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MODERATE DEVIATIONS OF MANY-SERVER QUEUES VIA IDEMPOTENT PROCESSES

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Abstract

This paper obtains logarithmic asymptotics of moderate deviations of the stochastic process of the number of customers in a many-server queue with generally distributed inter-arrival and service times under a heavy-traffic scaling akin to the Halfin–Whitt regime. The deviation function is expressed in terms of the solution to a Fredholm equation of the second kind. A key element of the proof is the large-deviation principle in the scaling of moderate deviations for the sequential empirical process. The techniques of large-deviation convergence and idempotent processes are used extensively.

Keywords: Halfin–Whitt regime; heavy traffic; large-deviation principle; sequential empirical process

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Secondary 60K25

1. Introduction

Many-server queues are important in applications, but their analysis beyond Markovian assumptions is difficult; see, e.g., [2]. Various heavy-traffic asymptotics have been explored when the arrival and service rates tend to infinity. Of particular interest for applications is the set-up proposed in [8], where the service time distributions are held fixed, whereas the number of servers, n, and the arrival rate, λ, grow without bound in such a way that $\sqrt{n}(1-\rho)\to\beta\in\mathbb{R}$, with ρ representing the traffic intensity: $\rho=\lambda/(n\mu)$, where μ represents the reciprocal mean service time. With $Q(t)$ denoting the number of customers present at time t, assuming the initial conditions are suitably chosen, in a fairly general situation the sequence of processes $(Q(t)-n)/\sqrt{n}$, considered as random elements of the associated Skorokhod space, converges in law to a continuous-path process; see [1, 8, 11, 21, 23]. Unless the service time distribution is exponential, the limit process is a process with memory, depends in an essential way on the service time cumulative distribution function (CDF), and is not well understood.

In order to gain additional insight, the paper [20] proposed the study of moderate deviations of $Q(t)$ and conjectured a large-deviation principle (LDP) for the process $(Q(t)-n)/(b_n\sqrt{n})$ under the heavy-traffic condition $\sqrt{n}/b_n(1-\rho)\to\beta$, where $b_n\to\infty$ and $b_n/\sqrt{n}\to 0$. (It has been observed that moderate-deviation asymptotics may capture exponents in the distributions of corresponding weak convergence limits; cf. [18].) The deviation function (a.k.a. rate function) was purported to solve a convex variational problem with a quadratic objective function. In this paper we verify the conjecture and prove the LDP in question. Furthermore, we express

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the deviation function in terms of the solution to a Fredholm equation of the second kind, and we propose a framework for evaluating it numerically.

The proofs are arguably of methodological value, as they systematically use weak convergence methods and the machinery of idempotent processes. As in [17, 19, 22], the LDP is viewed as an analogue of weak convergence, the cornerstone of the approach being the following analogue of the celebrated tightness theorem of Prokhorov: a sequence of probability measures on a complete separable metric space is exponentially tight if and only if every subsequence of it admits a further subsequence that satisfies an LDP (Theorem (P) in [16]). Consequently, once exponential tightness has been proved, the proof of the LDP is accomplished by proving the uniqueness of a subsequential large-deviation limit point. In order to take full advantage of weak convergence methods, it is convenient to recast the definition of the LDP for stochastic processes as large-deviation convergence (LD convergence) to idempotent processes; see [19] and Appendix A for more detail. With tools for the study of weak convergence properties of many-server queues in heavy traffic being well developed, this paper derives the moderate-deviation asymptotics by using similar ideas. The main limit theorem asserts LD convergence of the process $(Q(t)-n)/(b_n\sqrt{n})$ to a certain idempotent process, which is analogous to the stochastic-process limit in [21]. A key element of the proof is an LD limit for the sequential empirical process (see Lemma 1), a result that complements developments in [12] and in [21] and may be of interest in its own right. It identifies the limit idempotent process through finite-dimensional distributions. Whereas in weak convergence looking at second moments usually suffices to establish tightness, establishing the stronger property of exponential tightness calls for more intricate arguments and necessitates working with exponential martingales. In addition, a study of idempotent counterparts of the standard Wiener process, the Brownian bridge, and the Kiefer process is carried out. The properties of those idempotent processes are integral to the proofs.

The paper is organised as follows. Section 2 provides a precise specification of the model as well as the main result on the logarithmic asymptotics of moderate deviations of the number-in-the-system process. An added feature is the moderate-deviation asymptotics of the number of customers in an infinite-server queue in heavy traffic, which is also stated in the form of an LDP. The proofs of the LDPs in Section 2 are presented in Section 3. The techniques of LD convergence are employed. Section 4 is concerned with evaluating the deviation functions by reduction to solving Fredholm equations of the second kind. For the reader's convenience, Appendix A gives a primer on idempotent processes and the use of weak convergence methods for proving LD convergence. Appendix B is concerned with the absolute continuity of the solution to a nonlinear renewal equation which is needed in Section 4.

2. Trajectorial moderate-deviation limit theorems

Assume as given a sequence of many-server queues with unlimited waiting room indexed by n, where n represents the number of servers. Service is performed on a first-come-first-served basis. If, upon a customer's arrival, there are available servers, then the customer starts being served by one of the available servers, chosen arbitrarily. Otherwise, the customer joins the queue and awaits her turn to be served. When the service is complete, the customer leaves, relinquishing the server.

Let $Q_n(t)$ denote the number of customers present at time t. Of those customers, $Q_n(t)\wedge n$ customers are in service and $(Q_n(t)-n)^+$ customers are in the queue. The service times of the customers in the queue at time 0 and the service times of customers exogenously arriving after time 0 are denoted by $\eta_1, \eta_2, \ldots$ (in the order in which they enter service) and come

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from a sequence of independent and identically distributed (i.i.d.) positive unbounded random variables with continuous CDF F. It is thus assumed that

$$F(0) = 0, F(x) < 1, \text{ for all } x.$$

The mean service time $\mu^{-1} = \int_0^\infty x\, dF(x)$ is assumed to be finite. The residual service times of customers in service at time 0 are denoted by $\eta_1^{(0)}, \eta_2^{(0)}, \ldots$ and are assumed to be i.i.d. with CDF F_0, which is the CDF of the delay in a stationary renewal process with inter-renewal CDF F. Thus,

$$F_0(x) = \mu \int_0^x (1 - F(y))\, dy. \tag{2.1}$$

Let $A_n(t)$ denote the number of exogenous arrivals by time t, with $A_n(0) = 0$. It is assumed that the process $A_n(t)$ has unit jumps. The entities $Q_n(0)$, $\{\eta_1^{(0)}, \eta_2^{(0)}, \ldots\}$, $\{\eta_1, \eta_2, \ldots\}$, and $A_n = (A_n(t),\ t \in \mathbb{R}_+)$ are assumed to be independent. All stochastic processes are assumed to have right-continuous paths with left-hand limits. Let $\hat{A}_n(t)$ denote the number of customers that enter service after time 0 and by time t, with $\hat{A}_n(0) = 0$. Since the random variables η_i are continuous, the process $\hat{A}_n = (\hat{A}_n(t), t \in \mathbb{R}_+)$ has unit jumps almost surely. Balancing the arrivals and departures yields the equation

$$Q_n(t) = Q_n^{(0)}(t) + (Q_n(0) - n)^+ + A_n(t) - \int_0^t \int_0^t \mathbf{1}_{\{x+s\le t\}}\, d \sum_{i=1}^{\hat{A}_n(s)} \mathbf{1}_{\{\eta_i \le x\}}, \tag{2.2}$$

where

$$Q_n^{(0)}(t) = \sum_{i=1}^{Q_n(0)\wedge n} \mathbf{1}_{\{\eta_i^{(0)} > t\}}\,, \tag{2.3}$$

which represents the number of customers present at time t out of those in service at time 0, and

$$\int_0^t \int_0^t \mathbf{1}_{\{x+s\le t\}}\, d \sum_{i=1}^{\hat{A}_n(s)} \mathbf{1}_{\{\eta_i \le x\}} = \sum_{i=1}^{\hat{A}_n(t)} \mathbf{1}_{\{\eta_i + \hat{\tau}_{n,i} \le t\}}\,,$$

which represents the number of customers that enter service after time 0 and leave by time t, with $\hat{\tau}_{n,i}$ denoting the ith jump time of $\hat{A}_n$, i.e., $\hat{\tau}_{n,i} = \inf\{t: \hat{A}_n(t) \ge i\}$. In addition, since each customer that is either in the queue at time 0 or has arrived exogenously by time t must either be in the queue at time t or have entered service by time t,

$$(Q_n(0) - n)^+ + A_n(t) = (Q_n(t) - n)^+ + \hat{A}_n(t). \tag{2.4}$$

For the existence and uniqueness of a solution to (2.2)–(2.4), the reader is referred to [21].

Given $r_n \to \infty$, as $n \to \infty$, a sequence $\mathbb{P}_n$ of probability laws on the Borel σ-algebra of a metric space M, and a $[0, \infty]$-valued function I on M such that the sets $\{y \in M: I(y) \le \gamma\}$ are compact for all $\gamma \ge 0$, the sequence $\mathbb{P}_n$ is said to obey the LDP for rate r_n with deviation function I, also referred to as a rate function, provided $\lim_{n\to\infty} 1/r_n\ \ln \mathbb{P}_n(W) = -\inf_{y\in W} I(y)$, for all Borel sets W such that the infima of I over the interior and the closure of W agree.

We now introduce the deviation function for the number-in-the-system process. For $T > 0$ and $m \in \mathbb{N}$, let $\mathbb{D}([0, T], \mathbb{R}^m)$ and $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^m)$ represent the Skorokhod spaces of right-continuous $\mathbb{R}^m$-valued functions with left-hand limits defined on $[0, T]$ and $\mathbb{R}_+$, respectively.

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These spaces are endowed with metrics rendering them complete separable metric spaces; see [6, 9] for more detail. Given $q = (q(t),\, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ and $x_0 \in \mathbb{R}$, let

$$I^Q_{x_0}(q) = \frac{1}{2} \inf \left\{ \int_0^1 \dot{w}^0(x)^2\, dx + \int_0^\infty \dot{w}(t)^2\, dt + \int_0^\infty \int_0^1 \dot{k}(x, t)^2\, dx\, dt \right\}, \tag{2.5}$$

the infimum being taken over $w^0 = (w^0(x),\, x \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$, $w = (w(t),\, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$, and $k = ((k(x, t),\, x \in [0, 1]),\, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$ such that $w^0(0) = w^0(1) = 0$, $w(0) = 0$, $k(x, 0) = k(0, t) = k(1, t) = 0$; w^0, w, and k are absolutely continuous with respect to the Lebesgue measures on $[0, 1]$, $\mathbb{R}_+$, and $[0, 1] \times \mathbb{R}_+$, respectively; and, for all t,

$$q(t) = (1 - F(t))x_0^+ - (1 - F_0(t))x_0^- - \beta F_0(t) + \int_0^t q(t-s)^+\, dF(s) + w^0(F_0(t))$$
$$+ \int_0^t \big(1 - F(t-s)\big)\sigma\, \dot{w}(s)\, ds + \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, \dot{k}(F(x), \mu s)\, dF(x)\, \mu\, ds, \tag{2.6}$$

where $x_0^- = (-x_0)^+$, and $\dot{w}^0(x)$, $\dot{w}(t)$ and $\dot{k}(x, t)$ represent the respective Radon–Nikodym derivatives. If w^0, w, and k as indicated do not exist, then $I^Q_{x_0}(q) = \infty$. Note that $I^Q_{x_0}(q) = \infty$ unless $q(0) = x_0$ and $q(t)$ is a continuous function, as the right-hand side of (2.6) is a continuous function of t. It is proved in Lemma 8 that if F is, in addition, absolutely continuous with respect to Lebesgue measure, then $q(t)$ in (2.6) is absolutely continuous too. By Lemma B.1 in [21], the equation (2.6) has a unique solution $q(t)$ in the space of essentially locally bounded functions.

Let the process $X_n = (X_n(t),\, t \in \mathbb{R}_+)$ be defined by

$$X_n(t) = \frac{\sqrt{n}}{b_n}\left(\frac{Q_n(t)}{n} - 1\right). \tag{2.7}$$

The next theorem verifies and refines Conjecture 1 in [20]. Its proof is presented in Section 3.

Theorem 1. *Suppose, in addition, that A_n is a renewal process of rate λ_n. Let $\rho_n = \lambda_n/(n\mu)$, $\beta \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $\sigma > 0$. Suppose that, as $n \to \infty$,*

$$\frac{\sqrt{n}}{b_n}(1 - \rho_n) \to \beta \tag{2.8}$$

and the sequence of random variables $\sqrt{n}/b_n\,(Q_n(0)/n - 1)$ obeys the LDP in $\mathbb{R}$ for rate b_n^2 with deviation function $I_{x_0}(y)$ such that $I_{x_0}(x_0) = 0$ and $I_{x_0}(y) = \infty$, for $y \neq x_0$. Suppose that the sequence of processes $\big((A_n(t) - \lambda_n t)/(b_n\sqrt{n}),\, t \in \mathbb{R}_+\big)$ obeys the LDP in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ for rate b_n^2 with deviation function $I^A(a)$ such that $I^A(a) = 1/(2\sigma^2) \int_0^\infty \dot{a}(t)^2\, dt$, provided $a = (a(t),\, t \in \mathbb{R}_+)$ is an absolutely continuous function with $a(0) = 0$, and $I^A(a) = \infty$, otherwise. If, in addition,

$$b_n^6 n^{1/b_n^2 - 1} \to 0, \tag{2.9}$$

then the sequence X_n obeys the LDP in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ for rate b_n^2 with deviation function $I^Q_{x_0}(q)$.

Remark 1. In order that the LDP for $\big((A_n(t) - \lambda_n t)/(b_n\sqrt{n})\, t \geq 0\big)$ in the statement hold, it suffices that $\mathbb{E}(n\xi_n) \to 1/\mu$, $\mathrm{Var}(n\xi_n) \to \sigma^2/\mu^3$, and that either $\sup_n \mathbb{E}(n\xi_n)^{2+\epsilon} < \infty$, for some

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$\epsilon > 0$, and $\sqrt{\ln n}/b_n \to \infty$, or $\sup_n \mathbb{E} \exp(\alpha (n\xi_n)^\delta) < \infty$ and $n^{\delta/2}/b_n^{2-\delta} \to \infty$, for some $\alpha > 0$ and $\delta \in (0, 1]$, where ξ_n represents a generic inter-arrival time for the nth queue; see [18].

Remark 2. The condition (2.9) implies that $b_n^6/n \to 0$, so that the condition that $b_n/\sqrt{n} \to 0$ necessarily holds. On the other hand, if $b_n^6/n^{1-\epsilon} \to 0$ for some $\epsilon > 0$, then (2.9) holds.

As suggested by a referee, the next statement provides a version for the case of infinitely many servers. Consider a $GI/GI/\infty$ queue with renewal arrival process A_n of rate $\lambda_n = n\lambda$. All the assumptions and notation concerning the service times are the same as in Theorem 1. The arrival process, the initial number of customers, and the service times are independent. With $\overline{Q}_n(t)$ denoting the number of customers present at time t, the equations (2.2) and (2.3) are replaced with the respective equations

$$\overline{Q}_n(t) = \overline{Q}_n^{(0)}(t) + A_n(t) - \int_0^t \int_0^t \mathbf{1}_{\{x+s \le t\}}\, d \sum_{i=1}^{A_n(s)} \mathbf{1}_{\{\eta_i \le x\}} \tag{2.10}$$

and

$$\overline{Q}_n^{(0)}(t) = \sum_{i=1}^{\overline{Q}_n(0)} \mathbf{1}_{\{\eta_i^{(0)} > t\}}. \tag{2.11}$$

Given $q_0 \in \mathbb{R}_+$, let

$$\overline{q}(t) = q_0(1 - F_0(t)) + \lambda t - \lambda \int_0^t (t-s)\, dF(s) \tag{2.12}$$

and

$$\overline{X}_n(t) = \frac{\sqrt{n}}{b_n}\left(\frac{\overline{Q}_n(t)}{n} - \overline{q}(t)\right). \tag{2.13}$$

Theorem 2. *Suppose that the sequence of processes $\big((A_n(t) - \lambda_n t)/(b_n\sqrt{n}),\, t \in \mathbb{R}_+\big)$ obeys the LDP in the hypotheses of Theorem 1. Given $x_0 \in \mathbb{R}$, suppose that the sequence $\overline{X}_n(0)$ obeys the LDP with deviation function $\overline{I}_{x_0}(y)$ such that $\overline{I}_{x_0}(x_0) = 0$ and $\overline{I}_{x_0}(y) = \infty$, for $y \neq x_0$. If, in addition, (2.9) holds, then the sequence $\overline{X}_n$ obeys the LDP in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ for rate b_n^2 with deviation function $\overline{I}^Q_{q_0,x_0}(q)$ given by the right-hand side of (2.5), provided*

$$q(t) = (1 - F_0(t))x_0 + \sqrt{q_0}\, w^0(F_0(t)) + \int_0^t \big(1 - F(t-s)\big)\sigma\, \dot{w}(s)\, ds$$
$$+ \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s \le t\}}\; \dot{k}(F(x), \lambda s)\, dF(x)\, \lambda\, ds,$$

and equals ∞ otherwise.

