A GENERALIZATION OF DIFFERENCE SETS

ROBERT J. McELIECE

1. Introduction. A (v, k, λ) *difference set* D is a set of k distinct residues $\{a_1, a_2, \ldots, a_k\}$ modulo v such that every residue $b \not\equiv 0 \pmod{v}$ can be expressed in exactly λ ways in the form $b \equiv a_i - a_j \pmod{v}$. With each difference set we may associate a binary periodic sequence $(s_1, s_2, \ldots)$ with $s_i = 1$ if $i \pmod{v}$ is in D, and $s_i = 0$ otherwise. Since this sequence is periodic of period v, we need only consider one cycle from the sequence. Such cycles we agree to call (binary) *difference cycles*. Difference cycles (equivalently, difference sets) have been studied intensively **(2, 4)**. They have important applications to digital communications, mainly because they have 2-*level autocorrelation*. In this paper we shall point out certain other (equivalent) properties of difference cycles which seem susceptible to immediate generalization, but show that these generalizations are vacuous.

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2. Motivation. If s is a difference cycle, then the defining property of difference sets tells us that the number of ordered pairs (s_i, s_{i+b}) from s (subscripts taken modulo v) of the form $(1, 1)$ is λ for all values of $b \not\equiv 0 \pmod{v}$. More generally, let the number of ordered pairs (s_i, s_{i+b}) from s of the form (ϵ_1, ϵ_2) be denoted by $p_{\epsilon_1,\epsilon_2}(b)$. Thus $p_{1,1}(b) = \lambda$ for all $b \not\equiv 0 \pmod{v}$. A simple enumeration now shows that, in addition,

$$p_{0,1}(b) = p_{1,0}(b) = k - \lambda, \qquad p_{0,0}(b) = v - 2k + \lambda$$

whenever $b \not\equiv 0 \pmod{v}$. For let us represent s and its bth translate

$$s_b = (s_b, s_{b+1}, \ldots, s_v, s_1, \ldots, s_{b-1})$$

schematically as below:

$$\begin{array}{lcccc} & \overbrace{\qquad\qquad}^{k} & & \overbrace{\qquad\qquad\qquad}^{v-k} & \\ s: & 1\,1\ldots1 & 1\,1\ldots1 & 0\,0\ldots0 & 0\,0\ldots0 \\ s_b: & \underbrace{1\,1\ldots1}_{\lambda} & \underbrace{0\,0\ldots0}_{k-\lambda} & \underbrace{1\,1\ldots1}_{k-\lambda} & \underbrace{0\,0\ldots0}_{v-2k+\lambda} \end{array}$$

Since exactly λ of the ones from s match up with ones from s_b, the remaining $k - \lambda$ ones from s must be paired with zeros from s_b. This shows that $p_{1,0}(b) = k - \lambda$. The other relations may be verified similarly.

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Now let s be any binary cycle of length v (not necessarily associated with a difference set). The (unnormalized) autocorrelation of s, $R_s(b)$, has been defined as follows **(2)**: $R_s(b) = A_s(b) - D_s(b)$ where $A_s(b)$ is the number of agreements between s and s_b, i.e. the number of components in which s and s_b have the same entry, and $D_s(b)$ is the number of disagreements. With the notation introduced above,

$$R_s(b) = p_{1,1}(b) + p_{0,0}(b) - p_{1,0}(b) - p_{0,1}(b).$$

Our remarks then show in particular that for a difference cycle s, $R_s(b)$ is independent of b if $b \not\equiv 0 \pmod{v}$, and so s has two-level autocorrelation. (Of course $R_s(0) = v$.) Conversely, it is easy to show that any cycle with two-level autocorrelation is associated with a difference set **(2)**. In what follows, we shall refer to the trivial cycles of the form $(1, 0, 0, \ldots, 0)$ or $(0, 1, 1, \ldots, 1)$ or a translate of one of these as *pulses*. They correspond to the trivial difference sets with $k = 1$ or $k = v - 1$.

