b) we have either  $q_i^2 = m_i^2$  or  $\alpha_i = 0$  for  $i = 1, \dots, 4$  and

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4 = 0 \tag{1.12.9}$$

To take any  $\alpha_i = 0$  is equivalent to removing that line from consideration, so for example if  $\alpha_2, \alpha_4 = 0$  we have fig. 1.10(c). This requires  $q_2^2 = q_3^2 = m^2$  and  $\alpha_1 q_1 + \alpha_3 q_3 = 0$  so  $q_1 = -q_3$  and the singularity is at  $s = (q_1 - q_3)^2 = 4q_1^2 = 4m^2$ , i.e. at the threshold. If none of the  $\alpha$ 's vanish (1.12.9) must hold. Multiplying (1.12.9) successively by each of the  $q_i$  ( $i = 1, \dots, 4$ ) gives us four linear equations for the  $\alpha$ 's, and a solution with  $\alpha_i \neq 0$  is possible only if the determinant of the coefficients vanishes, i.e.

$$\det(q_i \cdot q_j) = 0, \quad i, j = 1, \dots, 4$$

Since  $s = (q_1 - q_3)^2$  and  $t = (q_2 - q_4)^2$  we find the singularity is at

$$(s - 4m^2)(t - 4m^2) = 4m^4 \tag{1.12.10}$$

This is the boundary of the Mandelstam double spectral function (1.11.3), because it gives us the curve where the discontinuity across the  $s$ -threshold cut has a discontinuity in  $t$  due to the  $t$ -threshold. Note that as  $s \rightarrow \infty$  this boundary moves to the threshold at  $t = 4m^2$ . More complex singularities, involving larger numbers of particles in the intermediate states, will occur at larger values of the invariants. We shall not pursue the subject further here, and readers seeking a more detailed discussion should consult Eden *et al.* (1966). We shall want to make use of some of these results below.

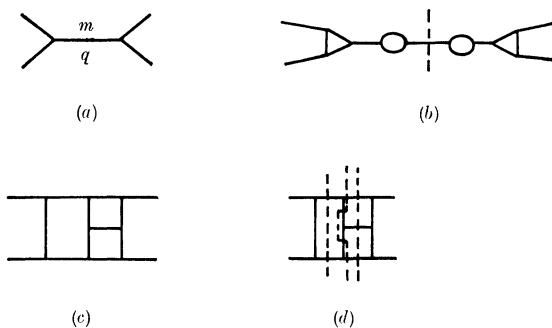


FIG. 1.11 (a) The unitarity diagram for single particle exchange giving a pole discontinuity of the form  $\delta(q^2 - m^2)$ . (b) One of the (infinite) set of Feynman diagrams which, when cut across the single-particle propagator as shown by the dashed line contributes to the discontinuity in (a). (c) A Feynman diagram. (d) Three different ways of cutting (c) showing that it contributes to the two-, three- and four-particle unitarity diagrams.

It should be noted that the correspondence between Feynman diagrams and unitarity diagrams is always many-to-one. Thus the single particle exchange unitarity diagram fig. 1.11 (a) corresponds to the discontinuity of the sum of the infinite sequence of Feynman diagrams like fig. 1.11 (b) which give the re-normalization of the vertices, and of the mass of the exchanged particle. And a more complicated Feynman diagram like fig. 1.11 (c) will contribute to several different unitary diagrams because the discontinuity across this diagram can be taken in different ways as in fig. 1.11 (d). This must be borne in mind when interpreting Feynman-diagram models for strong interaction processes.

### 1.13 Potential scattering

It is rather obvious that non-relativistic potential-scattering theory can have at most limited relevance to particle physics. This is not just a matter of the failure to incorporate relativistic kinematics, but because the very idea of a potential which is a function of the spatial co-ordinates is very difficult to generalize to the relativistic situation. In fact the occurrence of a local causal interaction through a potential field always implies, because of Lorentz invariance, radiation of the field quanta too. And in particle physics, except at very low energies, it is always likely that inelastic processes involving the production of new particles will occur, which clearly cannot readily be incorporated into the framework of potential scattering.