Remark 3. The parameter q_0 arises as a law-of-large-numbers limit for the scaled initial number of customers. The corresponding parameter for the many-server queue in Theorem 1 equals 1.

3. Large-deviation convergence and proofs of Theorems 1 and 2

It is convenient to recast Theorem 1 as a statement on LD convergence. Introduce

$$Y_n(t) = \frac{\sqrt{n}}{b_n}\left(\frac{A_n(t)}{n} - \mu t\right) \tag{3.1}$$

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and let $Y_n = (Y_n(t),\ t \in \mathbb{R}_+)$. For the statement and proof of the next theorem, Appendix A is recommended reading.

Theorem 3. *Suppose that, as $n \to \infty$, the sequence $X_n(0)$ LD converges in distribution in $\mathbb{R}$ at rate b_n^2 to an idempotent variable X(0), the sequence Y_n LD converges in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ at rate b_n^2 to an idempotent process Y with continuous paths, and (2.9) holds. Then the sequence X_n LD converges in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ at rate b_n^2 to the idempotent process $X = (X(t), t \in \mathbb{R}_+)$ that is the unique solution to the equation*

$$X(t) = (1 - F(t))X(0)^+ - (1 - F_0(t))X(0)^- + \int_0^t X(t-s)^+\, dF(s) + W^0(F_0(t)) + Y(t) - \int_0^t Y(t-s)\, dF(s) + \int_0^t \int_0^t \mathbf{1}_{\{x+s \le t\}}\ \dot{K}(F(x), \mu s)\, dF(x)\, \mu\, ds, \qquad (3.2)$$

where $W^0 = (W^0(x), x \in [0, 1])$ is a Brownian bridge idempotent process and $K = (K(x, t), (x, t) \in [0, 1] \times \mathbb{R}_+)$ is a Kiefer idempotent process, X(0), Y, W^0, and K being independent.

Theorem 1 is obtained as a special case. Suppose A_n is a renewal process of rate λ_n, the condition (2.8) holds, and the sequence $((A_n(t) - \lambda_n t)/(b_n\sqrt{n})\,, t \in \mathbb{R}_+)$ LD converges in distribution in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ at rate b_n^2 to σW, where $\sigma > 0$ and $W = (W(t), t \in \mathbb{R}_+)$ is a standard Wiener idempotent process. Then, in the statement of Theorem 3, $Y(t) = \sigma W(t) - \beta\mu t$, so that the limit idempotent process X solves the equation

$$X(t) = (1 - F(t))X(0)^+ - (1 - F_0(t))X(0)^- - \beta F_0(t) + \int_0^t X(t-s)^+\, dF(s) + W^0(F_0(t)) + \int_0^t \big(1 - F(t-s)\big)\sigma\ \dot{W}(s)\, ds + \int_0^t \int_0^t \mathbf{1}_{\{x+s \le t\}}\ \dot{K}(F(x), \mu s)\, dF(x)\, \mu\, ds,$$

with $X(0)$, W, W^0, and K being independent. The assertion of Theorem 1 follows on observing that $\exp(-I^Q(y))$, with $y \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$, is the deviability density of the idempotent distribution of X. To see the latter, note that the mapping $(w^0, w, k) \to q$, as specified by (2.6), is continuous when restricted to the set $\{(w^0, w, k) : \Pi^{W^0,W,K}(w^0, w, k) \ge a\}$, where $\Pi^{W^0,W,K}(w^0, w, k) = \Pi^{W^0}(w^0)\Pi^W(w)\Pi^K(k)$ and $a \in (0, 1]$, so that X is strictly Luzin on $\big(\mathbb{D}([0, 1], \mathbb{R}) \times \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{D}(\mathbb{R}_+, \mathbb{D}(\mathbb{R}_+, \mathbb{R})), \Pi^{W^0,W,K}\big)$; see Appendix A for the definition and properties of being strictly Luzin. Therefore,

$$\Pi^X(q) = \Pi^{W^0,W,K}(X = q) = \sup_{(w^0,w,k):\,(3.2)\text{ holds}} \Pi^{W^0}(w^0)\Pi^W(w)\Pi^K(k).$$

It is noteworthy that the limit idempotent process in (3.2) is analogous to the limit stochastic process on p. 139 in [21].

The proof of Theorem 3 relies on an analogue of the weak convergence of the sequential empirical process to the Kiefer process; see, e.g., [12]. Let random variables ζ_i be independent and uniform on [0, 1]. Define the centred and normalised sequential empirical process by

$$K_n(x, t) = \frac{1}{b_n\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \big(\mathbf{1}_{\{\zeta_i \le x\}} - x\big), \qquad (3.3)$$

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and let

$$B_n(x,t)=\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(\mathbf{1}_{\{\zeta_i\le x\}}-\int_0^{x\wedge\zeta_i}\frac{dy}{1-y}\right), \tag{3.4}$$

where $x\in[0,1]$ and $t\in\mathbb{R}_+$. It is a simple matter to check that

$$K_n(x,t)=-\int_0^x\frac{K_n(y,t)}{1-y}\,dy+B_n(x,t). \tag{3.5}$$

Let $K_n=\big((K_n(x,t),\ x\in[0,1]),t\in\mathbb{R}_+\big)$ and $B_n=\big((B_n(x,t),\ x\in[0,1]),t\in\mathbb{R}_+\big)$. Both processes are considered as random elements of $\mathbb{D}(\mathbb{R}_+,\mathbb{D}([0,1],\mathbb{R}))$. Let $B=((B(x,t),x\in[0,1]),t\in\mathbb{R}_+)$ represent a Brownian sheet idempotent process, which is the canonical coordinate process on $\mathbb{D}(\mathbb{R}_+,\mathbb{D}([0,1],\mathbb{R}))$, endowed with deviability Π. Let $K=((K(x,t),x\in[0,1]),t\in\mathbb{R}_+)$ be defined as the solution of the equation

$$K(x,t)=-\int_0^x\frac{K(y,t)}{1-y}\,dy+B(x,t). \tag{3.6}$$

It is a Kiefer idempotent process by Lemma 7.

Lemma 1. *Under (2.9), the sequence of stochastic processes* (K_n,B_n) *LD converges at rate* b_n^2 *in* $\mathbb{D}(\mathbb{R}_+,\mathbb{D}([0,1],\mathbb{R})^2)$ *to the idempotent process (K,B).*

Proof. The proof draws on the proof of Lemma 3.1 in [12]; see also [9, Chapter IX, §4c]. Given $t\in\mathbb{R}_+$, let $K_n(t)=(K_n(x,t),\ x\in[0,1])\in\mathbb{D}([0,1],\mathbb{R})$, $B_n(t)=(B_n(x,t),\ x\in[0,1])\in\mathbb{D}([0,1],\mathbb{R})$, $K(t)=(K(x,t),\ x\in[0,1])\in\mathbb{D}([0,1],\mathbb{R})$, and $B(t)=(B(x,t),\ x\in[0,1])\in\mathbb{D}([0,1],\mathbb{R})$. We prove first that, for $0\le t_1<t_2<\ldots<t_k$, the sequence of $\mathbb{D}([0,1],\mathbb{R})^{2k}$-valued stochastic processes $\big((K_n(t_1),B_n(t_1)),((K_n(t_2),B_n(t_2)),\ldots,(K_n(t_k),B_n(t_k))\big)$ LD converges to the $\mathbb{D}([0,1],\mathbb{R})^{2k}$-valued idempotent process $\big((K(t_1),B(t_1)),(K(t_2),B(t_2)),\ldots,(K(t_k),B(t_k))\big)$ in $\mathbb{D}([0,1],\mathbb{R})^{2k}$, as $n\to\infty$. Since both the stochastic processes $((K_n(t),B_n(t)),t\in\mathbb{R}_+)$ and the idempotent process $((K(t),B(t)),t\in\mathbb{R}_+)$ have independent increments in t (see Lemma 7), it suffices to prove convergence of one-dimensional distributions, so we work with $((K_n(x,t),B_n(x,t)),x\in[0,1])$ and $((K(x,t),B(x,t)),x\in[0,1])$, holding t fixed. By (3.4) and [9, Chapter II, §3c], the stochastic process $(B_n(x,t),x\in[0,1])$ is a martingale with respect to the natural filtration with the measure of jumps

$$\mu^{n,B}([0,x],\Gamma)=\mathbf{1}_{\{1/(b_n\sqrt{n})\in\Gamma\}}\sum_{i=1}^{\lfloor nt\rfloor}\mathbf{1}_{\{\zeta_i\le x\}}\,,$$

the predictable measure of jumps

$$\nu^{n,B}([0,x],\Gamma)=\mathbf{1}_{\{1/(b_n\sqrt{n})\in\Gamma\}}\sum_{i=1}^{\lfloor nt\rfloor}\int_0^{x\wedge\zeta_i}\frac{dy}{1-y},$$

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and the predictable quadratic variation process

$$\langle B_n\rangle(x,t) = \int_0^x \int_{\mathbb{R}} u^2 \nu^{n,B}(dy,du) = \frac{1}{b_n^2 n}\,\nu^{n,B}([0,x],\{1/(b_n\sqrt{n})\})$$
$$= \frac{1}{b_n^2 n}\sum_{i=1}^{\lfloor nt\rfloor}\int_0^{x\wedge\zeta_i}\frac{dy}{1-y} = \frac{\lfloor nt\rfloor}{b_n^2 n}\,x + \frac{1}{b_n\sqrt{n}}\,K_n(x,t) - \frac{1}{b_n\sqrt{n}}\,B_n(x,t), \quad (3.7)$$

where $\Gamma \subset \mathbb{R}\setminus\{0\}$.

We show next that

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{P}(\sup_{x\in[0,1]}|B_n(x,t)| > r)^{1/b_n^2} = 0. \quad (3.8)$$

Since the process

$$\exp\Big(b_n^2 B_n(x,t) - \int_0^x\int_{\mathbb{R}}(e^{b_n^2 u} - 1 - b_n^2 u)\nu^{n,B}(dy,du)\Big)$$
$$= \exp\left(b_n^2 B_n(x,t) - (e^{b_n/\sqrt{n}} - 1 - \frac{b_n}{\sqrt{n}})\sum_{i=1}^{\lfloor nt\rfloor}\int_0^{x\wedge\zeta_i}\frac{dy}{1-y}\right)$$

is a local martingale with respect to x (see, e.g., [19, Lemma 4.1.1, p. 294]), for any stopping time τ,

$$\mathbb{E}\exp\left(b_n^2 B_n(\tau,t) - (e^{b_n/\sqrt{n}} - 1 - \frac{b_n}{\sqrt{n}})\sum_{i=1}^{\lfloor nt\rfloor}\int_0^{\tau\wedge\zeta_i}\frac{dy}{1-y}\right) \le 1.$$

Lemma 3.2.6 on p. 282 in [19] implies that, for $r > 0$ and $\gamma > 0$,

$$\mathbb{P}(\sup_{x\in[0,1]} e^{b_n^2 B_n(x,t)} \ge e^{b_n^2 r}) \le e^{b_n^2(\gamma - r)} + \mathbb{P}\left(\exp\left(\left(e^{b_n/\sqrt{n}} - 1 - \frac{b_n}{\sqrt{n}}\right)\sum_{i=1}^{\lfloor nt\rfloor}\int_0^{\zeta_i}\frac{dy}{1-y}\right) \ge e^{b_n^2\gamma}\right)$$
$$\le e^{b_n^2(\gamma - r)} + e^{-b_n^2\gamma}\,\mathbb{E}\left(\exp\left(\left(e^{b_n/\sqrt{n}} - 1 - \frac{b_n}{\sqrt{n}}\right)\sum_{i=1}^{\lfloor nt\rfloor}\int_0^{\zeta_i}\frac{dy}{1-y}\right)\right)$$
$$= e^{b_n^2(\gamma - r)} + e^{-b_n^2\gamma}\big(1 - (e^{b_n/\sqrt{n}} - 1 - b_n/\sqrt{n})\big)^{-\lfloor nt\rfloor},$$

with the latter equality holding for all n large enough because $e^{b_n/\sqrt{n}} - 1 - b_n/\sqrt{n} \to 0$. Hence, assuming that $e^{b_n/\sqrt{n}} - 1 - b_n/\sqrt{n} \le 1/2$, we have

$$\mathbb{P}(\sup_{x\in[0,1]} e^{b_n^2 B_n(x,t)} \ge e^{b_n^2 r})^{1/b_n^2} \le e^{\gamma - r} + e^{-\gamma}2^{\lfloor nt\rfloor/b_n^2}.$$

Since $n/b_n^2 \to \infty$, it follows that

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{P}(\sup_{x\in[0,1]} B_n(x,t) \ge r)^{1/b_n^2} = 0. \quad (3.9)$$

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A similar convergence holds with $-B_n(x, t)$ substituted for $B_n(x, t)$. The limit (3.8) has been proved.

We next prove that, similarly,

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}(\sup_{x\in[0,1]}|K_n(x, t)| > r)^{1/b_n^2} = 0. \tag{3.10}$$

Since, by (3.3), $(K_n(x, t), x \in [0, 1])$ is distributed as $(- K_n(1 - x, t), x \in [0, 1])$, it suffices to prove that

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}(\sup_{x\in[0,1/2]}|K_n(x, t)| > r)^{1/b_n^2} = 0. \tag{3.11}$$

By (3.5) and the Gronwall–Bellman inequality,

$$\sup_{x\in[0,1/2]}|K_n(x, t)| \le e^2 \sup_{x\in[0,1/2]}|B_n(x, t)|, \tag{3.12}$$

so that (3.11) follows from (3.9).

By (3.7), (3.8), and (3.10), for $x \in [0, 1]$,

$$\lim_{n\to\infty} \mathbb{P}(|b_n^2\langle B_n\rangle(x, t) - tx| > \epsilon)^{1/b_n^2} = 0. \tag{3.13}$$

If we extend it past $x = 1$ by letting $B_n(x, t) = B_n(1, t)$, the process $(B_n(x, t), x \in \mathbb{R}_+)$ is a square-integrable martingale with predictable quadratic variation process $(\langle B_n\rangle(x \wedge 1, t), x \in \mathbb{R}_+)$, so, by (3.13) and Theorem 5.4.4 on p. 423 in [19], where one takes $\beta_\phi = b_n\sqrt{n}$, $\alpha_\phi = n$, and $r_\phi = b_n^2$, the sequence of the extended processes $(B_n(x, t), x \in \mathbb{R}_+)$ LD converges in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ to the idempotent process $(B(x \wedge 1, t), x \in \mathbb{R}_+)$. By (3.5), (3.6), and the continuous mapping principle, for $0 \le x_1 \le \ldots \le x_l < 1$, the $((K_n(x_i, t), B_n(x_i, t)), i \in \{1, 2, \ldots, l\})$ LD converge in $\mathbb{R}^{2l}$ to $((K(x_i, t), B(x_i, t)), i \in \{1, 2, \ldots, l\})$. Since $K_n(1, t) = 0$ and $K(1, t) = 0$ (see Appendix A), the latter convergence also holds if $x_l = 1$.