3. Generalization. The preceding discussion motivates the following formal generalization of a difference cycle which depends neither on the notion of autocorrelation function nor on group theory. If $s = (s_1, s_2, \ldots, s_v)$ is any binary cycle, and if b is an integer satisfying $0 < b < v$, we define a *bigram* $M(s, b)$ as follows:

$$M(s, b) = \{(s_i, s_{i+b}): i = 1, 2, \ldots, v\},$$

multiplicity included. We have seen that if s is a binary difference cycle, then $M(s, b) = M(s, b')$ (equality means that the two collections contain the same pairs with the same multiplicity) whenever $0 < b < v$, $0 < b' < v$. More generally, we define an m-gram for any n-ary cycle $s = (s_1, s_2, \ldots, s_v)$ as follows:

Definition. Let $b = (b_1, b_2, \ldots, b_{m-1})$ be an ordered $(m-1)$-tuple of integers with $0 < b_1 < b_2 < \ldots < b_{m-1} < v$. We define the *m-gram* $M(s, b)$ to be the collection

$$\{(s_i, s_{i+b_1}, s_{i+b_2}, \ldots, s_{i+b_{m-1}}), i = 1, 2, \ldots, v\},$$

multiplicity included. If $M(s, b) = M(s, b')$ for all such b and b', we say that s is *m-tuply regular.*

Thus in the new terminology, an ordinary difference cycle becomes a doubly-regular binary cycle. We have formally generalized in two directions; we allow both the degree of regularity of s and the number of symbols in s to increase. We remark that m-tuply regular unary cycles and singly regular n-ary cycles are trivial, and that a pulse binary cycle is m-tuply regular in a trivial way. It is surprising that binary difference cycles and the above trivial examples are the only examples of m-tuply regular n-ary cycles possible. We prove this result now.

4. Non-existence.

THEOREM 1. *If s is an m-tuply regular n-ary cycle, then one of the three alternatives below holds:*

(1) $n = 2$ *and either* $m = 2$ *or* s *is a pulse,*

(2) $m = 1$,

(3) $n = 1$.

The *proof* is in two parts: (1) we assume that $m > 2$ and conclude that $n = 1$ unless s is a pulse; (2) we assume that $n > 2$ and show that $m = 1$.

Suppose, then, that s is an m-tuply regular n-ary cycle with $m > 2$. It is clearly sufficient to prove this is impossible for $n = 2$, since if $n > 2$ we may identify certain of the symbols to obtain an m-tuply regular binary cycle. We now digress in order to place the problem in a wider context.

Definition **(3)**: Let S be a set of v distinct objects. A *tactical configuration* $C = C[k, m, \lambda, v]$ is a collection of b subsets (called *blocks*) B_i, $i = 1, 2, \ldots, b$, of S such that each block contains exactly k objects from S, and each (unordered) m-tuple from S occurs in exactly λ blocks. C is *symmetric* if $b = v$, and if each object in S occurs in exactly k blocks.

Our plan is to show that the existence of an m-tuply regular binary cycle implies the existence of a symmetric $C[k, m, \lambda, v]$. Theorem 2 shows that such configurations are trivial; we prove Theorem 2 first.

THEOREM 2. *There are no non-trivial symmetric $C[k, m, \lambda, v]$ configurations, if $m \geqslant 3$.* (*Trivial* means that k does not satisfy $m \leqslant k \leqslant v - m$.)

Proof. It is clear that a $C[k, m, \lambda, v]$ configuration is also a $C[k, m', \lambda, v]$ configuration for $m' \leqslant m$, since each unordered m'-tuple from S is a subset of exactly $\binom{v - m'}{m - m'}$ unordered m-tuples from S, and so each m'-tuple occurs

$$\lambda\binom{v - m'}{m - m'} \Big/ \binom{k - m'}{m - m'}$$

times in the configuration. Consequently it will be sufficient to prove Theorem 2 for $m = 3$.

For the moment let $\lambda = \lambda_3$, and let λ_2 represent the number of times each unordered pair from S occurs in C. Then counting in two different ways the number of times a triple involving a given pair occurs in the configuration, we see that

$$\lambda_2(k - 2) = \lambda_3(v - 2). \tag{1}$$

Note that (1) holds for any $C[k, 3, \lambda, v]$ configuration, symmetric or not.

If C is symmetric, let us count in two ways the number of times a pair involving a given element occurs:

$$k(k - 1) = \lambda_2(v - 1). \tag{2}$$

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We now perform the standard trick (see **1**) of deleting from C an arbitrary block, and all objects occurring in that block. Since C is in particular a symmetric block design, the derived design C' will also be a $C\ [k', 3, \lambda', v']$ configuration (but no longer symmetric) with $k' = k - \lambda_2$, $\lambda' = \lambda'_3 = \lambda_3$, $\lambda'_2 = \lambda_2$, $v' = v - k$. Equation (1) will now apply to the derived parameters; i.e.,

$$\lambda_2(k - \lambda_2 - 2) = \lambda_3(v - k - 2). \tag{3}$$

Combining (3) with (1), we see that $\lambda_2^2/k = \lambda_3$. Thus $\lambda_2/k = (k - 2)(v - 2)$ and so, from (2), $(k - 1)/(v - 1) = (k - 2)/(v - 2)$, which implies that $k = v$. But $k = v$ is a trivial design, and this completes the proof of Theorem 2.