None the less, potential scattering is a very useful theoretical laboratory in which to study many aspects of quantum scattering theory, and some of the models used in particle physics are founded on analogies with potential theory. For our purposes it is particularly important that the sort of dispersion relations which we have been discussing in this chapter can be proved to hold in potential scattering provided that the potentials are suitably behaved. And in chapter 3 we shall find that the validity of the basic ideas of Regge theory can be proved in potential scattering too. In this section we shall try to bring out the similarities between the singularity structure of Yukawa potential-scattering amplitudes and those of the strong-interaction  $S$ -matrix.

The Schroedinger equation for two particles interacting via a local potential  $V(r)$ , in the centre-of-mass system, is (Schiff 1968)

$$\left[ \frac{\hbar^2}{2M} \nabla^2 + \frac{\hbar^2}{2M} k^2 - V(r) \right] \psi(\mathbf{r}) = 0 \quad (1.13.1)$$

where  $k$  is the wave number (energy  $E = \hbar^2 k^2 / 2M$ ), and  $M$  is the reduced mass. It is convenient to introduce

$$U(r) = V(r) \frac{2M}{\hbar^2} \quad (1.13.2)$$

so that (1.13.1) becomes

$$(\nabla^2 + k^2 - U(r)) \psi(\mathbf{r}) = 0 \quad (1.13.3)$$

The initial state is represented by a plane wave, wave vector  $\mathbf{k}$ , along the  $z$  axis (fig. 1.12)

$$\psi(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} = e^{ikz} \quad (1.13.4)$$

and we seek a solution to this equation subject to the boundary condition that as  $r \rightarrow \infty$

$$\psi(\mathbf{r}) \rightarrow e^{ikz} + A(\mathbf{k}, \mathbf{k}') \frac{e^{i\mathbf{k}' \cdot \mathbf{r}}}{r} \quad (1.13.5)$$

where the second term is the outgoing scattered wave, with wave vector  $\mathbf{k}'$  in the direction of unit vector  $\hat{\mathbf{r}}$ , and  $A(\mathbf{k}, \mathbf{k}')$  is the scattering amplitude. For elastic scattering  $|\mathbf{k}| = |\mathbf{k}'| = k$ .

The solution to (1.13.3) with the boundary condition (1.13.5) is given by the Lippman–Schwinger equation

$$\psi(\mathbf{r}) = e^{ikz} + \int G_0(\mathbf{r}, \mathbf{r}') U(\mathbf{r}') \psi(\mathbf{r}') d\mathbf{r}' \quad (1.13.6)$$



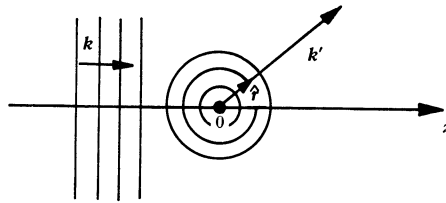


FIG. 1.12 Incident plane wave, wave vector  $k$  along  $z$  axis, scattered by a potential centred at  $z = 0$  into the direction  $\hat{r}$ , with wave vector  $k'$

where the Green's function is

$$G_0(\mathbf{r}, \mathbf{r}') \equiv -\frac{1}{4\pi} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{1.13.7}$$

That (1.13.6) is a solution of (1.13.3) may be checked by direct substitution, remembering that

$$\nabla^2 \left( \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -4\pi \delta(\mathbf{r}-\mathbf{r}') \tag{1.13.8}$$

And provided  $rV(r) \rightarrow 0$  we find, since  $|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}' \cdot \hat{r}$ ,  $r \rightarrow \infty$

$$\psi(\mathbf{r}) \rightarrow e^{ikr} - \frac{e^{ikr}}{4\pi r} \int e^{-ik' \cdot \mathbf{r}'} U(\mathbf{r}') \psi(\mathbf{r}') d\mathbf{r}' \tag{1.13.9}$$

which by comparison with (1.13.5) gives

$$A(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi} \int e^{ik' \cdot \mathbf{r}'} U(\mathbf{r}') \psi(\mathbf{r}') d\mathbf{r}' \tag{1.13.10}$$

The Born approximation, appropriate at high energies, is obtained by approximating  $\psi(\mathbf{r}')$  in (1.3.10) by the incoming plane wave (1.13.4), assuming the scattering to be small, giving

$$A^B(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi} \int e^{i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}'} U(\mathbf{r}') d\mathbf{r}' \tag{1.13.11}$$