We now show that the sequence $(K_n(x, t), x \in [0, 1])$ is $\mathbb{C}$-exponentially tight of order b_n^2 in $\mathbb{D}([0, 1], \mathbb{R})$. (The definition and basic properties of $\mathbb{C}$-exponential tightness are reviewed in Appendix A.) By Theorem 8, (3.10) needs to be complemented with

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{x\in[0,1]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x + y, t) - K_n(x, t)| \ge \eta)^{1/b_n^2} = 0, \tag{3.14}$$

for arbitrary $\eta > 0$, where $K_n(x, t) = 0$ when $x \ge 1$. We use an argument similar to that used in the proof of (3.10). Defining $\overline{K}_n(x, t) = -K_n(1 - x, t)$ for $x \in [0, 1]$ and $\overline{K}_n(x, t) = 0$ for $x \ge 1$, we have by (3.3) that

$$\begin{aligned}
&\sup_{x\in[0,1]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x + y, t) - K_n(x, t)| \ge \eta)\\
&\le \sup_{x\in[0,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x + y, t) - K_n(x, t)| \ge \eta)\\
&+ \sup_{x\in[1/2,1-\delta]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x + y, t) - K_n(x, t)| \ge \eta)\\
&+ \sup_{x\in[1-\delta,1]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x + y, t) - K_n(x, t)| \ge \eta)\\
&\le \sup_{x\in[0,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x + y, t) - K_n(x, t)| \ge \eta)\\
&+ \sup_{x\in[\delta,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(1 - x + y, t) - K_n(1 - x, t)| \ge \eta)
\end{aligned}$$

$$+\mathbb{P}(\sup_{u\in[1-\delta,1]}|K_n(u,t)|\geq\eta/2)\leq\sup_{x\in[0,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\geq\eta)$$
$$+\sup_{x\in[\delta,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|\overline{K}_n(x-y,t)-\overline{K}_n(x,t)|\geq\eta)$$
$$+\mathbb{P}(\sup_{u\in[0,\delta]}|\overline{K}_n(u,t)|\geq\eta/2)$$
$$\leq\sup_{x\in[0,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\geq\eta)$$
$$+\sup_{x\in[0,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|\overline{K}_n(x+y,t)-\overline{K}_n(x,t)|\geq\eta)$$
$$+\mathbb{P}(\sup_{u\in[0,\delta]}|\overline{K}_n(u,t)|\geq\eta/2). \tag{3.15}$$

Since the random variables ζ_i are independent and uniformly distributed on [0,1], $\overline{K}_n$ has the same finite-dimensional distributions as K_n, so that

$$\sup_{x\in[0,1]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\geq\eta)$$
$$\leq 2\sup_{x\in[0,1/2]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\geq\eta)+\mathbb{P}(\sup_{u\in[0,\delta]}|K_n(u,t)|\geq\eta/2). \tag{3.16}$$

Since $x+y\leq 2/3$ when $x\in[0,1/2]$ and $y\in[0,\delta]$ provided δ is small enough, by (3.5), for $x\in[0,1/2]$ and δ small enough,

$$\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\leq$$
$$3\delta\sup_{u\in[0,1]}|K_n(u,t)|+\sup_{y\in[0,\delta]}|B_n(x+y,t)-B_n(x,t)|. \tag{3.17}$$

Similarly,

$$\sup_{u\in[0,\delta]}|K_n(u,t)|\leq\frac{\delta}{1-\delta}\sup_{u\in[0,1]}|K_n(u,t)|+\sup_{u\in[0,\delta]}|B_n(u,t)|. \tag{3.18}$$

By (3.16), (3.17), (3.18), and the fact that $B_n(t)$ LD converges to $B(t)$,

$$\limsup_{n\to\infty}\sup_{x\in[0,1]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\geq\eta)^{1/b_n^2}$$
$$\leq\limsup_{n\to\infty}\mathbb{P}(3\delta\sup_{u\in[0,1]}|K_n(u,t)|\geq\eta/2)^{1/b_n^2}$$
$$+\limsup_{n\to\infty}\mathbb{P}(\frac{\delta}{1-\delta}\sup_{u\in[0,1]}|K_n(u,t)|\geq\eta/4)^{1/b_n^2}$$
$$+\Pi(\sup_{x\in[0,1/2]}\sup_{y\in[0,\delta]}|B(x+y,t)-B(x,t)|\geq\eta/2)$$
$$+\Pi(\sup_{u\in[0,\delta]}|B(u,t)|\geq\eta/4). \tag{3.19}$$

By (3.10),

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{x\in[0,1]}\mathbb{P}(\sup_{y\in[0,\delta]}|K_n(x+y,t)-K_n(x,t)|\geq\eta)^{1/b_n^2}$$
$$\leq\lim_{\delta\to 0}\Pi(\sup_{x\in[0,1/2]}\sup_{y\in[0,\delta]}|B(x+y,t)-B(x,t)|\geq\eta/2)$$
$$+\lim_{\delta\to 0}\Pi(\sup_{u\in[0,\delta]}|B(u,t)|\geq\eta/4). \tag{3.20}$$

The idempotent process $B = ((B(x, t), x \in [0, 1]), t \in \mathbb{R}_+)$ has trajectories from the space of continuous functions $\mathbb{C}(\mathbb{R}_+, \mathbb{C}([0, 1], \mathbb{R}))$; see Appendix A. Since the collections of sets $\{b \in \mathbb{C}(\mathbb{R}_+, \mathbb{C}([0, 1], \mathbb{R})) \colon \sup_{x\in[0,1/2]}\sup_{y\in[0,\delta]}|b(x + y, t) - b(x, t)| \geq \eta/2\}$ and $\{b \in \mathbb{C}(\mathbb{R}_+, \mathbb{C}([0, 1], \mathbb{R})) \colon \sup_{x\in[0,\delta]}|b(x, t)| \geq \eta/4\}$ are nested collections of closed sets as $\delta \downarrow 0$, the limit on the right of (3.20) is (see Appendix A)

$$\Pi(\sup_{x\in[0,1/2]}\sup_{y\in[0,0]}|B(x + y, t) - B(x, t)| \geq \eta/2) + \Pi(\sup_{u\in[0,0]}|B(u, t)| \geq \eta/4) = 0,$$

which concludes the proof of (3.14).

Since the sequence of stochastic processes $((K_n(x, t), B_n(x, t)), x \in [0, 1])$ LD converges to the idempotent process $((K(x, t), B(x, t)), x \in [0, 1])$ in the sense of finite-dimensional distributions and is $\mathbb{C}$-exponentially tight, the LD convergence holds in $\mathbb{D}([0, 1], \mathbb{R}^2)$; see Theorem 7 in Appendix A. It has thus been proved that the sequence of stochastic processes $\big(((K_n(x, t_1), B_n(x, t_1)),\ x \in [0, 1]), \ldots, ((K_n(x, t_l), B_n(x, t_l)),\ x \in [0, 1])\big)$ LD converges in $\mathbb{D}([0, 1], \mathbb{R}^2)^l$ to the idempotent process $\big(((K(x, t_1), B(x, t_1)),\ x \in [0, 1]), \ldots, ((K(x, t_l), B(x, t_l)),\ x \in [0, 1])\big)$, for all $t_1 \leq t_2 \leq \ldots \leq t_l$. The proof of the lemma will be complete if the sequence $\big(((K_n(x, t), B_n(x, t)), x \in [0, 1]), t \in \mathbb{R}_+\big)$ is shown to be $\mathbb{C}$-exponentially tight of order b_n^2 in $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}^2))$. The definition of exponential tightness implies that it is sufficient to prove that each of the sequences $\{K_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ is $\mathbb{C}$-exponentially tight of order b_n^2 in $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}))$. By (3.10) and Theorem 8, the $\mathbb{C}$-exponential tightness of $\{K_n, n \geq 1\}$ would follow if, for all $L > 0$ and $\eta > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \sup_{s\in[0,L]}\mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1]}|K_n(x, s + t) - K_n(x, s)| \geq \eta)^{1/b_n^2} = 0. \qquad (3.21)$$

Since, in analogy with the reasoning in (3.15),

$$\begin{aligned}
&\mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1]}|K_n(x, t + s) - K_n(x, s)| \geq \eta)\\
&\qquad \leq \mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1/2]}|K_n(x, t + s) - K_n(x, s)| \geq \eta)\\
&\qquad + \mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[1/2,1]}|K_n(x, t + s) - K_n(x, s)| \geq \eta)\\
&\qquad = \mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1/2]}|K_n(x, t + s) - K_n(x, s)| \geq \eta)\\
&\qquad + \mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1/2]}|\overline{K}_n(x, t + s) - \overline{K}_n(x, s)| \geq \eta)\\
&\qquad\qquad = 2\mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1/2]}|K_n(x, t + s) - K_n(x, s)| \geq \eta),
\end{aligned}$$

(3.21) is implied by

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \sup_{s\in[0,L]}\mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1/2]}|K_n(x, t + s) - K_n(x, s)| \geq \eta)^{1/b_n^2} = 0. \qquad (3.22)$$

By (3.3), with x being held fixed, the process $(K_n(x, t + s) - K_n(x, s), t \in \mathbb{R}_+)$ is a locally square-integrable martingale, so $(\sup_{x\in[0,1/2]}(K_n(x, t + s) - K_n(x, s)), t \in \mathbb{R}_+)$, is a submartingale; hence by Doob's inequality [14, Theorem 3.2, p. 60],

$$\begin{aligned}
&\mathbb{P}(\sup_{t\in[0,\delta]}\sup_{x\in[0,1/2]}|K_n(x, t + s) - K_n(x, s)| \geq \eta)\\
&\qquad \leq \frac{1}{\eta^{2b_n^2}}\, \mathbb{E}\sup_{x\in[0,1/2]}(K_n(x, s + \delta) - K_n(x, s))^{2b_n^2}.
\end{aligned} \qquad (3.23)$$

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As noted earlier, with t being held fixed, the process $(B_n(x, t), x \in [0, 1])$ is a square-integrable martingale [9, Chapter II, §3c]. Equation (3.5) yields, by the Gronwall–Bellman inequality, as in (3.12),

$$\sup_{x\in[0,1/2]}|K_n(x, s+\delta) - K_n(x, s)| \leq e^2 \sup_{x\in[0,1]}|B_n(x, s+\delta) - B_n(x, s)|. \tag{3.24}$$

Since $(B_n(x, s+\delta) - B_n(x, s), x \in [0, 1])$ is a square-integrable martingale, by another application of Doob's inequality (see [9, Theorem I.1.43] and [13, Theorem I.9.2]), as well as by Jensen's inequality,

$$\mathbb{E}\sup_{x\in[0,1]}(B_n(x, s+\delta) - B_n(x, s))^{2b_n^2} \leq \Big(\frac{2b_n^2}{2b_n^2 - 1}\Big)^{2b_n^2} \mathbb{E}(B_n(1, s+\delta) - B_n(1, s))^{2b_n^2}.$$

By (3.4), the fact that $1 - \zeta_1$ and ζ_1 have the same distribution, and the bound (5.6) in the proof of Theorem 19 in [15, Chapter III, §5],

$$\begin{aligned}\mathbb{E}(B_n(1, s+\delta) - B_n(1, s))^{2b_n^2} \leq (b_n\sqrt{n})^{-2b_n^2}\big((b_n^2+1)^{2b_n^2}(n\delta+1)\mathbb{E}(1+\ln\zeta_1)^{2b_n^2} \\ + 2b_n^2(b_n^2+1)^{b_n^2}e^{b_n^2+1}(n\delta+1)^{b_n^2}\big(\mathbb{E}(1+\ln\zeta_1)^2\big)^{b_n^2}\big).\end{aligned}$$

(More specifically, the following bound is used. Suppose $X_1, \ldots, X_n$ are i.i.d. with $\mathbb{E}X_1 = 0$. Then, provided $p \geq 2$ and $r > p/2$,

$$\mathbb{E}|\sum_{i=1}^n X_i|^p \leq r^p n\mathbb{E}|X_1|^p + pr^{p/2}e^r n^{p/2}(\mathbb{E}X_1^2)^{p/2}.$$

See also [25] for similar results.)

As ζ_1 is uniform on [0,1], $\mathbb{E}(\ln\zeta_1)^{2b_n^2} = (2b_n^2)!$, so that, with the use of Jensen's inequality,

$$\begin{aligned}&\limsup_{n\to\infty}\,(\mathbb{E}(B_n(1, s+\delta) - B_n(1, s))^{2b_n^2})^{1/b_n^2} \\ &\quad\leq \limsup_{n\to\infty}\,(b_n\sqrt{n})^{-2}\Big(4n^{1/b_n^2}(b_n^2+1)^2\big((2b_n^2)!\big)^{1/b_n^2} + (b_n^2+1)e(n\delta+1)\mathbb{E}(1+\ln\zeta_1)^2\Big),\end{aligned}$$

which implies, via Stirling's formula, on recalling that $b_n^6 n^{1/b_n^2 - 1} \to 0$, that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{s\in[0,L]}\Big(\mathbb{E}(B_n(1, s+\delta) - B_n(1, s))^{2b_n^2}\Big)^{1/b_n^2} = 0.$$

Recalling (3.23), (3.24), and (3.25) yields (3.22). The proof of the $\mathbb{C}$-exponential tightness of B_n is similar. (It is actually simpler.)

Going back to the set-up of Theorem 3, let

$$H_n(t) = Y_n(t) - \int_0^t Y_n(t-s)\,dF(s), \tag{3.25}$$

$$X_n^{(0)}(t) = \frac{\sqrt{n}}{b_n}\,\big(\frac{1}{n}\,Q_n^{(0)}(t) - (1 - F_0(t))\big), \tag{3.26}$$

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$$U_n(x,t)=\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\hat{A}_n(t)}\big(\mathbf{1}_{\{\eta_i\le x\}}-F(x)\big), \tag{3.27}$$

and

$$\Theta_n(t)=-\int_{\mathbb{R}_+^2}\mathbf{1}_{\{x+s\le t\}}\,dU_n(x,s). \tag{3.28}$$

As, owing to (2.1),

$$1-F_0(t)+\mu t-\mu\int_0^t(t-s)dF(s)=1, \tag{3.29}$$

(2.2), (2.4), and (2.7) imply that

$$X_n(t)=(1-F(t))X_n(0)^+ + X_n^{(0)}(t)+\int_0^t X_n(t-s)^+\,dF(s)+H_n(t)+\Theta_n(t). \tag{3.30}$$

The equation (3.2) is written in a similar way: introducing

$$H(t)=Y(t)-\int_0^t Y(t-s)\,dF(s), \tag{3.31}$$

$$X^{(0)}(t)=W^0(F_0(t))-(1-F_0(t))X(0)^-, \tag{3.32}$$

$$U(x,t)=K(F(x),\mu t), \tag{3.33}$$

and

$$\Theta(t)=-\int_{\mathbb{R}_+^2}\mathbf{1}_{\{x+s\le t\}}\,dU(x,s)=-\int_{\mathbb{R}_+^2}\mathbf{1}_{\{x+s\le t\}}\,\dot{K}(F(x),\mu s)\,dF(x)\,\mu ds \tag{3.34}$$

yields

$$X(t)=(1-F(t))X(0)^+ + X^{(0)}(t)+\int_0^t X(t-s)^+\,dF(s)+H(t)+\Theta(t). \tag{3.35}$$

Let $H=(H(t),\ t\in\mathbb{R}_+)$, $H_n=(H_n(t),\ t\in\mathbb{R}_+)$, $X^{(0)}=(X^{(0)}(t),\ t\in\mathbb{R}_+)$, $X_n^{(0)}=(X_n^{(0)}(t),\ t\in\mathbb{R}_+)$, $U=((U(x,t),x\in\mathbb{R}_+),t\in\mathbb{R}_+)$, $U_n=((U_n(x,t),x\in\mathbb{R}_+),t\in\mathbb{R}_+)$, $\Theta=(\Theta(t),t\in\mathbb{R}_+)$, and $\Theta_n=(\Theta_n(t),\ t\in\mathbb{R}_+)$.

