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A note on Fritz John sufficiency

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An elementary proof is given of a sufficient optimality condition recently proven by B.D. Craven. This proof avoids the use of a transposition theorem and this allows for a strengthening of Craven's result.

Recently Craven [2] has given a general sufficiency theorem for a Fritz John necessary condition [6] to imply optimality. This extended a sufficiency result for complex programmes given by Gulati [5] which was in turn stimulated by necessary conditions proved by Craven and Mond [3], [4].

It is the purpose of this note to correct an omission in the statements of the theorems in [2] and [5] and to provide a simpler proof of a more general result than in [2]. Our notation is as in [2]. Consider the non-linear programme

(P) $$\min_{x \in U} \{\operatorname{re} f(x) : -g(x) \in S,\ h(x) = 0,\ -k(x) \in N\} ,$$

where X, Y, Z, W are real or complex Banach spaces, U is open in X, $S \subset Y$, $T \subset Z$, $N \subset W$ are closed, convex cones, $f : U \to \mathrm{R}$ (or C), $g : U \to Y$, $h : U \to Z$ are Gâteaux differentiable, and $k : X \to W$ is affine and continuous. The dual cone of a convex cone S is

$$S^* = \{u \in Y' : \operatorname{re} u(s) \geq 0 \text{ for all } s \in S\} ,$$

where Y' is the topological dual of Y. Let R^+ denote the non-negative real axis, int S denote the interior of S.

The map $g : U \to Y$ is (strictly) S-convex at $a \in U$ if for each $x \in U/\{a\}$,

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$$g(x) - g(a) - g'(a)(x-a) \in S \ (\in \text{int } S) .$$

(This latter supposes $\text{int } S \neq \emptyset$.)

The map $f : X \to R$ is pseudoconvex at a if

$$x \in U \text{ and } f(x) < f(a) \text{ implies } f'(a)(x-a) < 0 .$$

We now present our result.

THEOREM. *Suppose that* $a \in U$, re f *is pseudoconvex,* g *is strictly* S*-convex, and* h *is strictly* T*-convex. Suppose there is a solution* r, v, w, m *to*

(F) *(i)* $\operatorname{re}\left(rf'(a)+vg'(a)+wh'(a)+mk'(a)\right) = 0$,

(ii) $\operatorname{re} vg(a) = 0$, $\operatorname{re} mk(a) = 0$,

with $r \in R^+$, $v \in S^*$, $w \in T^*$, $m \in N^*$, *and such that not all of* r, v, w *are zero.*

It follows that if a *is feasible for* (P) *it is optimal for* (P).

Proof. Suppose first that $r = 0$. If there is no $x \neq a$, feasible for (P), we are done since a is assumed feasible. Suppose $\bar{x} \neq a$ is feasible. Then

(1) $$g'(a)(\bar{x}-a) + g(a) \in -\text{int } S$$

and

(2) $$h'(a)(\bar{x}-a) + h(a) \in -\text{int } T .$$

Then, since one of v, w is non-zero, we have $(v \in S^*, w \in T^*)$

(3) $$\operatorname{re}\left(vg'(a)(\bar{x}-a)+vg(a)+wh'(a)(\bar{x}-a)+wh(a)\right) < 0 .$$

Since $\operatorname{re} vg(a) = 0$ by (ii) and $wh(a) = 0$ by the feasibility of a , we have

(4) $$\operatorname{re}\left(vg'(a)(\bar{x}-a)+wh'(a)(\bar{x}-a)\right) < 0 .$$

Also $\operatorname{re} mk(a) = 0$ by (ii), so

(5) $$\operatorname{re}\left(mk'(a)(\bar{x}-a)\right) = \operatorname{re}\left(mk(\bar{x})-mk(a)\right) \leq 0 ,$$

since k is affine, $\bar{x}$ is feasible, $m \in N^*$, and $k(\bar{x}) \in -N$. Adding (4) and (5) contradicts (F). Thus $r \neq 0$. We may assume that $r = 1$. The optimality of $a \in U$ now follows from the pseudoconvexity of re f and the convexity of $G(x) = \operatorname{re}\left(vg(x)+wh(x)+mk(x)\right)$ at a , since

$G(a) = 0$. □

REMARKS. (i) In both [2], [5], it is not assumed that a is feasible. This is clearly necessary as is shown by the real programme

$$\text{minimize } \left\{\frac{x^2}{2} : \frac{(x-1)^2}{2} \le 0\right\}$$

which satisfies the conditions of Theorem 1 of [2]. Now

$$\mathrm{re}\big(rf'(a)+vg'(a)\big) = 0 \ , \quad \mathrm{re}\ vg(a) = 0 \ , \quad r \in R^+ \ , \quad v \in S^* \ ,$$

is solved by $r = 1$, $v = 0$, $a = 0$, or $r = 0$, $v = 1$, $a = 1$, and the former is not feasible; hence not optimal.

(ii) The proof presented here removes Craven's condition that either $[k(a)k'(a)]$ is surjective or that $k^T(N^*)$ is weak star closed by avoiding the use of a Transposition Theorem [2].

(iii) In the same manner as in Theorem 1 we can remove the extraneous condition on k in Theorems 2 and 3 of [2]. In the latter case this is just the observation that if one of r or v is nonzero we need only assume h is T-convex.

(iv) It seems to the author that Theorem 1 is more properly a Kuhn-Tucker Sufficiency Condition [1] than a Fritz John condition since it essentially gives a constraint qualification to force r to be nonzero. It would be interesting to see a "true" Fritz John condition that gave necessary and sufficient conditions for optimality in absence of any added convexity hypotheses.

References

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