It is convenient to introduce (like our previous notation)  $s = k^2$  for the total energy (in units where  $\hbar^2 = 2M = 1$ ), and

$$t = -K^2 = -(\mathbf{k} - \mathbf{k}')^2 = -2k^2(1 - \cos \theta)$$

where  $K$  is the momentum transfer vector. Then

$$A^B(\mathbf{k}, \mathbf{k}') \equiv A^B(s, t) = -\frac{1}{4\pi} \int e^{i\mathbf{K} \cdot \mathbf{r}'} U(\mathbf{r}') d\mathbf{r}' \tag{1.13.12}$$

Then putting  $\int d\mathbf{r}' = \int_0^\infty r'^2 dr' \int_0^\pi \sin \alpha d\alpha \int_0^{2\pi} d\beta$  (1.13.13)

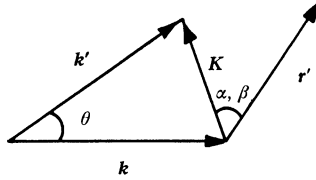


FIG. 1.13 The wave vectors  $|k| = |k'|$  so  $|K| = 2|k| \sin \frac{1}{2}\theta$ . The angles  $\alpha, \beta$  are the polar angles of  $r'$  with respect to the  $K$  axis.

and  $K \cdot r' = Kr' \cos \alpha$ , where  $\alpha, \beta$  are polar angles about the  $K$  axis (fig. 1.13), the angular integration is readily performed, since  $U = U(r')$  only, giving

$$A^B(s, t) = -\frac{1}{K} \int_0^\infty \sin(Kr') U(r') r' dr' \tag{1.13.14}$$

The simplest form of potential which has the short-range character appropriate to strong interactions is the Yukawa potential

$$U(r) = g^2 \frac{e^{-\mu r}}{r} \tag{1.13.15}$$

where  $g^2$  is the coupling strength and  $\mu^{-1}$  is the range, for which we find

$$A^B(s, t) = \frac{g^2}{\mu^2 + K^2} = \frac{g^2}{\mu^2 - t} \tag{1.13.16}$$

So the Born approximation to the Yukawa scattering amplitude is just a pole at  $t = \mu^2$  whose residue is given by the coupling strength. Of course if we have more complicated potentials the analyticity properties will not be so simple, but a large class of potentials can be represented by a superposition of Yukawa's

$$U(r) = \frac{1}{r} \int_m^\infty \rho(\mu) e^{-\mu r} d\mu \tag{1.13.17}$$

where  $\rho$  is a weight function, giving

$$A^B(s, t) = \int_m^\infty d\mu \frac{\rho(\mu)}{\mu^2 - t} \tag{1.13.18}$$

which is obviously holomorphic in  $s$ , and cut in  $t$  for  $t = m^2 \rightarrow \infty$ .

To proceed further we note that since

$$(\nabla^2 + k^2) e^{ik \cdot r} = 0 \tag{1.13.19}$$

(1.13.3) can be written

$$(\nabla^2 + k^2) \psi = (\nabla^2 + k^2) e^{ik \cdot r} + U \psi \tag{1.13.20}$$

and so formally 
$$\psi = e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{\nabla^2 + k^2} U \psi \tag{1.13.21}$$

which by successive re-substitution becomes

$$\psi = e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{\nabla^2 + k^2} U e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{\nabla^2 + k^2} U \frac{1}{\nabla^2 + k^2} U e^{i\mathbf{k}\cdot\mathbf{r}} + \dots \tag{1.13.22}$$

and so in (1.13.10) we get

$$A(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi} \int e^{-i\mathbf{k}'\cdot\mathbf{r}'} U \left( e^{i\mathbf{k}\cdot\mathbf{r}'} + \frac{1}{\nabla^2 + k^2} U e^{i\mathbf{k}\cdot\mathbf{r}'} + \dots \right) d\mathbf{r}' \tag{1.13.23}$$