Theorem 4. *As $n\to\infty$, the sequence $(X_n(0),X_n^{(0)},H_n,\Theta_n)$ LD converges in distribution at rate b_n^2 in $\mathbb{R}\times\mathbb{D}(\mathbb{R}_+,\mathbb{R})^3$ to $(X(0),X^{(0)},H,\Theta)$.*

The groundwork needs to be laid first. Let

$$L_n(x,t)=\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\hat{A}_n(t)}\left(\mathbf{1}_{\{\eta_i\le x\}}-\int_0^{\eta_i\wedge x}\frac{dF(u)}{1-F(u)}\right). \tag{3.36}$$

Since the random variables $F(\eta_i)$ are i.i.d. and uniform on [0, 1], in view of (3.3), (3.4), and (3.27), it may be assumed that

$$L_n(x,t)=B_n(F(x),\frac{\hat{A}_n(t)}{n}) \tag{3.37}$$

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and that

$$U_n(x,t) = K_n\left(F(x), \frac{\hat{A}_n(t)}{n}\right). \tag{3.38}$$

By (3.5),

$$U_n(x,t) = -\int_0^x \frac{U_n(y,t)}{1-F(y)}\,dF(y) + L_n(x,t). \tag{3.39}$$

By (3.28),

$$\Theta_n(t) = J_n(t) - M_n(t), \tag{3.40}$$

with

$$J_n(t) = \int_0^t \frac{U_n(x,t-x)}{1-F(x)}\,dF(x) \tag{3.41}$$

and

$$M_n(t) = \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, dL_n(x,s). \tag{3.42}$$

Lemma 2. *Under the hypotheses,*

$$\lim_{r\to\infty}\limsup_{n\to\infty} \mathbb{P}(\sup_{s\in[0,t]}|U_n(s,t-s)|>r)^{1/b_n^2} = 0 \tag{3.43}$$

and

$$\lim_{r\to\infty}\limsup_{n\to\infty} \mathbb{P}(\sup_{s\in[0,t]}|M_n(s)|>r)^{1/b_n^2} = 0. \tag{3.44}$$

Proof. Note that

$$\lim_{r\to\infty}\limsup_{n\to\infty} \mathbb{P}\left(\frac{A_n(t)}{n}>r\right)^{1/b_n^2} = 0, \tag{3.45}$$

which is a consequence of the LD convergence at rate b_n^2 of the Y_n to Y (see (3.1)). Similarly, the LD convergence of $X_n(0)$ to $X(0)$ implies that

$$\lim_{r\to\infty}\limsup_{n\to\infty} \mathbb{P}\left(\frac{Q_n(0)}{n}>r\right)^{1/b_n^2} = 0. \tag{3.46}$$

By (2.4), $\hat{A}_n(t) \le (Q_n(0)-n)^+ + A_n(t)$, so that (3.45) and (3.46) imply that

$$\lim_{L\to\infty}\limsup_{n\to\infty} \mathbb{P}\left(\frac{\hat{A}_n(t)}{n}>L\right)^{1/b_n^2} = 0. \tag{3.47}$$

By (3.38),

$$\mathbb{P}(\sup_{s\in[0,t]}|U_n(s,t-s)|>r) \le \mathbb{P}\left(\frac{\hat{A}_n(t)}{n}>L\right) + \mathbb{P}(\sup_{\substack{s\in[0,L],\\ x\in[0,1]}}|K_n(x,s)|>r). \tag{3.48}$$

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By the LD convergence of K_n to K in Lemma 1, and since the trajectories of K are continuous,

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(\sup_{\substack{s\in[0,L],\\ x\in[0,1]}}|K_n(x,s)|>r\big)^{1/b_n^2}=0.$$

Combined with (3.47) and (3.48), this proves (3.43).

Lemma 3.1 in [21] implies that the process $M_n=(M_n(t),\ t\in\mathbb{R}_+)$ is a local martingale with respect to the filtration $\mathbf{G}_n$ defined as follows. For $t\in\mathbb{R}_+$, let $\hat{\mathcal{G}}_n(t)$ denote the complete σ-algebra generated by the random variables $\mathbf{1}_{\{\hat{\tau}_{n,i}\le s\}}\ \mathbf{1}_{\{\eta_i\le x\}}$, where $x+s\le t$ and $i\in\mathbb{N}$, and by the $\hat{A}_n(s)$ (or, equivalently, by the $\mathbf{1}_{\{\hat{\tau}_{n,i}\le s\}}$ for $i\in\mathbb{N}$), where $s\le t$. Define $\mathcal{G}_n(t)=\cap_{\epsilon>0}\hat{\mathcal{G}}_n(t+\epsilon)$ and $\mathbf{G}_n=(\mathcal{G}_n(t),\,t\in\mathbb{R}_+)$. By (3.36) and (3.42),

$$M_n(t)=\frac{1}{b_n\sqrt{n}}\sum_{i=1}^{\hat{A}_n(t)}\bigg(\mathbf{1}_{\{\eta_i+\hat{\tau}_{n,i}\le t\}}-\int_0^{\eta_i\wedge(t-\hat{\tau}_{n,i})}\frac{dF(u)}{1-F(u)}\bigg). \tag{3.49}$$

Thus, the measure of jumps of M_n is

$$\mu_n([0,t],\Gamma)=\mathbf{1}_{\{1/(b_n\sqrt{n})\in\Gamma\}}\sum_{i=1}^{\hat{A}_n(t)}\mathbf{1}_{\{\eta_i+\hat{\tau}_{n,i}\le t\}} \tag{3.50}$$

and the associated $\mathbf{G}_n$-predictable measure of jumps is

$$\nu_n([0,t],\Gamma)=\mathbf{1}_{\{1/(b_n\sqrt{n})\in\Gamma\}}\sum_{i=1}^{\hat{A}_n(t)}\int_0^{\eta_i\wedge(t-\hat{\tau}_{n,i})}\frac{dF(u)}{1-F(u)}. \tag{3.51}$$

Note that it is a continuous process. (For $\hat{A}_n$ being predictable, see Lemma C.1 in [21].) The associated stochastic cumulant is (see, e.g., [19, p. 293])

$$G_n(\alpha,t)=\bigg(e^{\alpha/(b_n\sqrt{n})}-1-\frac{\alpha}{b_n\sqrt{n}}\bigg)\sum_{i=1}^{\hat{A}_n(t)}\int_0^{\eta_i\wedge(t-\hat{\tau}_{n,i})}\frac{dF(u)}{1-F(u)}. \tag{3.52}$$

By Lemma 4.1.1 on p. 294 in [19], the process $(e^{\alpha M_n(t)-G_n(\alpha,t)},\,t\in\mathbb{R}_+)$ is a local martingale, so that $\mathbb{E}e^{\alpha M_n(\tau)-G_n(\alpha,\tau)}\le 1$, for arbitrary stopping time τ. Lemma 3.2.6 on p. 282 in [19] implies that, for $\gamma>0$,

$$\begin{aligned}\mathbb{P}(\sup_{s\in[0,t]}e^{\alpha b_n^2M_n(s)}\ge e^{\alpha b_n^2 r})&\le e^{\alpha b_n^2(\gamma-r)}+\mathbb{P}(e^{G_n(\alpha b_n^2,t)}\ge e^{\alpha b_n^2\gamma})\\ &\le e^{\alpha b_n^2(\gamma-r)}+\mathbb{P}(e^{\hat{G}_n(\alpha b_n^2,t)}\ge e^{\alpha b_n^2\gamma}),\end{aligned}$$

where

$$\hat{G}_n(\alpha,t)=\bigg(e^{\alpha/(b_n\sqrt{n})}-1-\frac{\alpha}{b_n\sqrt{n}}\bigg)\sum_{i=1}^{\hat{A}_n(t)}\int_0^{t-\hat{\tau}_{n,i}}\frac{dF(u)}{1-F(u)}. \tag{3.53}$$

Hence, for $\alpha>0$,

$$\mathbb{P}(\sup_{s\in[0,t]}M_n(s)\ge r)^{1/b_n^2}\le e^{\alpha(\gamma-r)}+\mathbb{P}(\hat{G}_n(\alpha b_n^2,t)\ge\alpha b_n^2\gamma)^{1/b_n^2}. \tag{3.54}$$

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On writing

$$\hat{G}_n(\alpha b_n^2, t) = \Big(e^{\alpha b_n/\sqrt{n}} - 1 - \frac{\alpha b_n}{\sqrt{n}}\Big) \int_0^t \int_0^{t-s} \frac{dF(u)}{1-F(u)}\, d\hat{A}_n(s)$$

$$= \Big(e^{\alpha b_n/\sqrt{n}} - 1 - \frac{\alpha b_n}{\sqrt{n}}\Big) \int_0^t \hat{A}_n(t-u)\, \frac{dF(u)}{1-F(u)} \qquad (3.55)$$

and noting that $(n/b_n^2)(e^{\alpha b_n/\sqrt{n}} - 1 - \alpha b_n/\sqrt{n}) \to \alpha^2/2$, one can see, thanks to (3.47), that

$$\lim_{n\to\infty} \mathbb{P}(\hat{G}_n(\alpha b_n^2, t) \geq \alpha b_n^2 \gamma)^{1/b_n^2} = 0,$$

provided α is small enough, which proves that

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}(\sup_{s\in[0,t]} M_n(s) > r)^{1/b_n^2} = 0.$$

The argument for $\sup_{s\in[0,t]}(-M_n(s))$ is similar. The convergence (3.44) has been proved.

Lemma 3. *For arbitrary $\epsilon > 0$ and $t > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} \Big|\frac{\hat{A}_n(s)}{n} - \mu s\Big| > \epsilon\Big)^{1/b_n^2} = 0.$$

Proof. By (2.4), (2.2), (3.27), and (3.28),

$$\frac{1}{n}\, Q_n(t) = \Big(\frac{1}{n}\, Q_n(0) - 1\Big)^+ (1 - F(t)) + \frac{1}{n}\, Q_n^{(0)}(t) + \frac{1}{n}\, A_n(t) - \frac{1}{n} \int_0^t A_n(t-s)\, dF(s)$$

$$+ \frac{1}{n} \int_0^t (Q_n(t-s) - n)^+\, dF(s) + \frac{b_n}{\sqrt{n}}\, \Theta_n(t). \qquad (3.56)$$

By (3.40), (3.41), (3.43), and (3.44), on recalling that $F(t) < 1$, we have

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}(\sup_{s\in[0,t]} |\Theta_n(s)| > r)^{1/b_n^2} = 0. \qquad (3.57)$$

The LD convergence at rate b_n^2 of Y_n to Y implies that, for $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} \Big|\frac{A_n(s)}{n} - \mu s\Big| > \epsilon\Big)^{1/b_n^2} = 0. \qquad (3.58)$$

By (2.3), (3.46), and Lemma 1,

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} \Big|\frac{1}{n}\, Q_n^{(0)}(s) - (1 - F_0(s))\Big| > \epsilon\Big)^{1/b_n^2} = 0.$$

Recalling (3.29) implies that

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} \Big|\frac{1}{n}\, Q_n^{(0)}(s) + \frac{1}{n}\, A_n(s) - \frac{1}{n} \int_0^s A_n(s-x)\, dF(x) - 1\Big| > \epsilon\Big)^{1/b_n^2} = 0.$$

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In addition, the LD convergence of $X_n(0)$ to $X(0)$ implies that (3.46) can be strengthened as follows:

$$\limsup_{n\to\infty} \mathbb{P}\left(|\frac{Q_n(0)}{n} - 1| > \epsilon\right)^{1/b_n^2} = 0. \tag{3.59}$$

Hence, by (3.56) and (3.57),

$$\frac{1}{n} Q_n(t) - 1 = \int_0^t \left(\frac{1}{n} Q_n(t-s) - 1\right)^+ dF(s) + \theta_n(t),$$

where

$$\theta_n(t) = \left(\frac{1}{n} Q_n(0) - 1\right)^+ (1 - F(t)) + \frac{1}{n} Q_n^{(0)}(t) + \frac{1}{n} A_n(t) - \frac{1}{n} \int_0^t A_n(t-s)\, dF(s) + \frac{b_n}{\sqrt{n}}\, \Theta_n(t)$$

and

$$\lim_{n\to\infty} \mathbb{P}(\sup_{s\in[0,t]}|\theta_n(s)| > \epsilon)^{1/b_n^2} = 0.$$

Lemma B.1 in [21] implies that there exists a function ρ, which depends only on the function F, such that

$$\sup_{s\in[0,t]}|\frac{1}{n} Q_n(s) - 1| \leq \rho(t) \sup_{s\in[0,t]}|\theta_n(s)|.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in[0,t]}|\frac{1}{n} Q_n(s) - 1| > \epsilon\right)^{1/b_n^2} = 0. \tag{3.60}$$

When combined with (2.4) and (3.58), this yields the assertion of the lemma.

Let $L_n = \big((L_n(x, t), x \in \mathbb{R}_+), t \in \mathbb{R}_+\big)$ and $L = \big((L(x, t), x \in \mathbb{R}_+), t \in \mathbb{R}_+\big)$, with $L(x, t) = B(F(x), \mu t)$.

Lemma 4. *As $n \to \infty$, the sequence $(X_n(0), X_n^{(0)}, U_n, L_n)$ LD converges in distribution in $\mathbb{R} \times \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{D}(\mathbb{R}_+, \mathbb{D}(\mathbb{R}_+, \mathbb{R}))^2$ to $(X(0), X^{(0)}, U, L)$.*

Proof. Let

$$\tilde{X}_n^{(0)}(x, t) = \frac{1}{b_n\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} (\mathbf{1}_{\{\eta_i^{(0)} > x\}} - (1 - F_0(x))), \tag{3.61}$$

$\tilde{X}_n^{(0)}(t) = (\tilde{X}_n^{(0)}(x, t), x \in \mathbb{R}_+)$ and $\tilde{X}_n^{(0)} = (\tilde{X}_n^{(0)}(t), t \in \mathbb{R}_+)$. By the hypotheses of Theorem 3, $X_n(0)$ LD converges to $X(0)$, by Lemma 1 and because F_0 is strictly increasing, and $\tilde{X}_n^{(0)}$ LD converges to $((\tilde{K}(F_0(x), t), x \in \mathbb{R}_+), t \in \mathbb{R}_+)$, where $\tilde{K}$ represents a Kiefer idempotent process that is independent of $(X(0),K, B)$. Also (K_n, B_n) LD converges to (K, B). By independence assumptions, these convergences hold jointly; cf. Appendix A. Since $Q_n(0)/n \to 1$ and $\hat{A}_n(t)/n \to \mu t$ super-exponentially in probability by (3.59) and (3.58), respectively, 'Slutsky's theorem' (Lemma 6) yields joint LD convergence of $(X_n(0), \tilde{X}_n^{(0)}, K_n, B_n, Q_n(0)/n, (\hat{A}_n(t)/n, t \in \mathbb{R}_+))$ to $(X(0), ((\tilde{K}(F_0(x), t), x \in \mathbb{R}_+), t \in \mathbb{R}_+), K, B, 1, (\mu t, t \in \mathbb{R}_+))$. In addition, by (2.3), (3.26), and (3.61),

$$X_n^{(0)}(t) = \tilde{X}_n^{(0)}\left(t, \frac{Q_n(0)}{n} \wedge 1\right) - (1 - F_0(t))X_n(0)^-.$$

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In order to deduce the LD convergence of $(X_n(0), X_n^{(0)}, U_n, L_n)$ to $(X(0), X^{(0)}, U, L)$, it remains to recall (3.32), (3.33), (3.37), and (3.38); note that, by Lemma 7, $(\tilde{K}(t, 1), t \in [0, 1])$ is a Brownian bridge idempotent process; and apply the continuous mapping principle, the associated composition mappings being continuous at continuous limits. (See [24] for more background on continuous functions in the Skorokhod space context.)

Lemma 5. *The sequence $\{\Theta_n, n \in \mathbb{N}\}$ is $\mathbb{C}$-exponentially tight of order b_n^2 in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$.*

Proof. Let

$$J(t) = \int_0^t \frac{U(x, t-x)}{1-F(x)}\, dF(x). \tag{3.62}$$

By Lemma 4, (3.41), (3.62), and the continuous mapping principle, $J_n = (J_n(t), t \in \mathbb{R}_+)$ LD converges to $J = J(t), t \in \mathbb{R}_+)$, so the sequence J_n is $\mathbb{C}$-exponentially tight. By (3.40), it remains to check that the sequence M_n is $\mathbb{C}$-exponentially tight, which, according to Theorem 8, is implied by the following convergences:

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}(\sup_{s\in[0,t]}|M_n(s)| > r)^{1/b_n^2} = 0$$

and

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{s\in[0,t]}\mathbb{P}(\sup_{s'\in[0,\delta]}|M_n(s+s') - M_n(s)| > \epsilon)^{1/b_n^2} = 0, \tag{3.63}$$

where $t > 0$ and $\epsilon > 0$. The former convergence has already been proved; see (3.44). The proof of (3.63) proceeds along similar lines. Since, with $\alpha \in \mathbb{R}$, the process $(\exp\big(\alpha(M_n(s+s') - M_n(s)) - (G_n(\alpha, s+s') - G_n(\alpha, s))\big), s' \in \mathbb{R}_+)$ is a local martingale, so that, for arbitrary stopping time τ,

$$\mathbb{E}e^{\alpha(M_n(s+\tau)-M_n(s))-(G_n(\alpha,s+\tau)-G_n(\alpha,s))} \le 1,$$

by Lemma 3.2.6 on p. 282 in [19], for arbitrary $\gamma > 0$, in analogy with (3.54), for $b_n \ge 1$, we have

$$\begin{aligned}
&\mathbb{P}(\sup_{s'\in[0,\delta]}(M_n(s+s') - M_n(s)) \\
&\qquad \ge \epsilon)^{1/b_n^2} \le e^{\alpha(\gamma-\epsilon)} + \mathbb{P}(\hat{G}_n(\alpha b_n^2, s+\delta) - \hat{G}_n(\alpha b_n^2, s) \ge \alpha b_n^2\gamma)^{1/b_n^2}.
\end{aligned}$$

By (3.55) and Lemma 3,

$$\frac{1}{b_n^2}\,\hat{G}_n(\alpha b_n^2, t) \to \frac{\alpha^2}{2}\,\mu \int_0^t (t-u)\frac{dF(u)}{1-F(u)}$$

super-exponentially in probability at rate b_n^2. The fact that the latter super-exponential convergence in probability is locally uniform in t, as the limit is a monotonic continuous function starting at 0, implies that, for δ small enough, depending on α,

$$\limsup_{n\to\infty} \sup_{s\in[0,t]}\mathbb{P}(\sup_{s'\in[0,\delta]}(M_n(s'+s) - M_n(s)) \ge \epsilon)^{1/b_n^2} \le e^{\alpha(\gamma-\epsilon)}.$$

Now, one chooses $\gamma < \epsilon$ and sends $\alpha \to \infty$. A similar convergence holds with $-M_n(s')$ substituted for $M_n(s')$. The limit (3.63) has been proved.