To complete the first part of the proof of Theorem 1, it remains to show that the existence of an m-tuply regular binary cycle ($m \geqslant 3$) which is not a pulse implies the existence of a non-trivial symmetric $C\ [k, m, \lambda, v]$ configuration. Let k be the number of *ones* in the cycle. First of all, it is clear that if $1 < k < m$, then no such cycle exists, since for certain b's, $M(s, b)$ will contain

$$\overbrace{(1\ 1\ \ldots\ 1}^{k}\ \overbrace{0\ 0\ \ldots\ 0)}^{m-k},$$

while others will not. Similarly $v - m > k > v - 1$ is impossible. Thus, except for pulses, all m-tuply regular binary sequences with k ones satisfy $m \leqslant k \leqslant v - m$.

If now s is an m-tuply regular binary cycle of length v (we assume the two symbols are 0 and 1), let $S = \{a_1, a_2, \ldots, a_v\}$ be any set containing v distinct objects. We define blocks B_i, $i = 0, 1, 2, \ldots, v - 1$, as follows: $a_j \in B_i$ if and only if $s_{i+j} = 1$. To show that these blocks form a symmetric $C\ [k, m, \lambda, v]$ configuration, we need only verify the m-tuple condition.

Thus, let

$$(a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \tag{*}$$

be an m-tuple from S, and assume that $i_1 < i_2 < \ldots < i_m$. Let $b_j = i_{j+1} - i_j$, $j = 1, 2, \ldots, m - 1$, and set $\mathbf{b} = (b_1, b_2, \ldots, b_{m-1})$. Since s is m-tuply regular, the m-tuple $(1, 1, \ldots, 1)$ will occur in $M(s, b)$ a certain number of times, say λ_m, and λ_m is independent of b. It is clear that if $(s_i, s_{i+b_1}, \ldots, s_{i+b_{m-1}})$ is such an m-tuple from $M(s, b)$, then B_{i-i_1} will contain the m-tuple (*) and conversely. Hence every m-tuple from S occurs in exactly λ_m blocks, and so the blocks B_i do form a (non-trivial) symmetric $C\ [k, m, \lambda, v]$ configuration. But this is impossible by Theorem 2, and so every m-tuply regular binary cycle is a pulse. This completes the first part of the proof of Theorem 1.

Our attention has recently been drawn to the fact that the second half of Theorem 1 was proved independently by R. Titsworth **(5)** several years ago. (The reader who consults that report will see that Titsworth's "perfect" sequences are precisely the doubly regular sequences discussed here.) We

present here a new proof which makes use of the highly developed theory of difference sets.

In order to prove the second half of Theorem 1, we shall assume that s is a doubly regular n-ary cycle, and show that $n > 2$ is impossible. It will be sufficient to prove that there are no doubly regular ternary cycles, since a doubly regular n-ary cycle ($n > 3$) can be transformed into a doubly regular ternary cycle by a simple identification of certain symbols.

If s is a doubly regular ternary cycle in the symbols 0, 1, 2, let s contain k_0 zeros, k_1 ones, and k_2 twos. We observe that each k_i must be $\geqslant 2$, since if (say) $k_0 = 1$, then some bigrams would contain (0, 1) but not (0, 2), while others would contain (0, 2) but not (0, 1). Let us now identify the symbols 0 and 1; s then becomes a doubly regular *binary* cycle (which is not a pulse by the observation above), and so the set $D_2 = \{i\colon s_i = 2\}$ is a difference set. Similarly, D_0 and D_1 are also difference sets. But also $D_{0,1} = \{i\colon s_i = 0 \text{ or } s_i = 1\}$, being the complement of D_2, is a difference set. Also, $D_{0,2}$ and $D_{1,2}$, defined similarly, are also difference sets.

We remark at this stage that $D_0 \cup D_1 = D_{0,1}$, $D_0 \cap D_1 = \emptyset$. This means that if we could prove that the union of two disjoint difference sets is never a non-trivial difference set, the second part of Theorem 1 would follow as an immediate corollary. But although there is some evidence that this is so (see the discussion at the end of this paper), we are as yet unable to prove (or disprove) it. So we must use another route.