The first term is just the Born approximation (1.13.11) which we can denote by

$$A^B(\mathbf{k}, \mathbf{k}') = \langle \mathbf{k}' | U | \mathbf{k} \rangle \tag{1.13.24}$$

where the states  $|\mathbf{k}\rangle$  are momentum eigenstates such that

$$\nabla^2 |\mathbf{k}\rangle = -k^2 |\mathbf{k}\rangle.$$

Then using the completeness relation to write

$$\frac{1}{\nabla^2 + k^2} = \frac{1}{(2\pi)^3} \int |\mathbf{p}\rangle \frac{d^3\mathbf{p}}{k^2 - p^2} \langle \mathbf{p}| \tag{1.13.25}$$

the Born series (1.13.23) becomes

$$A(\mathbf{k}, \mathbf{k}') = \langle \mathbf{k}' | U | \mathbf{k} \rangle + \frac{1}{(2\pi)^3} \int \langle \mathbf{k}' | U | \mathbf{p} \rangle \frac{d^3\mathbf{p}}{k^2 - p^2} \{ \langle \mathbf{p} | U | \mathbf{k} \rangle + \dots \} \tag{1.13.26}$$

Since the term in brackets  $\{ \}$  is just the Born expansion of  $A(\mathbf{k}, \mathbf{p})$  we can rewrite (1.13.26) as the Lippman-Schwinger equation for the scattering amplitude

$$A(\mathbf{k}, \mathbf{k}') = A^B(\mathbf{k}, \mathbf{k}') + \frac{1}{(2\pi)^3} \int A(\mathbf{k}, \mathbf{p}) \frac{d^3\mathbf{p}}{k^2 - p^2} A^B(\mathbf{p}, \mathbf{k}') \tag{1.13.27}$$

which is represented diagrammatically in fig. 1.14.

For our Yukawa potential, using (1.13.16) for (1.13.24), (1.13.26) gives

$$A(\mathbf{k}, \mathbf{k}') = \frac{g^2}{\mu^2 + (k' - k)^2} + \frac{g^4}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{[(k' - p^2 + \mu^2)[k^2 - p^2][(p - k)^2 + \mu^2]} + \dots \tag{1.13.28}$$

a power series in the coupling constant which is reminiscent of the

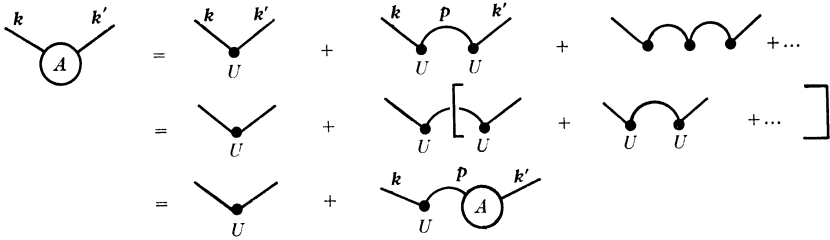


FIG. 1.14 Diagrammatic representation of the Lippman-Schwinger equation as a Born series in which the potential acts an arbitrary number of times.

Feynman rules for the diagrams in fig. 1.10, but of course in three dimensions. The second term has a cut in  $k^2 = s$  for  $k^2 > 0$  where the denominator  $(p^2 - k^2)^{-1}$  vanishes. The first term has a pole at  $t = \mu^2$ ; the second has a cut beginning at  $t = 4\mu^2$ , and in fact has a Mandelstam double spectral function boundary at

$$t = b(s) = 4\mu^2 + \frac{\mu^4}{s} \tag{1.13.29}$$

Thus Yukawa potential scattering, or simple generalizations like (1.13.18), have a singularity structure very similar to that of  $\phi^3$  quantum field theory. The principal differences are of course the absence of  $u$ -channel singularities (which would correspond to a Majorana type of exchange potential), the absence of inelastic thresholds in  $s$ , and the fact that the elastic threshold branch point is at  $s = 0$  because we are using the non-relativistic kinematics  $s = E = k^2$ , rather than the relativistic  $s = E^2 = k^2 + m^2$ .

**1.14 The eikonal expansion\***

A useful approximation method, which we shall make use of in chapter 8, is the so-called ‘eikonal’ expansion of the scattering amplitude. It can readily be derived in potential scattering where it is appropriate for energies much greater than the interaction potential, i.e.  $E \gg V$ , or  $k^2 \gg U$  in (1.13.3) (see Glauber 1959, Jochain and Quigg 1974).