Lemma 6. *The sequence $(X_n(0), X_n^{(0)}, H_n, J_n, L_n, U_n, \Theta_n)$ LD converges in distribution in $\mathbb{R} \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})^3 \times \mathbb{D}(\mathbb{R}_+, \mathbb{D}(\mathbb{R}_+, \mathbb{R}))^2 \times \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ to $(X(0), X^{(0)}, H, J, L, U, \Theta)$, as $n \to \infty$.*

Proof. Let

$$\Theta_n^{(l)}(t) = -\int_{\mathbb{R}_+^2} I_t^{(l)}(x, s) dU_n(x, s), \tag{3.64}$$

where

$$I_t^{(l)}(x, s) = \sum_{i=1}^{\infty} \mathbf{1}_{\{s \in (s_{i-1}^{(l)}, s_i^{(l)}]\}} \mathbf{1}_{\{x \in [0, t - s_{i-1}^{(l)}]\}},$$

for some $0 = s_0^{(l)} < s_1^{(l)} < \ldots$ such that $s_i^{(l)} \to \infty$, as $i \to \infty$, and $\sup_{i \geq 1}(s_i^{(l)} - s_{i-1}^{(l)}) \to 0$, as $l \to \infty$. Evidently,

$$\Theta_n^{(l)}(t) = -\sum_{i=1}^{\infty} (U_n(t - s_{i-1}^{(l)}, s_i^{(l)}) - U_n(t - s_{i-1}^{(l)}, s_{i-1}^{(l)})) \mathbf{1}_{\{s_{i-1}^{(l)} \leq t\}}.$$

Similarly, let

$$\Theta^{(l)}(t) = -\int_{\mathbb{R}_+^2} I_t^{(l)}(x, s)\, dU(x, s) = \sum_{i=1}^{\infty} (U(t - s_{i-1}^{(l)}, s_i^{(l)}) - U(t - s_{i-1}^{(l)}, s_{i-1}^{(l)})) \mathbf{1}_{\{s_{i-1}^{(l)} \leq t\}}.$$

By the LD convergence of Y_n to Y in the hypotheses of Theorem 3, Lemma 4, (3.25), (3.31), (3.41), (3.62), and the continuous mapping principle, the sequence $(X_n(0), X_n^{(0)}, H_n, J_n, L_n, U_n)$ LD converges to $(X(0), X^{(0)}, H, J, L, U)$. Hence, the sequence $(X_n(0), X_n^{(0)}, H_n, J_n, L_n, U_n, \Theta_n^{(l)})$ LD converges to $(X(0), X^{(0)}, H, J, L, U, \Theta^{(l)})$. In addition, by Lemma 5, the sequence Θ_n is $\mathbb{C}$-exponentially tight. Since the idempotent processes $X^{(0)}, H, J, L, U$ are seen to have continuous trajectories, the sequence $(X_n^{(0)}, H_n, J_n, L_n, U_n, \Theta_n)$ is $\mathbb{C}$-exponentially tight. That a limit point of $(X_n(0), X_n^{(0)}, H_n, J_n, L_n, U_n, \Theta_n)$ has the same idempotent distribution as $(X(0), X^{(0)}, H, J, L, U, \Theta)$ would follow from the LD convergence of finite-dimensional distributions of $(X_n(0), X_n^{(0)}, H_n, J_n, L_n, U_n, \Theta_n)$ to finite-dimensional distributions of $(X(0), X^{(0)}, H, J, L, U, \Theta)$. Owing to (3.33),

$$\Theta^{(l)}(t) = -\int_{\mathbb{R}_+^2} I_t^{(l)}(x, s) \dot{K}(F(x), \mu s)\, dF(x)\, \mu ds,$$

which implies, by (3.34) and the Cauchy–Schwarz inequality, that $\Theta^{(l)} \to \Theta$ locally uniformly, as $l \to \infty$. Since the sequence $(X_n(0), X_n^{(0)}, H_n, J_n, L_n, U_n)$ LD converges to $(X(0), X^{(0)}, H, J, L, U)$, to prove the finite-dimensional LD convergence it suffices to prove that

$$\lim_{l \to \infty} \limsup_{n \to \infty} \mathbb{P}(|\Theta_n^{(l)}(t) - \Theta_n(t)| > \epsilon)^{1/b_n^2} = 0. \tag{3.65}$$

Let

$$\hat{\Theta}_n^{(l)}(s, t) = \frac{1}{b_n \sqrt{n}} \sum_{i=1}^{\hat{A}_n(s)} \sum_{j=1}^{\infty} \mathbf{1}_{\{s_{j-1}^{(l)} \leq t\}} \mathbf{1}_{\{\hat{\tau}_{n,i} \in (s_{j-1}^{(l)}, s_j^{(l)}]\}} \Big(\mathbf{1}_{\{\eta_i \in (t - \hat{\tau}_{n,i}, t - s_{j-1}^{(l)}]\}} - (F(t - s_{j-1}^{(l)}) - F(t - \hat{\tau}_{n,i})) \Big).$$

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By (3.28) and (3.64),

$$\Theta_n(t) - \Theta_n^{(l)}(t) = \hat{\Theta}_n^{(l)}(t, t). \tag{3.66}$$

Let $\mathcal{F}_n(s)$ represent the complete σ-algebra generated by the random variables $\hat{\tau}_{n,j} \wedge \hat{\tau}_{n,\hat{A}_n(s)+1}$ and $\eta_{j\wedge\hat{A}_n(s)}$, where $j \in \mathbb{N}$. By Part 4 of Lemma C.1 in [21], with t held fixed, the process $(\hat{\Theta}_n^{(l)}(s, t), s \in \mathbb{R}_+)$ is an $\mathbb{F}_n$-locally square-integrable martingale. Its measure of jumps is

$$\mu_n^{(l)}([0, s], \Gamma) = \sum_{i=1}^{\hat{A}_n(s)} \sum_{j=1}^{\infty} \mathbf{1}_{\{s_{j-1}^{(l)} \le t\}} \mathbf{1}_{\{\hat{\tau}_{n,i} \in (s_{j-1}^{(l)}, s_j^{(l)}]\}} \big(\mathbf{1}_{\{(1-(F(t-s_{j-1}^{(l)})-F(t-\hat{\tau}_{n,i}))/(b_n\sqrt{n})\in\Gamma\}}$$
$$\mathbf{1}_{\{\eta_i \in (t-\hat{\tau}_{n,i}, t-s_{j-1}^{(l)}]\}} + \mathbf{1}_{\{(-(F(t-s_{j-1}^{(l)})-F(t-\hat{\tau}_{n,i}))/(b_n\sqrt{n})\in\Gamma\}} \mathbf{1}_{\{\eta_i \notin (t-\hat{\tau}_{n,i}, t-s_{j-1}^{(l)}]\}} \big),$$

where $\Gamma \subset \mathbb{R} \setminus \{0\}$. Accordingly, the $\mathbb{F}_n$-predictable measure of jumps is

$$\nu_n^{(l)}([0, s], \Gamma) = \sum_{i=1}^{\hat{A}_n(s)} \sum_{j=1}^{\infty} \mathbf{1}_{\{s_{j-1}^{(l)} \le t\}} \mathbf{1}_{\{\hat{\tau}_{n,i} \in (s_{j-1}^{(l)}, s_j^{(l)}]\}} \big(\mathbf{1}_{\{(1-(F(t-s_{j-1}^{(l)})-F(t-\hat{\tau}_{n,i}))/(b_n\sqrt{n})\in\Gamma\}}$$
$$(F(t-s_{j-1}^{(l)}) - F(t-\hat{\tau}_{n,i})) + \mathbf{1}_{\{-(F(t-s_{j-1}^{(l)})-F(t-\hat{\tau}_{n,i}))/(b_n\sqrt{n})\in\Gamma\}} (1 - (F(t-s_{j-1}^{(l)}) - F(t-\hat{\tau}_{n,i}))) \big).$$

For $\alpha \in \mathbb{R}$, as on p. 214 in [19], define the stochastic cumulant

$$\begin{aligned} G_n^{(l)}(\alpha, s) &= \int_0^s \int_{\mathbb{R}} (e^{\alpha x} - 1 - \alpha x)\nu_n^{(l)}(ds', dx) \\ &= \sum_{i=1}^{\hat{A}_n(s)} \sum_{j=1}^{\infty} \mathbf{1}_{\{s_{j-1}^{(l)} \le t\}} \mathbf{1}_{\{\hat{\tau}_{n,i} \in (s_{j-1}^{(l)}, s_j^{(l)}]\}} \big((e^{\alpha(1-(F(t-s_{j-1}^{(l)})-F(t-\hat{\tau}_{n,i})))/(b_n\sqrt{n})} \\ &\quad - 1 - \frac{\alpha}{b_n\sqrt{n}}(1 - (F(t-s_{j-1}^{(l)}) - F(t-\hat{\tau}_{n,i}))))(F(t-s_{j-1}^{(l)}) - F(t-\hat{\tau}_{n,i})) \\ &\quad + (e^{-\alpha(F(t-s_{j-1}^{(l)})-F(t-\hat{\tau}_{n,i}))/(b_n\sqrt{n})} - 1 \\ &\quad + \frac{\alpha}{b_n\sqrt{n}}(F(t-s_{j-1}^{(l)}) - F(t-\hat{\tau}_{n,i})))(1 - (F(t-s_{j-1}^{(l)}) - F(t-\hat{\tau}_{n,i}))) \big). \end{aligned} \tag{3.67}$$

The associated stochastic exponential is defined by

$$\mathcal{E}_n^{(l)}(\alpha, s) = e^{G_n^{(l)}(\alpha,s)} \prod_{0<s'\le s} (1 + \Delta G_n^{(l)}(\alpha, s'))e^{-\Delta G_n^{(l)}(\alpha,s')}, \tag{3.68}$$

where $\Delta G_n^{(l)}(s')$ represents the jump of $G_n^{(l)}(s'')$ with respect to s" evaluated at $s'' = s'$ and the product is taken over the jumps. By Lemma 4.1.1 on p. 294 in [19], the process $\big(e^{\alpha b_n^2 \hat{\Theta}_n^{(l)}(s,t)} \mathcal{E}_n(\alpha b_n^2, s)^{-1}, s \in \mathbb{R}_+\big)$ is a well-defined local martingale, so that, for any stopping time τ,

$$\mathbb{E} e^{\alpha b_n^2 \hat{\Theta}_n^{(l)}(\tau,t)} \mathcal{E}_n^{(l)}(\alpha b_n^2, \tau)^{-1} \le 1.$$

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Lemma 3.2.6 on p. 282 in [19] and (3.68) imply that, for $\alpha > 0$ and $\gamma > 0$,

$$\mathbb{P}(\sup_{s\in[0,t]}\hat{\Theta}_n^{(l)}(s,t)\ge\epsilon)\le e^{\alpha b_n^2(\gamma-\epsilon)}+\mathbb{P}(\mathcal{E}_n^{(l)}(\alpha b_n^2,t)\ge e^{\alpha b_n^2\gamma})$$
$$\le e^{\alpha b_n^2(\gamma-\epsilon)}+\mathbb{P}(G_n^{(l)}(\alpha b_n^2,t)\ge\alpha b_n^2\gamma). \tag{3.69}$$

By (3.67),

$$G_n^{(l)}(\alpha b_n^2,t)\le\hat{A}_n(t)\bigg(\sup_{|y|\le1}\bigg(e^{\alpha b_n y/\sqrt{n}}-1-\alpha\frac{b_n y}{\sqrt{n}}\bigg)\sup_j\big(F(t-s_{j-1}^{(l)})-F(t-s_j^{(l)})\big)$$
$$+\sup_j\sup_{y\in[F(t-s_j^{(l)}),F(t-s_{j-1}^{(l)})]}\bigg(e^{-\alpha b_n y/\sqrt{n}}-1+\alpha\frac{b_n}{\sqrt{n}}y\bigg)\bigg).$$

As $n/b_n^2(e^{\alpha b_n y/\sqrt{n}}-1-\alpha b_n y/\sqrt{n})\to\alpha^2y^2/2$ uniformly on bounded intervals, $\hat{A}_n(t)/n\to\mu t$ super-exponentially as $n\to\infty$, and $\sup_j(s_j^l-s_{j-1}^l)\to0$ as $l\to\infty$, it follows that

$$\lim_{l\to\infty}\limsup_{n\to\infty}\mathbb{P}(G_n^{(l)}(\alpha b_n^2,t)\ge\alpha b_n^2\gamma)^{1/b_n^2}=0,$$

which implies, thanks to (3.69), that

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\mathbb{P}(\sup_{s\in[0,t]}\hat{\Theta}_n^{(l)}(s,t)\ge\epsilon)^{1/b_n^2}\le e^{\alpha(\gamma-\epsilon)}.$$

Picking $\gamma<\epsilon$ and sending α to ∞ shows that the latter left-hand side equals zero. A similar argument proves the convergence

$$\lim_{l\to\infty}\limsup_{n\to\infty}\mathbb{P}(\sup_{s\in[0,t]}(-\hat{\Theta}_n^{(l)}(s,t))\ge\epsilon)^{1/b_n^2}=0.$$

Recalling (3.66) yields the convergence (3.65).

Theorem 4 has thus been proved. In order to obtain the assertion of Theorem 3, note that $-K$ has the same idempotent distribution as K and invoke the continuous mapping principle in (3.30), which applies by Lemma B.2 in [21].

The proof of Theorem 2 proceeds along similar lines. First we prove an analogue of Theorem 3 to the effect that if the random variables $\overline{X}_n(0)$ LD converge to an idempotent variable $\overline{X}(0)$, then the processes $\overline{X}_n$ LD converge to the idempotent process $\overline{X}$ that solves the following analogue of (3.2):

$$\overline{X}(t)=(1-F_0(t))\overline{X}(0)+\sqrt{q_0}\,W^0(F_0(t))+\int_0^t\big(1-F(t-s)\big)\sigma\,\dot{W}(s)\,ds$$
$$+\int_{\mathbb{R}_+^2}\mathbf{1}_{\{x+s\le t\}}\,\dot{K}(F(x),\lambda s)\,dF(x)\,\lambda\,ds. \tag{3.70}$$

The proof is a simpler version of the proof of Theorem 3. Essentially, one replaces $\hat{A}_n$ with A_n and μ with λ. A key element of the proof of Theorem 3 is the property, asserted in the statement of Lemma 3, that $\hat{A}_n(t)/n\to\mu t$ super-exponentially in probability. This property takes some effort to establish. Its counterpart for the infinite-server queue is that $A_n(t)/n\to\lambda t$ super-exponentially in probability; this is a direct consequence of the hypotheses.