To continue with our proof, write

$$D_0 = \{a_1, a_2, \ldots, a_{k_0}\} \quad \text{and} \quad D_1 = \{b_1, b_2, \ldots, b_{k_1}\}.$$

We define the polynomials θ_0 and θ_1 as follows (cf. **4**):

$$\theta_0(x) \equiv x^{a_1} + x^{a_2} + \ldots + x^{a_{k_0}} \pmod{x^v - 1},$$
$$\theta_1(x) \equiv x^{b_1} + x^{b_2} + \ldots + x^{b_{k_1}} \pmod{x^v - 1}.$$

Then since D_0 and D_1 are difference sets, we have, as in **(4)**,

$$\theta_0(x)\theta_0(x^{-1}) \equiv n_0 + \lambda_0 T(x) \pmod{x^v - 1},$$
$$\theta_1(x)\theta_1(x^{-1}) \equiv n_1 + \lambda_1 T(x) \pmod{x^v - 1}, \tag{4}$$

where

$$n_0 = k_0 - \lambda_0, \qquad \lambda_0 = k_0(k_0 - 1)/(v - 1),$$
$$n_1 = k_1 - \lambda_1, \qquad \lambda_1 = k_1(k_1 - 1)/(v - 1),$$

and

$$T(x) \equiv 1 + x + x^2 + \ldots + x^{v-1} \pmod{x^v - 1}.$$

Now since s is doubly regular, the pair (1, 0) occurs in each bigram $M(s, b)$ equally often, say μ times. Hence every residue $d \not\equiv 0 \pmod{v}$ can be written in exactly μ ways in the form $d \equiv a_i - b_j \pmod{v}$. In terms of the θ's, this condition becomes

$$\theta_0(x)\theta_1(x^{-1}) \equiv -\mu + \mu T(x) \pmod{x^v - 1}. \tag{5}$$

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If we multiply both sides of (5) by $\theta_1(x)$, and use (4), we obtain

$$n_1\,\theta_0(x) + \mu\theta_1(x) \equiv (\mu k_1 + \lambda_1 k_0)T(x) \pmod{x^v - 1}. \tag{6}$$

(Observe that $R(x)T(x) \equiv R(1)T(x) \pmod{x^v - 1}$; see **(4)**.) But the left-hand side of (6) cannot contain powers of x higher than $v - 1$, and so

$$n_1\,\theta_0(x) + \mu\theta_1(x) = (\mu k_1 - \lambda_1 k_0)T(x). \tag{7}$$

But this is impossible: the left-hand side of (7) cannot contain all the powers of x less than v, since some s_i are equal to 2. This contradiction completes the proof of Theorem 1.

5. Conclusion. We remark finally that there is a certain amount of arbitrariness in our definition of multiple regularity. With hindsight at least, we might regard the fact that the pairs (0, 1) and (1, 0) occur evenly distributed among the bigrams of a difference cycle as chance, peculiar to the case of binary cycles. We could then define multiple regularity by requiring only that m-tuples of the form $(aa\ldots a)$ be evenly distributed. If we had done this, the first part of the proof of Theorem 1 would still have worked, since we only needed the even distribution of $(11\ldots1)$ anyway. But a new proof of the second part of the theorem would be needed. In fact, it is easy to see that with the new definition the existence of a doubly-regular n-ary cycle is equivalent to the existence of $n - 1$ disjoint difference sets (mod v) whose union is also a non-trivial difference set. If the parameters of these difference sets are (v, k_i, λ_i), $i = 1, 2, \ldots, n - 1$, then an obvious necessary condition for their union to be a difference set is that $v - 1 | k_i(k_i - 1)$ for $i = 1, 2, \ldots, n - 1$, and $v - 1 | k_n(k_n - 1)$, where

$$k_n = v - \sum_{i=1}^{n-1} k_i.$$

More symmetrically, we require a solution to the number-theoretic problem of writing some v as $v = k_1 + \ldots + k_n$ so that $v - 1 | k_i(k_i - 1)$ for $i = 1, 2, \ldots, n$. For the case $n = 3$, computer search shows that there are exactly 51 solutions to this problem with $v \leqslant 300$; and in every case it is quite easy to show that there is at least one i for which there is no (v, k_i, λ_i) difference set, so that the problem has no solution for $n = 3$ and $v \leqslant 300$.

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