In this situation we expect that there will be very little scattering in the backward direction, and so we can write the solution of (1.13.3)

as 
$$\psi(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \phi(\mathbf{r}) \tag{1.14.1}$$

\* This section may be omitted at first reading.

where  $\phi(\mathbf{r})$  represents the modulation of the incoming wave caused by the potential. When (1.14.1) is substituted in (1.13.6) the equation for  $\phi(\mathbf{r})$  becomes

$$\begin{aligned} \phi(\mathbf{r}) &= 1 - \frac{1}{4\pi} \int e^{ik|\mathbf{r}-\mathbf{r}'| - ik \cdot (\mathbf{r}-\mathbf{r}')} U(\mathbf{r}') \phi(\mathbf{r}') (|\mathbf{r}-\mathbf{r}'|)^{-1} d\mathbf{r}' \\ &= 1 - \frac{1}{4\pi} \int e^{ikr''(1-\cos\theta'')} U(\mathbf{r}-\mathbf{r}'') \phi(\mathbf{r}-\mathbf{r}'') r'' dr'' d(\cos\theta'') d\phi'' \end{aligned} \tag{1.14.2}$$

where in the last step we have introduced the vector  $\mathbf{r}'' \equiv \mathbf{r} - \mathbf{r}'$ , and  $\theta''$ ,  $\phi''$  are the polar angles of  $\mathbf{r}''$  with respect to the direction of  $\mathbf{r}$ .

At high energies we can assume that the range over which  $U\phi$  varies appreciably is much greater than the wavelength of the beam,  $\lambda$ , so we can perform the  $\cos\theta''$  integration by parts, and neglect the second term, giving

$$\phi \approx 1 - \frac{1}{4\pi} \int \left( \frac{e^{ikr''(1-\cos\theta'')}}{-ikr''} U(\mathbf{r}-\mathbf{r}'') \phi(\mathbf{r}-\mathbf{r}'') \right)_{\cos\theta''=-1}^{\cos\theta''=1} r'' dr'' d\phi'' \tag{1.14.3}$$

However, the term with  $\cos\theta'' = -1$  is very rapidly oscillating, and hence makes a very small contribution when we perform the integration over  $r''$ , and neglecting it we get a contribution only when  $\mathbf{r}''$  is parallel to  $\mathbf{k}$ , i.e. along the  $z$  axis, and so (since  $\int d\phi'' = 2\pi$ ) (1.14.3) becomes

$$\phi \approx 1 - \frac{i}{2k} \int_{-\infty}^z U(x, y, z'') \phi(x, y, z'') dz'' \tag{1.14.4}$$

for which the solution is

$$\phi(x, y, z) = \exp \left( -\frac{i}{2k} \int_{-\infty}^z U(x, y, z'') dz'' \right) \tag{1.14.5}$$

So if we resolve  $\mathbf{r}$  into (see fig. 1.15)

$$\mathbf{r} = \mathbf{b} + \hat{\mathbf{k}}z$$

where  $\mathbf{b}$  is a two-dimensional vector perpendicular to the unit vector  $\hat{\mathbf{k}}$ , we have

$$\psi(\mathbf{r}) = \exp \left[ i\mathbf{k} \cdot \mathbf{r} - \frac{i}{2k} \int_{-\infty}^z U(\mathbf{b} + \hat{\mathbf{k}}z'') dz'' \right] \tag{1.14.6}$$

which in (1.13.10) gives

$$\begin{aligned} A(\mathbf{k}, \mathbf{k}') &= -\frac{1}{4\pi} \int e^{-ik' \cdot \mathbf{r}'} U(\mathbf{b}' + \hat{\mathbf{k}}z') \\ &\times \exp \left( i\mathbf{k} \cdot \mathbf{r}' - \frac{i}{2k} \int_{-\infty}^z U(\mathbf{b}' + \hat{\mathbf{k}}z'') dz'' \right) dz' d^2\mathbf{b}' \end{aligned} \tag{1.14.7}$$

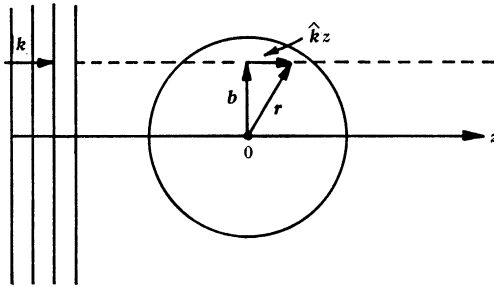


FIG. 1.15 Plane wave incident on a potential.  $\mathbf{b}$  is the two-dimensional impact-parameter vector, perpendicular to  $z$ .