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In some more detail, since, in analogy with (2.2),

$$\overline{Q}_n(t) = Q_n^{(0)}(t) + A_n(t) - \int_0^t \int_0^t \mathbf{1}_{\{x+s\le t\}}\, d\sum_{i=1}^{A_n(s)} \mathbf{1}_{\{\eta_i \le x\}},$$

grouping terms appropriately and recalling (2.12) yields

$$\frac{1}{n}\overline{Q}_n(t) - \overline{q}(t) = \Big(\frac{1}{n}\overline{Q}_n(0) - q_0\Big)(1 - F_0(t)) + \frac{1}{n}A_n(t) - \lambda t - \int_0^t \Big(\frac{1}{n}A_n(t-s) - \lambda(t-s)\Big)dF(s)$$
$$+ \frac{1}{n}\sum_{i=1}^{\overline{Q}_n(0)} \big(\mathbf{1}_{\{\eta_i^{(0)} > t\}} - (1 - F_0(t))\big) - \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, d\sum_{i=1}^{A_n(s)} \big(\mathbf{1}_{\{\eta_i \le x\}} - F(x)\big).$$

On introducing

$$\overline{Y}_n(t) = \frac{\sqrt{n}}{b_n}\Big(\frac{A_n(t)}{n} - \lambda t\Big),$$

$$\overline{X}_n^{(0)}(t) = \frac{1}{b_n\sqrt{n}} \sum_{i=1}^{\overline{Q}_n(0)} \big(\mathbf{1}_{\{\eta_i^{(0)} > t\}} - (1 - F_0(t))\big),$$

$$\overline{H}_n(t) = \overline{Y}_n(t) - \int_0^t \overline{Y}_n(t-s)\, dF(s),$$

$$\overline{U}_n(x, t) = \frac{1}{b_n\sqrt{n}} \sum_{i=1}^{A_n(t)} \big(\mathbf{1}_{\{\eta_i \le x\}} - F(x)\big),$$

and

$$\overline{\Theta}_n(t) = -\int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, d\overline{U}_n(x, s),$$

we obtain the following analogue of (3.30):

$$\overline{X}_n(t) = (1 - F_0(t))\overline{X}_n(0) + \overline{X}_n^{(0)}(t) + \overline{H}_n(t) + \overline{\Theta}_n(t). \tag{3.71}$$

The hypotheses imply that $\overline{Y}_n$ LD converges to σW, with the notation of the proof of Theorem 3 being reused. Since, in analogy with (3.38), $\overline{U}_n(x, t) = K_n(F(x), A_n(t)/n)$ and $A_n(t)/n$ converges to λt super-exponentially in probability, the process $\overline{U}_n$ LD converges to $\overline{U}$, where $\overline{U}(x, t) = K(F(x), \lambda t)$. Furthermore, similarly to Lemma 4, it is proved that $(\overline{X}_n(0), \overline{X}_n^{(0)}, \overline{U}_n, \overline{L}_n)$ LD converges to $(\overline{X}(0), \overline{X}^{(0)}, \overline{U}, \overline{L})$, where $\overline{X}^{(0)}(t) = \sqrt{q_0}W^0(F_0(t))$ and $\overline{L}(x, t) = B(F(x), \lambda t)$. Put together, these properties imply the analogue of Theorem 4: that $(\overline{X}_n(0), \overline{X}_n^{(0)}, \overline{H}_n, \overline{\Theta}_n)$ LD converges to $(\overline{X}(0), \overline{X}^{(0)}, \overline{H}, \overline{\Theta})$, where

$$\overline{H}(t) = \sigma W(t) - \sigma \int_0^t W(t-s)\, dF(s) = \sigma W(t) - \sigma \int_0^t (1 - F(t-s))\, \dot{W}(s)\, ds$$

and

$$\overline{\Theta}(t) = -\int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, d\overline{U}(x, s) = -\int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, \dot{K}(F(x), \lambda s)\, dF(x)\, \lambda ds.$$

An application of the continuous mapping principle to (3.71) concludes the proof.

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4. Evaluating the deviation functions

This section is concerned with solving for $I_{x_0}^Q(q)$ and $\overline{I}_{q_0,x_0}^Q(q)$.

Theorem 5. *Suppose that the CDF F is an absolutely continuous function and $I_{x_0}^Q(q) < \infty$. Then q is absolutely continuous, $(\dot{q}(t) - \int_0^t \dot{q}(s)\, \mathbf{1}_{\{q(s)>0\}}\, F'(t-s)\, ds,\, t \in \mathbb{R}_+) \in \mathbb{L}_2(\mathbb{R}_+)$, the infimum in (2.5) is attained uniquely, and*

$$I_{x_0}^Q(q) = \frac{1}{2} \int_0^\infty \hat{p}(t)\Big(\dot{q}(t) - \int_0^t \dot{q}(s)\, \mathbf{1}_{\{q(s)>0\}}\, F'(t-s)\, ds + (\beta - x_0^-) F_0{}'(t)\Big)\, dt,$$

where $\hat{p}(t)$ represents the unique $\mathbb{L}_2(\mathbb{R}_+)$-solution p(t) of the Fredholm equation of the second kind

$$\begin{aligned}(\mu + \sigma^2) p(t) = \dot{q}(t) - \int_0^t \dot{q}(s)\, \mathbf{1}_{\{q(s)>0\}}\, F'(t-s)\, ds + (\beta - x_0^-) F_0{}'(t) \\ + \sigma^2 \int_0^\infty F'(|s-t|) p(s)\, ds - \sigma^2 \int_0^\infty \int_0^{s\wedge t} F'(s-r)\, F'(t-r)\, dr\, p(s)\, ds, \quad (4.1)\end{aligned}$$

with $\dot{q}$, $F_0{}'$, and F' denoting derivatives.

Proof. Writing

$$\int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, \dot{k}(F(x),\, \mu s)\, dF(x)\, \mu\, ds = \int_0^t \int_0^{F(t-s)} \dot{k}(x,\, \mu s)\, dx\, \mu\, ds, \quad (4.2)$$

we see that the equation (2.6) is of the form

$$q(t) = f(t) + \int_0^t q(t-s)^+\, dF(s),\ t \in \mathbb{R}_+,$$

with the functions $f(t)$ and $F(t)$ being absolutely continuous. The function $q(t)$ is absolutely continuous by Lemma 8. In addition, (4.2) implies that, almost everywhere (a.e.),

$$\frac{d}{dt} \int_{\mathbb{R}_+^2} \mathbf{1}_{\{x+s\le t\}}\, \dot{k}(F(x),\, \mu s)\, dF(x)\, \mu\, ds = \int_0^t \dot{k}(F(s),\, \mu(t-s)) F'(s)\, \mu ds. \quad (4.3)$$

The infimum in (2.5) is attained uniquely by coercitivity and strict convexity of the function being minimised; cf. [5, Proposition II.1.2]. Differentiation in (2.6) with the account of (4.3) implies that, a.e.,

$$\begin{aligned}\dot{w}^0(F_0(t)) F_0{}'(t) + \sigma\, \dot{w}(t) - \int_0^t F'(t-s)\sigma\, \dot{w}(s)\, ds + \int_0^t \dot{k}(F(s),\, \mu(t-s)) F'(s)\, \mu\, ds \\ - \left(\dot{q}(t) - \int_0^t \dot{q}(s)\, \mathbf{1}_{\{q(s)>0\}}\, F'(t-s)\, ds + (\beta - x_0^-) F_0{}'(t)\right) = 0.\end{aligned}$$

Introduce the map

$$\begin{aligned}\Phi\colon (\dot{w}^0,\, \dot{w},\, \dot{k}) \to \Big(\dot{w}^0(F_0(t)) F_0{}'(t) + \sigma\, \dot{w}(t) - \int_0^t F'(t-s)\sigma\, \dot{w}(s)\, ds \\ + \int_0^t \dot{k}(F(s),\, \mu(t-s)) F'(s)\, \mu\, ds,\ t \in \mathbb{R}_+\Big).\end{aligned}$$

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Since ${F_0}'(t)$ is bounded by (2.1), Φ maps $V = \mathbb{L}_2([0, 1]) \times \mathbb{L}_2(\mathbb{R}_+) \times \mathbb{L}_2([0, 1] \times \mathbb{R}_+)$ to $\mathbb{L}_2(\mathbb{R}_+)$. For instance, using the fact that $\int_0^\infty F'(s)\, ds = 1$, we have

$$\int_0^\infty \Big(\int_0^t F'(t-s)\dot{w}(s)\, ds\Big)^2\, dt \le \int_0^\infty \int_0^t F'(t-s)\dot{w}(s)^2\, ds\, dt = \int_0^\infty \dot{w}(s)^2\, ds < \infty$$

and

$$\int_0^\infty \Big(\int_0^t \dot{k}(F(s), \mu(t-s))F'(s)\, \mu ds\Big)^2\, dt \le \int_0^\infty \int_0^t \dot{k}(F(s), \mu(t-s))^2 F'(s)\, \mu^2 ds\, dt$$
$$= \mu^2 \int_0^\infty \int_0^1 \dot{k}(x, t)^2 dx\, dt < \infty.$$

The method of Lagrange multipliers, more specifically, [5, Proposition III.5.2] with $Y = \mathbb{L}_2(\mathbb{R}_+)$ and the set of nonnegative functions as the cone $\mathcal{C}$, yields

$$I^Q_{x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)} \inf_{\substack{(\dot{w}^0,\dot{w},\dot{k})\in\mathbb{L}_2([0,1])\\ \times\mathbb{L}_2(\mathbb{R}_+)\times\mathbb{L}_2([0,1]\times\mathbb{R}_+)}} \Big(\frac{1}{2}\int_0^1 \dot{w}^0(x)^2\, dx + \frac{1}{2}\int_0^\infty \dot{w}(t)^2\, dt$$
$$+ \frac{1}{2}\int_0^\infty \int_0^1 \dot{k}(x, t)^2\, dx\, dt + \int_0^\infty p(t)\big(\dot{q}(t) + F'(t)x_0^+ + (\beta - x_0^-){F_0}'(t)$$
$$- \int_0^t \dot{q}(s)\, \mathbf{1}_{\{q(s)>0\}}\, F'(t-s)\, ds - \dot{w}^0(F_0(t)){F_0}'(t) - \sigma\, \dot{w}(t) + \int_0^t F'(t-s)\sigma\, \dot{w}(s)\, ds$$
$$- \int_0^t \dot{k}(F(s), \mu(t-s))F'(s)\, \mu\, ds\big)\, dt\Big). \tag{4.4}$$

Minimising in (4.4) yields, with $(\dot{\hat{w}}^0(t), \dot{\hat{w}}(t), \dot{\hat{k}}(x, t))$, being optimal,

$$\dot{\hat{w}}^0(x) - p(F_0^{-1}(x)) = 0,$$
$$\dot{\hat{w}}(t) - \sigma p(t) + \sigma\int_0^\infty p(t+s)F'(s)\, ds = 0,$$
$$\dot{\hat{k}}(x, t) - p(\frac{t}{\mu} + F^{-1}(x)) = 0.$$

(For the latter, note that

$$\int_0^\infty p(t)\int_0^t \dot{k}(F(s), \mu(t-s))F'(s)\, \mu ds\, dt = \int_0^\infty \int_s^\infty p(t)\dot{k}(F(s), \mu(t-s))F'(s)\, \mu\, dt\, ds$$
$$= \int_0^\infty \int_0^\infty p(t+s)\dot{k}(F(s), \mu t)F'(s)\, \mu\, dt\, ds = \int_0^\infty \int_0^1 p(\frac{t}{\mu} + F^{-1}(x))\dot{k}(x, t)\, dx\, dt.)$$

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Hence,

$$I^Q_{x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)}\bigg(\int_0^\infty p(t)\big(\dot q(t) - \int_0^t \dot q(s)\,\mathbf{1}_{\{q(s)>0\}}\,F'(t-s)\,ds + (\beta - x_0^-)F_0{}'(t)\big)\,dt$$
$$-\frac{1}{2}\bigg(\int_0^\infty p(s)^2 F_0{}'(s)\,ds + \int_0^\infty (\sigma p(t) - \sigma\int_0^\infty p(t+s)F'(s)\,ds)^2\,dt$$
$$+\mu\int_0^\infty\int_0^\infty p(t+s)^2\,F'(s)ds\,dt\bigg)\bigg).$$

Noting that

$$\int_0^\infty p(s)^2 F_0{}'(s)\,ds + \mu\int_0^\infty\int_0^\infty p(t+s)^2\,F'(s)ds\,dt = \mu\int_0^\infty p(s)^2\,ds \tag{4.5}$$

yields

$$I^Q_{x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)}\bigg(\int_0^\infty p(t)\bigg(\dot q(t) - \int_0^t \dot q(s)\,\mathbf{1}_{\{q(s)>0\}}\,F'(t-s)\,ds + (\beta - x_0^-)F_0{}'(t)\bigg)\,dt$$
$$-\frac{1}{2}\bigg(\mu\int_0^\infty p(s)^2\,ds + \int_0^\infty (\sigma p(t) - \sigma\int_0^\infty p(t+s)F'(s)\,ds)^2\,dt\bigg)\bigg). \tag{4.6}$$

The existence and uniqueness of a maximiser in (4.6) follows from Proposition II.1.2 in [5] because the expression in the supremum tends to $-\infty$ as $\|p\|_{\mathbb{L}_2(\mathbb{R}_+)}\to\infty$. Varying p in (4.6) implies (4.1). As the maximiser in (4.6) is unique, so is an $\mathbb{L}_2(\mathbb{R}_+)$-solution of the Fredholm equation.

It is noteworthy that the integral operator on $\mathbb{L}_2(\mathbb{R}_+)$ with kernel

$$\tilde K(t, s) = F'(|s-t|) - \int_0^{s\wedge t} F'(s-r)\,F'(t-r)\,dr$$

is not generally either Hilbert–Schmidt or compact, so the existence and uniqueness of $\hat p(t)$ is not a direct consequence of the general theory.

The numerical solution of Fredholm equations, such as (4.1), is discussed at quite some length in the literature; see, e.g., [3] and references therein. For instance, the collocation method with a basis of 'hat' functions may be tried: for $i\in\mathbb{N}$ and $n\in\mathbb{N}$, let $t_i = i/n$ and $\ell_i(t) = (1-|t-t_i|)\,\mathbf{1}_{\{t_{i-1}\le t\le t_i\}}$, with $t_0 = 0$. Then an approximate solution is

$$p_n(t) = \sum_{i=1}^{n^2} p_n(t_i)\ell_i(t),$$

where the $p_n(t_i)$, $i = 1, \ldots, n^2$, satisfy the linear system

$$(\mu+\sigma^2)p_n(t_i) - \sigma^2\sum_{j=1}^{n^2} p_n(t_j)\int_0^{n^2}\tilde K(t_i, s)\ell_j(s)\,ds$$
$$= \dot q(t_i) - \int_0^{t_i}\dot q(s)\,\mathbf{1}_{\{q(s)>0\}}\,F'(t_i-s)\,ds + (\beta - x_0^-)F_0{}'(t_i).$$

For more background, see [3].

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The evaluation of $\overline{I}^{Q}_{q_0,x_0}$ is carried out similarly:

$$\overline{I}^{Q}_{q_0,x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)} \inf_{\substack{(\dot w^0,\dot w,\dot k)\in\mathbb{L}_2([0,1])\\ \times\mathbb{L}_2(\mathbb{R}_+)\times\mathbb{L}_2([0,1]\times\mathbb{R}_+)}} \Big(\frac{1}{2}\int_0^1 \dot w^0(x)^2\,dx + \frac{1}{2}\int_0^\infty \dot w(t)^2\,dt$$
$$+\frac{1}{2}\int_0^\infty\int_0^1 \dot k(x,\,t)^2\,dx\,dt + \int_0^\infty p(t)\big(\dot q(t) + x_0 F_0{}'(t) - \sqrt{q_0}\dot w^0(F_0(t))F_0{}'(t) - \sigma\,\dot w(t)$$
$$+\int_0^t F'(t-s)\sigma\,\dot w(s)\,ds - \int_0^t \dot k(F(s),\,\lambda(t-s))F'(s)\,\lambda\,ds\big)\,dt\Big).$$

The infimum is attained at $(\dot{\overline{w}}^0(t),\,\dot{\overline{w}}(t),\,\dot{\overline{k}}(x,\,t))$ such that

$$\dot{\overline{w}}^0(x) - \sqrt{q_0}p(F_0^{-1}(x)) = 0,$$
$$\dot{\overline{w}}(t) - \sigma p(t) + \sigma\int_0^\infty p(t+s)F'(s)\,ds = 0,$$
$$\dot{\overline{k}}(x,\,t) - p(\frac{t}{\lambda} + F^{-1}(x)) = 0.$$

Consequently, taking into account (2.1) and (4.5), we have

$$\overline{I}^{Q}_{q_0,x_0}(q) = \sup_{p\in\mathbb{L}_2(\mathbb{R}_+)}\Big(\int_0^\infty p(t)(\dot q(t) + x_0F_0{}'(t))\,dt - \frac{1}{2}\Big(q_0\mu\int_0^\infty p(s)^2\,(1-F(s))\,ds$$
$$+\lambda\int_0^\infty p(s)^2F(s)\,ds + \int_0^\infty (\sigma p(t) - \sigma\int_0^\infty p(t+s)F'(s)\,ds)^2\,dt\Big)\Big)$$
$$= \frac{1}{2}\int_0^\infty \overline{p}(t)(\dot q(t) + x_0F_0{}'(t))\,dt,$$

with $\overline{p}(t)$ being the $\mathbb{L}_2(\mathbb{R}_+)$-solution $p(t)$ to the Fredholm equation of the second kind

$$(q_0\mu(1-F(t)) + \lambda F(t) + \sigma^2)p(t) = \dot q(t) + x_0F_0'(t) + \sigma^2\int_0^\infty \tilde K(t,\,s)p(s)\,ds.$$

Appendix A. Large-deviation convergence and idempotent processes

This section reviews the basics of LD convergence and idempotent processes; see, e.g., [19]. Let $\mathbf{E}$ represent a metric space. Let $\mathcal{P}(\mathbf{E})$ denote the power set of $\mathbf{E}$. The set function $\Pi\colon \mathcal{P}(\mathbf{E}) \to [0,\,1]$ is said to be a *deviability* if $\Pi(E) = \sup_{y\in E}\Pi(\{y\}),\ E\subset\mathbf{E}$, where the function $\Pi(y) = \Pi(\{y\})$ is such that $\sup_{y\in\mathbf{E}}\Pi(y) = 1$ and the sets $\{y\in\mathbf{E}\colon \Pi(y)\geq\gamma\}$ are compact for all $\gamma\in(0,\,1]$. (One can also refer to Π as a maxi-measure or an idempotent probability.) A deviability is a tight set function in the sense that $\inf_{K\in\mathcal{K}(\mathbf{E})}\Pi(\mathbf{E}\setminus K) = 0$, where $\mathcal{K}(\mathbf{E})$ stands for the collection of compact subsets of $\mathbf{E}$. If Ξ is a directed set and F_ξ, $\xi\in\Xi$, is a net of closed subsets of $\mathbf{E}$ that is nonincreasing with respect to the partial order on Ξ by inclusion, then $\Pi(\cap_{\xi\in\Xi}F_\xi) = \lim_{\xi\in\Xi}\Pi(F_\xi)$. A property pertaining to elements of $\mathbf{E}$ is said to hold Π-a.e. if the value of Π of the set of elements that do not have this property equals 0.

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A function f from $\mathbf{E}$ to metric space $\mathbf{E}'$ is called an idempotent variable. The idempotent distribution of the idempotent variable f is defined as the set function $\Pi \circ f^{-1}(\Gamma) = \Pi(f \in \Gamma)$, $\Gamma \subset \mathbf{E}'$. If f is the canonical idempotent variable defined by $f(y) = y$, then it has Π as its idempotent distribution. The continuous images of deviabilities are deviabilities; i.e., if $f: \mathbf{E} \to \mathbf{E}'$ is continuous, then $\Pi \circ f^{-1}$ is a deviability on $\mathbb{E}'$. Furthermore, this property extends to the situation where $f: \mathbf{E} \to \mathbf{E}'$ is strictly Luzin, i.e., continuous when restricted to the set $\{y \in \mathbf{E}: \Pi(y) \geq \gamma\}$, for arbitrary $\gamma \in (0, 1]$. Thus, the idempotent distribution of a strictly Luzin idempotent variable is a deviability. More generally, f is said to be a Luzin idempotent variable if the idempotent distribution of f is a deviability. If $f = (f_1, f_2)$, with f_i assuming values in $\mathbf{E}'_i$, then the (marginal) idempotent distribution of f_1 is defined by $\Pi^{f_1}(y'_1) = \Pi(f_1 = y'_1) = \sup_{y: f_1(y) = y'_1} \Pi(y)$. The idempotent variables f_1 and f_2 are said to be independent if $\Pi(f_1 = y'_1, f_2 = y'_2) = \Pi(f_1 = y'_1)\Pi(f_2 = y'_2)$ for all $(y'_1, y'_2) \in \mathbf{E}'_1 \times \mathbf{E}'_2$, so the joint distribution is the product of the marginal ones. Independence of finite collections of idempotent variables is defined similarly.

A sequence Q_n of probability measures on the Borel σ-algebra of $\mathbf{E}$ is said to large-deviation (LD) converge at rate r_n to deviability Π if for every bounded continuous non-negative function f on $\mathbb{E}$

$$\lim_{n\to\infty} \left(\int_{\mathbb{E}} f(x)^n \, Q_n(dx) \right)^{1/r_n} = \sup_{x\in\mathbf{E}} f(x)\Pi(x).$$

Equivalently, one may require that $\lim_{n\to\infty} Q_n(\Gamma)^{1/r_n} = \Pi(\Gamma)$ for every Borel set Γ such that Π of the interior of Γ and Π of the closure of Γ agree. If the sequence Q_n LD converges to Π, then $\Pi(y) = \lim_{\delta\to 0} \liminf_{n\to\infty} (Q_n(B_\delta(y)))^{1/r_n} = \lim_{\delta\to 0} \limsup_{n\to\infty} (Q_n(B_\delta(y)))^{1/r_n}$, for all $y \in \mathbf{E}$, where $B_\delta(y)$ represents the open ball of radius δ about y. (Closed balls may be used as well.) The sequence Q_n is said to be exponentially tight of order r_n if $\inf_{K\in\mathcal{K}(\mathbf{E})} \limsup_{n\to\infty} Q_n(\mathbf{E} \setminus K)^{1/r_n} = 0$. If the sequence Q_n is exponentially tight of order r_n, then there exists a subsequence $Q_{n'}$ that LD converges at rate $r_{n'}$ to a deviability. Any such deviability will be referred to as a large-deviation (LD) limit point of Q_n. Given $\tilde{E} \subset \mathbf{E}$, the sequence Q_n is said to be $\tilde{E}$-exponentially tight if it is exponentially tight and $\tilde{\Pi}(\mathbf{E} \setminus \tilde{E}) = 0$, for any LD limit point $\tilde{\Pi}$ of Q_n.

It is immediate that Π is a deviability if and only if $I(x) = -\ln \Pi(x)$ is a tight deviation function, i.e., the sets $\{x \in \mathbf{E}: I(x) \leq \gamma\}$ are compact for all $\gamma \geq 0$ and $\inf_{x\in\mathbf{E}} I(x) = 0$, and that the sequence Q_n LD converges to Π at rate r_n if and only if it obeys the LDP for rate r_n with deviation function I, i.e., $\liminf_{n\to\infty} (1/r_n) \ln Q_n(G) \geq -\inf_{x\in G} I(x)$ for all open sets G, and $\limsup_{n\to\infty} (1/r_n) \ln Q_n(F) \leq -\inf_{x\in F} I(x)$ for all closed sets F.

LD convergence of probability measures can be also expressed as LD convergence in distribution of the associated random variables to idempotent variables. A sequence $\{X_n,\ n \in \mathbb{N}\}$ of random variables with values in $\mathbf{E}'$ is said to LD converge in distribution at rate r_n as $n \to \infty$ to an idempotent variable X with values in $\mathbf{E}'$ if the sequence of the probability laws of the X_n LD converges at rate r_n to the idempotent distribution of X. If the random variables X'_n and X''_n are independent, X'_n LD converges to X', and X''_n LD converges to X", then the sequence (X'_n, X''_n) LD converges to (X',X") and X' and X" are independent. If a sequence $\{Q_n,\ n \in \mathbb{N}\}$ of probability measures LD converges to a deviability Π, then one has LD convergence in distribution of the canonical idempotent variables. A continuous mapping principle holds: if the random variables X_n LD converge at rate r_n to an idempotent variable X and $f: \mathbf{E}' \to \mathbf{E}''$ is a continuous function, then the random variables $f(X_n)$ LD converge at rate r_n to $f(X)$, where $\mathbf{E}''$ is a metric space. The following version of Slutsky's theorem holds; see, e.g., [19, Lemma 3.1.42,

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p. 275]. Certainly, a direct proof along the lines of the argument in [4, Chapter 1, Theorem 4.1] is possible.

Theorem 6. *Suppose that* $\mathbf{E}'$ *and* $\mathbf{E}''$ *are separable metric spaces. Let* Y_n *be random variables with values in* $\mathbb{E}''$, *and let* $a \in \mathbf{E}''$. *If the sequence* X_n *LD converges to* X *and* $Y_n \to a$ *super-exponentially in probability for rate* r_n, *i.e.,* $\mathbb{P}(d(Y_n, a) > \epsilon)^{1/r_n} \to 0$ *for arbitrary* $\epsilon > 0$, *then the sequence* (X_n, Y_n) *LD converges at rate* r_n *to (X,a) in* $\mathbf{E}' \times \mathbf{E}''$, *where d denotes the metric on* $\mathbf{E}''$.

A collection $(X_t,\ t \in \mathbb{R}_+)$ of idempotent variables on $\mathbf{E}$ is called an idempotent process. The functions $(X_t(y),\ t \in \mathbb{R}_+)$ for various $y \in \mathbf{E}$ are called trajectories (or paths) of X. Idempotent processes are said to be independent if they are independent as idempotent variables with values in the associated function spaces. The concepts of idempotent processes with independent and/or stationary increments mimic those for stochastic processes. Since this paper deals with stochastic processes having right-continuous trajectories with left-hand limits, the underlying space $\mathbf{E}$ may be assumed to be a Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^m)$, for suitable m.

Suppose $X_n = (X_n(t), t \in \mathbb{R}_+)$ is a sequence of stochastic processes that assume values in metric space $\mathbf{E}'$ with metric d' and have right-continuous trajectories with left-hand limits. The sequence X_n is said to be exponentially tight of order r_n if the sequence of the distributions of X_n as measures on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbf{E}')$ is exponentially tight of order r_n. It is said to be $\mathbb{C}$-exponentially tight if any LD limit point is the law of an idempotent process with continuous trajectories, i.e., $\Pi(\mathbb{D}(\mathbb{R}_+, \mathbf{E}') \setminus \mathbb{C}(\mathbb{R}_+, \mathbf{E}')) = 0$, whenever Π is an LD limit point of the laws of X_n.

The method of finite-dimensional distributions for LD convergence of stochastic processes is summarised in the next theorem [16]. The proof mimics the one used in weak convergence theory.

Theorem 7. *If, for all tuples* $t_1 < t_2 < \ldots < t_l$ *with the* t_i *coming from a dense subset of* $\mathbb{R}_+$, *the sequence* $(X_n(t_1), \ldots, X_n(t_l))$ *LD converges in* $\mathbb{R}^l$ *at rate* r_n *to* $(X(t_1), \ldots, X(t_l))$ *and the sequence* X_n *is* $\mathbb{C}$*-exponentially tight of order* r_n, *then* X *is a continuous-path idempotent process and the sequence* X_n *LD converges in* $\mathbb{D}(\mathbb{R}_+, \mathbf{E}')$ *at rate* r_n *to X.*

The form of the conditions in the next theorem, which is essentially due to [7], is at odds with what is common in weak convergence theory, so a proof is warranted, although the argument is standard.

Theorem 8. *Suppose* $\mathbf{E}'$ *is, in addition, complete and separable. The sequence* X_n *is* $\mathbb{C}$*-exponentially tight of order* r_n *if and only if the following hold:*

(i) *the sequence* $X_n(t)$ *is exponentially tight of order* r_n *for all t from a dense subset of* $\mathbb{R}_+$, *and*

(ii) *for all* $\epsilon > 0$ *and* $L > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{t \in [0,L]} \mathbb{P}(\sup_{s \in [0,\delta]} d'(X_n(t+s), X_n(t)) > \epsilon)^{1/r_n} = 0. \tag{A.1}$$

Proof. The necessity of the conditions follows from the continuity of the projection mapping and the continuous mapping principle. The sufficiency is proved next. For $L > 0$, $\delta > 0$ and X from the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbf{E}')$, let

$$w_L(X, \delta) = \sup_{t,s \in [0,L]:\, |t-s| \le \delta} d'(X(t), X(s)).$$

Since

$$\mathbb{P}(w_L(X_n,\delta)>\epsilon)\le \mathbb{P}(\bigcup_{i=0}^{\lfloor L/\delta\rfloor}\{3\sup_{t\in[i\delta,(i+1)\delta]\cap[0,L]}d'(X_n(t),X_n(i\delta))>\epsilon\})$$

$$\le \sum_{i=0}^{\lfloor L/\delta\rfloor}\mathbb{P}(3\sup_{t\in[i\delta,(i+1)\delta]\cap[0,L]}d'(X_n(t),X_n(i\delta))>\epsilon)$$

$$\le (\lfloor \frac{L}{\delta}\rfloor+1)\sup_{t\in[0,L]}\mathbb{P}(\sup_{s\in[t,t+\delta]}d'(X_n(s),X_n(t))>\frac{\epsilon}{3}),$$

the hypotheses imply that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}(w_L(X_n,\delta)>\epsilon)^{1/r_n}=0. \tag{A.2}$$

Let

$$w'_L(X,\delta)=\inf_{\substack{0=t_0<t_1<\ldots<t_k=L:\\ t_j-t_{j-1}>\delta}}\max_{j=1,\ldots,k}\sup_{u,v\in[t_{j-1},t_j)}d'(X(u),X(v)).$$

Since $w'_L(X,\delta)\le w_L(X,2\delta)$, provided $\delta<L/2$, by (A.2),

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}(w_L{}'(X_n,\delta)>\epsilon)^{1/r_n}=0. \tag{A.3}$$

As each X_n is a member of the Skorokhod space $\mathbb{D}(\mathbb{R}_+,\mathbb{E}')$,

$$\lim_{\delta\to 0}\mathbb{P}(w_L{}'(X_n,\delta)>\epsilon)=0,$$

so, by (A.3),

$$\lim_{\delta\to 0}\sup_n\mathbb{P}(w'_L(X_n,\delta)>\epsilon)^{1/r_n}=0.$$

Let $\{t_1,t_2,\ldots\}$ represent a dense subset of $\mathbb{R}_+$ such that $X_n(t_1),X_n(t_2),\ldots$ are exponentially tight of order r_n. Since $\mathbf{E}'$ is complete and separable, every probability measure on $\mathbf{E}'$ is tight, so it may be assumed that there exist compact subsets $K_1,K_2,\ldots$ such that, for all n,

$$\mathbb{P}(X_n(t_i)\notin K_i)^{1/r_n}<\frac{\epsilon}{2^i},\qquad i\in\mathbb{N}.$$

Choose positive $\delta_1,\delta_2,\ldots$ such that

$$\mathbb{P}(w_L{}'(X_n,\delta_i)>\frac{1}{2^i})^{1/r_n}<\frac{\epsilon}{2^i},\qquad i\in\mathbb{N}.$$

The set

$$A=\bigcap_{i\in\mathbb{N}}\{X\colon w'_L(X,\delta_i)\le\frac{1}{2^i},X(t_i)\in K_i\}$$

has compact closure (see, e.g., [10, Theorem A2.2, p. 563]), and

$$\mathbb{P}(X_n\notin A)^{1/r_n}\le\sum_{i=1}^{\infty}\mathbb{P}(w'_L(X_n,\delta_i)>\frac{1}{2^i})^{1/r_n}+\sum_{i=1}^{\infty}\mathbb{P}(X_n(t_i)\notin K_i)^{1/r_n}<2\epsilon.$$

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Thus, the sequence X_n is exponentially tight of order r_n in $\mathbb{D}(\mathbb{R}_+, \mathbf{E}')$. Let Π represent a deviability on $\mathbb{D}(\mathbb{R}_+, \mathbf{E}')$ that is an LD limit point of the distributions of X_n. It is proved next that $\Pi(X) = 0$ if X is a discontinuous function. Suppose X has a jump at t, i.e., $d'(X(t), X(t-)) > 0$. Let ρ denote a metric in $\mathbb{D}(\mathbb{R}_+, \mathbf{E}')$. It may be assumed that if $\rho(X', X) < \delta$, then there exists a continuous nondecreasing function $\lambda(t)$ such that $\sup_{s\le t+1} d'(X'(\lambda(s)), X(s)) < \delta$ and $\sup_{s\le t+1} |\lambda(s) - s| < \delta$; see, e.g., [6, 9]. Then, assuming that $2\delta < t$ and $\delta < 1$, we have $\inf_{s'\in[s-\delta,s+\delta]} d'(X(s'), X(s)) < \delta$, where $s \in [0, t+1]$. Note that

$$d'(X(t), X(t-)) \le d'(X(t), X'(s_1)) + d(X'(s_1), X'(t)) + d'(X'(t), X'(s_2)) \\ + d'(X'(s_2), X(s_3)) + d'(X(s_3), X(t-)),$$

so that, with $s_1 \in [t-\delta, t+\delta]$ such that $d'(X'(s_1), X(t)) < \delta$, $s_3 \in [t-\delta, t]$ such that $d'(X(s_3), X(t-)) < \delta$, and $s_2 \in [s_3-\delta, s_3+\delta]$ such that $d'(X'(s_2), X(s_3)) < \delta$, we have

$$d'(X(t), X(t-)) < 3\delta + 2\sup_{s\in[t-2\delta,t+\delta]} d(X'(s), X'(t)),$$

which implies that, for δ small enough,

$$\sup_{s\in[t-2\delta,t+\delta]} d'(X'(s), X'(t)) > \frac{d'(X(t), X(t-))}{3}.$$