For small-angle scattering  $(\mathbf{k} - \mathbf{k}')$ .  $\hat{\mathbf{k}} \approx 0$ , and in this approximation the  $z'$  integration is over an exact differential. That is because

$$\frac{\partial}{\partial z'} \left( \exp \left[ - \int^{z'} U dz'' \right] \right) = - \left( \exp \left[ - \int^{z'} U dz'' \right] \right) U dz'$$

And so we obtain

$$A(\mathbf{k}, \mathbf{k}') = \frac{ik}{2\pi} \int e^{i\mathbf{k} \cdot \mathbf{b}'} (1 - e^{\chi(\mathbf{b}')}) d^2\mathbf{b}' \tag{1.14.8}$$

where we have introduced the ‘eikonal function’ defined by

$$\chi(\mathbf{b}) \equiv - \frac{1}{2k} \int_{-\infty}^{\infty} U(\mathbf{b} + \hat{\mathbf{k}}z'') dz'' \tag{1.14.9}$$

For spherically symmetric potentials we can perform the angular integration in (1.14.8), since

$$d^2\mathbf{b}' = b' db' d\phi$$

$$\mathbf{K} \cdot \mathbf{b}' = (2k \sin \frac{1}{2}\theta) b' \cos \phi = (\sqrt{t}) b' \cos \phi$$

and (Magnus and Oberhettinger (1949) p. 26)

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \phi} d\phi = J_0(x) \tag{1.14.10}$$

where  $J_0$  is the zeroth order Bessel function, and obtain

$$A(\mathbf{k}, \mathbf{k}') = -ik \int_0^{\infty} J_0(b'\sqrt{-t}) (e^{i\chi(\mathbf{b}')} - 1) b' db' \tag{1.14.11}$$

If the exponent is expanded powers of  $\chi$  we get the eikonal series

$$A(\mathbf{k}, \mathbf{k}') = -ik \sum_n \int_0^{\infty} J_0(b'\sqrt{-t}) \frac{(i\chi)^n}{n!} b' db' \tag{1.14.12}$$

The eikonal function (1.14.9) can be expressed as the two-dimensional Fourier transform of the Born approximation (1.13.12) i.e.

$$\begin{aligned}\chi(\mathbf{b}) &= \frac{1}{2\pi k} \int d^2 \mathbf{k} e^{-i\mathbf{k} \cdot \mathbf{b}} A^{\text{B}}(\mathbf{k}, \mathbf{k}') \\ &= \frac{1}{2k} \int_{-\infty}^0 J_0(b\sqrt{-t}) A^{\text{B}}(s, t) dt\end{aligned}\quad (1.14.13)$$

and inverting (1.14.13) using (Magnus and Oberhettinger (1949) p. 35)

$$\int_0^{\infty} J_0(xy) J_0(x'y) dy = \delta(x - x') \quad (1.14.14)$$

we find 
$$A^{\text{B}}(s, t) = k \int_0^{\infty} \chi(b) J_0(b\sqrt{-t}) b db \quad (1.14.15)$$

which is just the first term in the series (1.14.12)

Thus the first term in the eikonal series is identical to the first term in the Born series (1.13.26) at high energies. The relationship between the higher order terms of the two series is more complicated (see Jochain and Quigg 1974) because for real potentials the eikonal series contains alternating real and imaginary terms, while in general all the terms of the Born series (except the first) are complex. But in the large  $k$ , fixed  $K$ , limit the two series agree. Thus the eikonal series can be regarded as an approximation to the sum of ladder diagrams (fig. 1.14) when each successive scattering is restricted to small angles only. We shall find that this is a very useful approximation in later work (see section 8.4).