Since

$$\begin{aligned}\sup_{s\in[t-2\delta,t+\delta]} d'(X'(s), X'(t)) &\le \sup_{s\in[t,t+\delta]} d'(X'(s), X'(t)) \\ &+ \sup_{s\in[t-2\delta,t-\delta]} d'(X'(s), X'(t)) + \sup_{s\in[t-\delta,t]} d'(X'(s), X'(t)) \\ &\le \sup_{s\in[t,t+\delta]} d'(X'(s), X'(t)) + \sup_{s\in[t-2\delta,t-\delta]} d'(X'(s), X'(t-2\delta)) \\ &+ \sup_{s\in[t-\delta,t]} d'(X'(s), X'(t-\delta)) + d'(X'(t-\delta), X'(t-2\delta)) + 2d'(X'(t), X'(t-\delta)),\end{aligned}$$

for $\epsilon = d'(X(t), X(t-))/18$,

$$\begin{aligned}\mathbb{P}(\rho(X_n, X) < \delta) &\le \mathbb{P}(\sup_{s\in[t,t+\delta]} d'(X_n(s), X_n(t)) > \epsilon) \\ &+ \mathbb{P}(\sup_{s\in[t-2\delta,t-\delta]} d'(X_n(s), X_n(t-2\delta)) > \epsilon) \\ &+ \mathbb{P}(\sup_{s\in[t-\delta,t]} d'(X_n(s), X_n(t-\delta)) > \epsilon).\end{aligned}$$

Therefore, assuming $L \ge t$ and $r_n \ge 1$,

$$\begin{aligned}\mathbb{P}(\rho(X_n, X) < \delta)^{1/r_n} &\le \mathbb{P}(\sup_{s\in[t,t+\delta]} d'(X_n(s), X_n(t)) > \epsilon)^{1/r_n} \\ &+ \mathbb{P}(\sup_{s\in[t-\delta,t]} d'(X_n(s), X_n(t-\delta)) > \epsilon)^{1/r_n} \\ &+ \mathbb{P}(\sup_{s\in[t-2\delta,t-\delta]} d'(X_n(s), X_n(t-2\delta)) > \epsilon)^{1/r_n} \\ &\le 3\sup_{t'\in[0,L]} \mathbb{P}(\sup_{s\in[0,\delta]} d'(X_n(t'+s), X_n(t')) > \epsilon)^{1/r_n}.\end{aligned}$$

Since

$$\Pi(X) = \lim_{\delta\to 0} \liminf_{n\to\infty} \mathbb{P}(\rho(X_n, X) < \delta)^{1/r_n},$$

(A.1) implies that $\Pi(X) = 0$.

The discussion below concerns the properties of the idempotent processes that feature prominently in the paper. All the processes assume values in $\mathbb{R}$. The standard Wiener idempotent process, denoted by $W = (W(t),\, t \in \mathbb{R}_+)$, is defined as an idempotent process with idempotent distribution

$$\Pi^W(w) = \exp\left(-\frac{1}{2}\int_0^\infty \dot{w}(t)^2\, dt\right),$$

provided $w = (w(t),\, t \in \mathbb{R}_+) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is absolutely continuous and $w(0) = 0$, and $\Pi^W(w) = 0$ otherwise. It is straightforward to show that W has stationary independent increments. The restriction to $[0, t]$ produces a standard Wiener idempotent process on $[0, t]$ which is specified by the idempotent distribution $\Pi_t^W(w) = \exp\big(-1/2 \int_0^t \dot{w}(s)^2\, ds\big)$. The Brownian bridge idempotent process on $[0, 1]$, denoted by $W^0 = (W^0(x),\ x \in [0, 1])$, is defined as an idempotent process with the idempotent distribution

$$\Pi^{W^0}(w^0) = \exp\left(-\frac{1}{2}\int_0^1 \dot{w}^0(x)^2\, dx\right),$$

provided $w^0 = (w^0(x),\, x \in [0, 1]) \in \mathbb{D}([0, 1], \mathbb{R})$ is absolutely continuous and $w^0(0) = w^0(1) = 0$, and $\Pi^{W^0}(w^0) = 0$ otherwise. The Brownian sheet idempotent process on $[0, 1] \times \mathbb{R}_+$ denoted by $(B(x, t),\, x \in [0, 1],\ t \in \mathbb{R}_+)$ is defined as a two-parameter idempotent process with the distribution

$$\Pi^B(b) = \exp\left(-\frac{1}{2}\int_{[0,1]\times\mathbb{R}_+} \dot{b}(x, t)^2\, dx\, dt\right),$$

provided $b = (b(x, t),\, x \in [0, 1],\, t \in \mathbb{R}_+)$ is absolutely continuous with respect to the Lebesgue measure on $[0, 1] \times \mathbb{R}_+$ and $b(x, 0) = b(0, t) = 0$, and $\Pi^B(b) = 0$ otherwise. The Kiefer idempotent process on $[0, 1] \times \mathbb{R}_+$, denoted by $(K(x, t),\, x \in [0, 1],\, t \in \mathbb{R}_+)$, is defined as a two-parameter idempotent process with the idempotent distribution

$$\Pi^K(k) = \exp\big(-\frac{1}{2}\int_{[0,1]\times\mathbb{R}_+} \dot{k}(x, t)^2\, dx\, dt\big),$$

provided $k = (k(x, t),\, x \in [0, 1],\, t \in \mathbb{R}_+)$ is absolutely continuous with respect to the Lebesgue measure on $[0, 1] \times \mathbb{R}_+$ and $k(0, t) = k(1, t) = k(x, 0) = 0$, and $\Pi^K(k) = 0$ otherwise. It is considered as an element of $\mathbb{D}(\mathbb{R}_+, \mathbb{D}([0, 1], \mathbb{R}_+))$. Furthermore, as the deviabilities that the idempotent processes W, W^0, B, and K have discontinuous paths are equal to zero, these idempotent processes can be considered as having paths from $\mathbb{C}(\mathbb{R}_+, \mathbb{R})$, $\mathbb{C}([0, 1], \mathbb{R})$, $\mathbb{C}(\mathbb{R}_+, \mathbb{C}([0, 1], \mathbb{R}))$, and $\mathbb{C}(\mathbb{R}_+, \mathbb{C}([0, 1], \mathbb{R}))$, respectively. Being LD limits of their stochastic prototypes, the idempotent processes introduced here have similar properties, as summarised in the next lemma.

Lemma 7.

(i) *For $x > 0$, the idempotent process $(B(x, t)/\sqrt{x},\, t \in \mathbb{R}_+)$ is a standard Wiener idempotent process.*

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(ii) *For $t > 0$, $(K(x, t)/\sqrt{t}, x \in [0, 1])$ is a Brownian bridge idempotent process. For $x \in (0, 1)$, the idempotent process $(K(x, t)/\sqrt{x(1-x)}, t \in \mathbb{R}_+)$ is a standard Wiener idempotent process.*

(iii) *The Kiefer idempotent process can be written as*

$$K(x, t) = -\int_0^x \frac{K(y, t)}{1-y}\, dy + B(x, t),\ x \in [0, 1], \tag{A.4}$$

where B(x,t) is a Brownian sheet idempotent process. Conversely, if B(x,t) is a Brownian sheet idempotent process and (A.4) holds, then K(x,t) is a Kiefer idempotent process. Similarly,

$$W^0(x) = -\int_0^x \frac{W^0(y)}{1-y}\, dy + W'(x),\ x \in [0, 1],$$

where W'(x) is a standard Wiener idempotent process.

Proof. Parts 1 and 2 are elementary. For instance,

$$\Pi\left(\left(\frac{K(x, t)}{\sqrt{t}}, x \in [0, 1]\right) = (w^0(x), x \in [0, 1])\right)$$
$$= \sup_{k:\, k(x,t)=\sqrt{t}w^0(x), x\in[0,1]} \exp\left(-\frac{1}{2}\int_{\mathbb{R}_+}\int_0^1 \dot{k}(x, s)^2\, dx\, ds\right).$$

An application of the Cauchy–Schwarz inequality shows that the optimal k is

$$k(x, s) = \frac{s \wedge t}{\sqrt{t}}\, w^0(x).$$

As for $(K(x, t)/\sqrt{x(1-x)}, t \in \mathbb{R}_+)$, the optimal trajectory $(k(y, t), y \in [0, 1])$ to get to $w(t)\sqrt{x(1-x)}$ at x is

$$k(y, t) = \begin{cases} w(t)\sqrt{x(1-x)}\,\dfrac{y}{x} & \text{if } y \in [0, x], \\[2mm] w(t)\sqrt{x(1-x)}\,\dfrac{1-y}{1-x} & \text{if } y \in [x, 1]. \end{cases}$$

To prove Part 3, it suffices to show that if

$$k(x, t) = -\int_0^x \frac{k(y, t)}{1-y}\, dy + b(x, t), \tag{A.5}$$

with k and b being absolutely continuous and with $\Pi^B(b) > 0$, then $k(1, t) = 0$ and

$$\int_0^\infty \int_0^1 \dot{k}(x, t)^2\, dx\, dt = \int_0^\infty \int_0^1 \dot{b}(x, t)^2\, dx\, dt. \tag{A.6}$$

Solving (A.5) yields, for $x < 1$,

$$k(x, t) = (1-x)\int_0^x \frac{b_y(y, t)}{1-y}\, dy, \tag{A.7}$$

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where $b_y(y,t) = \int_0^t \dot b(s,y)\,ds$. By the Cauchy–Schwarz inequality,

$$|\int_0^x \frac{b_y(y,t)}{1-y}\,dy| \le \sqrt{\int_0^x \frac{1}{(1-y)^2}\,dy}\sqrt{\int_0^1 b_y(y,t)^2 dy} \le \frac{\sqrt{t}}{\sqrt{1-x}}\sqrt{\int_{[0,1]\times\mathbb{R}_+} \dot b(y,s)^2\,dy\,ds}.$$

It follows that $k(x,t) \to 0$ as $x \to 1$, provided $\Pi^B(b) > 0$.

Let $k_t(x,t) = \int_0^x \dot k(y,t)\,dy$. We show first that if $\Pi^B(b) > 0$, then, a.e.,

$$\frac{1}{1-x} k_t(x,t)^2 \to 0 \text{ as } x \to 1. \tag{A.8}$$

Given arbitrary $a > 0$, by (A.7),

$$\begin{aligned}\frac{k_t(x,t)^2}{1-x} &= (1-x)\Big(\int_0^x \frac{\dot b(y,t)\,dy}{1-y}\Big)^2 \le 2(1-x)a^2\Big(\int_0^x \frac{dy}{1-y}\Big)^2 \\ &\quad + 2(1-x)\int_0^x \frac{dy}{(1-y)^2}\int_0^x \dot b(y,t)^2\,\mathbf{1}_{\{|\dot b(y,t)|>a\}}\,dy \\ &\le 2(1-x)|\ln(1-x)|^2 a^2 + 2\int_0^x \dot b(y,t)^2\,\mathbf{1}_{\{|\dot b(y,t)|>a\}}\,dy.\end{aligned}$$

Since $\int_0^1 \dot b(y,t)^2\,dy < \infty$ a.e., the latter right-hand side tends to 0 a.e., as $x \to 1$ and $a \to \infty$.

Next we prove (A.6). By (A.5),

$$\int_0^\infty \int_0^1 \dot b(x,t)^2\,dx\,dt = \int_0^\infty \int_0^1 \left(\dot k(x,t)^2 + 2\dot k(x,t)\frac{k_t(x,t)}{1-x} + \left(\frac{k_t(x,t)}{1-x}\right)^2\right)dx\,dt. \tag{A.9}$$

Integration by parts with the account of (A.8) yields, for almost all t,

$$\int_0^1 \dot k(x,t)\frac{k_t(x,t)}{1-x}\,dx = -\int_0^1 k_t(x,t)\left(\frac{\dot k(x,t)}{1-x} + \frac{k_t(x,t)}{(1-x)^2}\right)dx,$$

so that

$$\int_0^1 \dot k(x,t)\frac{k_t(x,t)}{1-x}\,dx = -\frac{1}{2}\int_0^1 \frac{k_t(x,t)^2}{(1-x)^2}\,dx.$$

Recalling (A.9) implies (A.6).

Appendix B. A nonlinear renewal equation

This section is concerned with the properties of the equation

$$g(t) = f(t) + \int_0^t g(t-s)^+\,dF(s),\; t \in \mathbb{R}_+. \tag{B.1}$$

It is assumed that $f(t)$ is a locally bounded measurable function and that $F(t)$ is a continuous distribution function on $\mathbb{R}_+$ with $F(0) = 0$. The existence and uniqueness of an essentially locally bounded solution $g(t)$ to (B.1) follows from Lemma B.2 in [21].

Lemma 8. *If the functions f and F are absolutely continuous with respect to Lebesgue measure, then the function g is absolutely continuous too.*

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Proof. Use Picard iterations. Let $g_0(t) = f(t)$ and

$$g_k(t) = f(t) + \int_0^t g_{k-1}(t-s)^+ \, dF(s). \tag{B.2}$$

The functions g_k are seen to be continuous. Let $\epsilon > 0$, $T > 0$, and $0 \le t_0 \le t_1 \le \ldots \le t_l \le T$. Since $g_k \to g$ locally uniformly (see [21, Lemma B.1]), the function g is continuous and $\sup_k \sup_{t\in[0,T]} |g_k(t)| \le M$ for some $M > 0$. Note that

$$\begin{aligned}
&|\int_0^{t_i} g_{k-1}(t_i - s)^+ \, dF(s) - \int_0^{t_{i-1}} g_{k-1}(t_{i-1} - s)^+ \, dF(s)| \\
&\quad \le \int_{t_{i-1}}^{t_i} |g_{k-1}(t_i - s)| \, dF(s) + \int_0^T \mathbf{1}_{\{s \le t_{i-1}\}} \, |g_{k-1}(t_i - s) - g_{k-1}(t_{i-1} - s)| \, dF(s).
\end{aligned} \tag{B.3}$$

Let

$$\psi(\delta) = \sup_{0\le t_0 \le t_1 \le \ldots \le t_l \le T} \{ \sum_{i=1}^{l} |f(t_i) - f(t_{i-1})| + M \sum_{i=1}^{l} |F(t_i) - F(t_{i-1})| \colon \sum_{l=1}^{l} (t_i - t_{i-1}) \le \delta \}$$

and

$$\phi_k(\delta) = \sup_{0\le t_0 \le t_1 \le \ldots \le t_l \le T} \{ \sum_{i=1}^{l} |g_k(t_i) - g_k(t_{i-1})| \colon \sum_{l=1}^{l} (t_i - t_{i-1}) \le \delta \}.$$

By (B.2) and (B.3), for $k \ge 1$,

$$\phi_k(\delta) \le \psi(\delta) + \phi_{k-1}(\delta) F(T). \tag{B.4}$$

Let

$$\phi(\delta) = \sup_{0\le t_0 \le t_1 \le \ldots \le t_l \le T} \{ |g(t_i) - g(t_{i-1})| \colon \sum_{l=1}^{l} (t_i - t_{i-1}) \le \delta \}.$$

Suppose that $F(T) < 1$. Since $g_k \to g$ locally uniformly, as $k \to \infty$, $\phi_k(\delta) \to \phi(\delta)$. Letting $k \to \infty$ in (B.4) implies that $\phi(\delta) \le \psi(\delta)/(1 - F(T))$, so that $\phi(\delta) \to 0$, as $\delta \to 0$. Hence $g(t)$ is absolutely continuous on $[0, T]$. Next, as in [21], write, for $t \in [0, T]$,

$$g(t+T) = f(t+T) + \int_t^{t+T} g(t+T-s) \, dF(s) + \int_0^t g(t+T-s) \, dF(s).$$

By what has been proved, the sum of the first two terms on the right-hand side is an absolutely continuous function of t on $[0, T]$. The preceding argument implies that $g(t+T)$ is absolutely continuous in t on $[0, T]$. Iterating the argument proves the absolute continuity of $g(t)$ on $\mathbb{R}_+$.

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